CONTEST CORNER SOLUTIONS

CC26. A function f is defined in such a way that $f(1) = 2$, and for each positive integer $n > 1$,

$$f(1) + f(2) + f(3) + \cdots + f(n) = n^2 f(n).$$

Determine the value of $f(2013)$.
(*Inspired by question 5 from the 1994 CMC Invitational Mathematics Challenge, Grade 11.*)

Solved by George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain; Greg Cook, Angelo State University, San Angelo, TX, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA;
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We present Heuver's solution below.

For $n \geq 2$, we can write

$$f(1) + f(2) + f(3) + \cdots + f(n) = n^2 f(n) \tag{1}$$

and

$$f(1) + f(2) + f(3) + \cdots f(n-1) = (n-1)^2 f(n-1). \tag{2}$$

Subtracting (2) from (1) and rearranging for $f(n)$ we get

$$f(n) = \frac{(n-1)^2}{n^2-1} f(n-1) = \frac{n-1}{n+1} f(n-1). \tag{3}$$

By considering the recurrence relation in (3), we obtain

$$f(n) = \frac{2(n-1)!}{(n+1)!} f(1) = \frac{4}{n(n+1)}. \tag{4}$$

It follows from (4) that $f(2013) = \dfrac{4}{2013 \cdot 2014} = \dfrac{2}{2027091}$.

CC27. A $n \times n \times n$ cube has its faces ruled into n^2 unit squares. A path is to be traced on the surface of the cube starting at $(0, 0, 0)$ and ending at (n, n, n) moving only in a positive sense along the ruled lines. Determine the number of distinct paths.
(*Originally question 10 b) from the 1986 Descartes Contest.*)

No solutions were received.

CC28. The quartic polynomial $P(x)$ satisfies $P(1) = 0$ and attains its maximum value of 3 at both $x = 2$ and $x = 3$. Compute $P(5)$.
(*Originally question 5 from the 2012 Stanford Math Tournament, Algebra Problems.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Leonard Giugiuc, Romania; Richard I. Hess, Rancho Palos Verdes, CA, USA; David Jonathan, SMA Xaverius 1, Palembang, Indonesia; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. Two incorrect solutions were received.

We present two solutions. The first solution is a composite of solutions by Hess, Jonathan, Manes, Wang and Zvonaru.

Let $P(x) = ax^4 + bx^3 + cx^2 + dx + e$. We know, from the conditions of the problem, that $P(1) = 0$, $P(2) = P(3) = 3$ and $P'(2) = P'(3) = 0$. Substituting into $P(x)$ and $P'(x)$ we obtain the system of equations

$$\begin{aligned} a + b + c + d + e &= 0 \\ 16a + 8b + 4c + 2d + e &= 3 \\ 81a + 27b + 9c + 3d + e &= 3 \\ 32a + 12b + 4c + d &= 0 \\ 108a + 27b + 6c + d &= 0. \end{aligned}$$

Solving the system yields $a = -\frac{3}{4}$, $b = \frac{15}{2}$, $c = -\frac{111}{4}$, $d = 45$, and $e = -24$, so

$$P(x) = \frac{3}{4}x^4 + \frac{15}{2}x^3 - \frac{111}{4}x^2 + 45x - 24$$

and hence $P(5) = -24$.

Next we present a composite of the solutions from Curtis and Giugiuc.

