OLYMPIAD SOLUTIONS

OC76. For any positive integer n, let a_n be the exponent of the largest power of 2 which occurs as a factor of $5^n - 3^n$. Also, let b_n be the exponent of the largest power of 2 which divides n. Show that

$$a_n \leq b_n + 3$$

for all n.
(*Originally question 1 from the 2011 British IMO selection, Day 2.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Norvald Midttun, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; Daniel Văcaru, Piteşti, Romania; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Midttun.

If n is odd, then

$$5^n - 3^n \equiv 1 - 3 \equiv 2 \pmod 4 .$$

Therefore $a_n = 1$ and $b_n = 0$, thus the inequality holds.

Now, let $n = 2^m \cdot q$ with q odd and $m \geq 1$. Then

$$\begin{aligned} 5^{2^m \cdot q} - 3^{2^m \cdot q} &= \left(5^{2^{m-1}\cdot q} + 3^{2^{m-1}\cdot q}\right)\left(5^{2^{m-1}\cdot q} - 3^{2^{m-1}\cdot q}\right) \\ &= \left(5^{2^{m-1}\cdot q} + 3^{2^{m-1}\cdot q}\right)\left(5^{2^{m-2}\cdot q} + 3^{2^{m-2}\cdot q}\right)\left(5^{2^{m-2}\cdot q} - 3^{2^{m-2}\cdot q}\right) \\ &\vdots \\ &= \left(5^{2^{m-1}\cdot q} + 3^{2^{m-1}\cdot q}\right)\left(5^{2^{m-2}\cdot q} + 3^{2^{m-2}\cdot q}\right)\left(5^{2^{m-3}\cdot q} + 3^{2^{m-3}\cdot q}\right) \times \\ &\qquad \cdots \left(5^{2\cdot q} + 3^{2\cdot q}\right)\left(5^q + 3^q\right)\left(5^q - 3^q\right) \quad (1) \end{aligned}$$

As all odd perfect squares are congruent to 1 (mod 4) we have

$$\begin{aligned} 5^{2^{m-1}\cdot q} + 3^{2^{m-1}\cdot q} &\equiv 5^{2^{m-2}\cdot q} + 3^{2^{m-2}\cdot q} \\ &\equiv 5^{2^{m-3}\cdot q} + 3^{2^{m-3}\cdot q} \equiv \cdots \equiv 5^{2\cdot q} + 3^{2\cdot q} \equiv 2 \pmod 4 \end{aligned}$$

Therefore, the exponent of the largest power of 2 that divides

$$\left(5^{2^{m-1}\cdot q} + 3^{2^{m-1}\cdot q}\right)\left(5^{2^{m-2}\cdot q} + 3^{2^{m-2}\cdot q}\right)\left(5^{2^{m-3}\cdot q} + 3^{2^{m-3}\cdot q}\right)\cdots\left(5^{2\cdot q} + 3^{2\cdot q}\right)$$

is $m - 2$.

Since q is odd, we have exactly as in the first part of the proof

$$5^q - 3^q \equiv 2 \pmod 4 .$$

We claim that

$$5^q + 3^q \equiv 8 \pmod{16} .$$

Indeed, let $q = 2k+1$. Then

$$5^{2k+1} + 3^{2k+1} \equiv 5 \cdot 25^k + 3 \cdot 9^k \equiv 5 \cdot 9^k + 3 \cdot 9^k \equiv 8 \cdot 9^k \equiv 8 \pmod{16} .$$

This shows that in this case $a_n = b_n + 3$.

OC77. Find all functions $f : (0, \infty) \to (0, \infty)$ so that for all $x, y \in (0, \infty)$ we have

$$f(x)f(y) = f(y)f\left(xf(y)\right) + \frac{1}{xy} .$$

(*Originally question 6 from the 2011 Czech Republic Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France.

It is easy to check that $f(x) = x + \dfrac{1}{x}$ is a solution. We show that there is no other solution.

Let $f : (0, \infty) \to (0, \infty)$ be any solution. Replacing x by $\dfrac{x}{f(y)}$ we get

$$f\left(\frac{x}{f(y)}\right) = f(x) + \frac{1}{xy} \tag{2}$$

therefore

$$f\left(\frac{1}{f(y)}\right) = a + \frac{1}{y} \tag{3}$$

where $a = f(1)$. It follows from (3) that f is injective.

Setting $y = 1$ in (3) we get

$$f\left(\frac{1}{a}\right) = a + 1 \tag{4}$$

while setting $y = 1, x = \dfrac{1}{a}$ in (2) yields

$$f\left(\frac{1}{a^2}\right) = f\left(\frac{1}{a}\right) + a \tag{5}$$

therefore

$$f\left(\frac{1}{a^2}\right) = 2a + 1 .$$

On another hand, setting $y = \dfrac{1}{a+1}$ in (3) yields

$$f\left(\frac{1}{f(\frac{1}{a+1})}\right) = 2a + 1 .$$

As f is injective, we get

$$f\left(\frac{1}{a+1}\right) = a^2$$

therefore

$$f\left(\frac{1}{f(a)}\right) = a^2\,.$$

Setting $y = \dfrac{1}{a}$ in (3) yields

$$f\left(\frac{1}{f(\frac{1}{a})}\right) = 2a\,. \tag{6}$$

This shows that $a^2 = 2a$ and hence, as $a = f(1) > 0$ we get $a = 2$.

Now, replacing x by $\dfrac{1}{f(x)}$ and y by 1 in the original relation we get

$$f\left(\frac{1}{f(x)}\right) = f\left(\frac{2}{f(x)}\right) + \frac{f(x)}{2}\,.$$

Now, combining (3) with $a = 2$ we have

$$f\left(\frac{1}{f(x)}\right) = 2 + \frac{1}{x}\,.$$

Moreover, from (2) we get

$$f\left(\frac{2}{f(x)}\right) = f(2) + \frac{1}{2x}\,.$$

Therefore

$$2 + \frac{1}{x} = f(2) + \frac{1}{2x} + \frac{f(x)}{2}$$

or

$$f(x) = 4 - 2f(2) + \frac{1}{x}\,.$$

Now, setting in the given relation $x = y = 1$ we get

$$2^2 = 2f(2) + 1 \Rightarrow 2f(2) = 3\,,$$

which shows that

$$f(x) = 1 + \frac{1}{x}\,.$$

This completes the proof.

OC78. Let $a_1 = 1, a_2 = 5, a_3 = 14, a_4 = 19, \ldots$ be the sequence of positive integers starting with 1, followed by all integers with the sum of the digits divisible by 5. Prove that for all n we have

$$a_n \leq 5n\,.$$

(*Originally question 4 from 2011 Kazahstan National Olympiad, Grade 9.*)

Solved by Oliver Geupel, Brühl, NRW, Germany; and Daniel Văcaru, Piteşti, Romania. There was one incomplete solution. We give the writeup from Geupel.

For any positive integer n, define the set

$$A_n = \{5n-5,\ 5n-4,\ 5n-3,\ 5n-2,\ 5n-1\}.$$

We have $a_1 \in A_1$.

Moreover, for each $n \geq 2$, the elements in A_n only differ in the last digit, thus the sums of digits of the members of A_n are in distinct residue classes modulo 5.

Therefore, exactly one member of each A_n has a sum of digits that is divisible by 5. Consequently, $a_n \in A_n$ for $n = 1, 2, \ldots$, which implies

$$a_n \leq \max A_n = 5n - 1.$$

This completes the proof.

OC79. Let D be a point different from the vertices on the side BC of a ΔABC. Let I, I_1 and I_2 be the incenters of $\Delta ABC, \Delta ABD$ respectively ΔADC. Let E be the second intersection point of the circumcircles of ΔAI_1I and ΔADI_2, and let F be the second intersection point of the circumcircles of ΔAII_2 and ΔAI_1D. If $AI_1 = AI_2$, prove that

$$\frac{EI}{FI} \cdot \frac{ED}{FD} = \frac{{EI_1}^2}{{FI_2}^2}\,.$$

(*Originally question 1 from the 2011 Turkey Team Selection Test, Day 2.*)

No solution to this problem was received.

OC80. Let G be a simple graph with $3n^2$ vertices ($n \geq 2$), such that the degree of each vertex of G is not greater than $4n$, there exists at least one vertex of degree one, and between any two vertices, there is a path of length ≤ 3. Prove that the minimum number of edges that G might have is equal to $\dfrac{7n^2 - 3n}{2}$. (*Originally question 3 from 2011 China Team Selection Test, Quiz 3, Day 1.*)

No solution to this problem was received.