Let $Q(x) = P(x) - 3$, then Q is a quartic polynomial that satisfies

$$Q(2) = Q(3) = 0, \quad \text{and} \quad Q'(2) = Q'(3) = 0.$$

Hence, 2 and 3 are double roots of Q, so

$$Q(x) = k(x - 2)^2(x - 3)^2$$

for some constant k, and thus

$$P(x) = k(x-2)^2(x-3)^2 + 3.$$

Using $P(1) = 0$ we obtain $k = -\frac{3}{4}$, so that

$$P(x) = -\frac{3}{4}(x-2)^2(x-3)^2 + 3,$$

whence

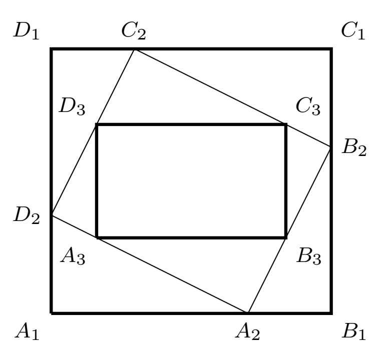
$$P(5) = -24.$$

CC29. Consider three parallelograms P_1, P_2, P_3. Parallelogram P_3 is inside parallelogram P_2, and the vertices of P_3 are on the edges of P_2. Parallelogram P_2 is inside parallelogram P_1, and the vertices of P_2 are on the edges of P_1. The sides of P_3 are parallel to the sides of P_1. Prove that one side of P_3 has length at least half the length of the parallel side of P_1.
(*Originally question 8 from the 2010 Sun Life Financial Repêchage Competition.*)

Solved by J. Chris Fisher, University of Regina, Regina, SK; and Titu Zvonaru, Cománeşti, Romania. No other solutions were received. We use the solution of Zvonaru, modified by the editor.

Let $P_i = A_iB_iC_iD_i$ for $i = 1, 2, 3$. Since affine transformations preserve the ratios of segment lengths along parallel lines, we may suppose that P_1, P_2, and P_3 are rectangles. [First map the outer parallelogram P_1 to a rectangle, in which case P_3 would also become a rectangle; then adjust the height A_1D_1 so that the diagonals A_2C_2 and B_2D_2 become equal, which forces P_2 to become a rectangle. This can always be accomplished since A_2C_2 grows from small to large and B_2D_2 remains fixed as A_1D_1 increases from 0 to infinity; thus, at some point in between they will be equal.]

Let $A_iB_iC_iD_i$ be the vertices of the rectangle P_i for $i = 1, 2, 3$.



We choose a system of coordinates such that $A_3(0, 0)$, $B_3(a, 0)$, $C_3(a, b)$, $D_3(0, b)$, where we assume that, without loss of generality, $a \geq b$.

Let $m > 0$ be the slope of the line A_2B_2, then the slope of the line A_2D_2 is $-\dfrac{1}{m}$. In order to find the coordinates of A_2, B_2, D_2, we have to solve the following systems:

$$A_2 : \begin{cases} y = -\dfrac{1}{m}x \\ y = m(x-a) \end{cases} ; \qquad B_2 : \begin{cases} y - b = -\dfrac{1}{m}(x-a) \\ y = m(x-a) \end{cases} ;$$

$$D_2 : \begin{cases} y - b = mx \\ y = -\dfrac{1}{m}x \end{cases} .$$

We obtain

$$A_2\left(\frac{am^2}{m^2+1}, -\frac{am}{m^2+1}\right),\ B_2\left(\frac{am^2+bm+a}{m^2+1}, \frac{bm^2}{m^2+1}\right),\ D_2\left(-\frac{bm}{m^2+1}, \frac{b}{m^2+1}\right).$$

It follows that

$$A_1\left(-\frac{bm}{m^2+1}, -\frac{am}{m^2+1}\right), \qquad B_1\left(\frac{am^2+bm+a}{m^2+1}, -\frac{am}{m^2+1}\right)$$

Since $A_3B_3 = a$ and $A_1B_1 = \dfrac{am^2+bm+a}{m^2+1} + \dfrac{bm}{m^2+1}$, it remains to prove that

$$\frac{am^2+2bm+a}{m^2+1} \le 2a \Leftrightarrow 2mb \le a(m^2+1),$$

which is true because $b \le a$ and $2m \le m^2+1$.

CC30. Two polite but vindictive children play a game as follows. They start with a bowl containing N candies, the number known to both contestants. In turn, each child takes (if possible) one or more candies, subject to the rule that no child may take, on any one turn, more than half of what is left. The winner is not the child who gets most candy, but the last child who gets to take some.
Thus, if there are 3 candies, the first player may only take one, as two would be more than half. The second player may take one of the remaining candies; and the first player cannot move and loses.
(a) Show that if the game begins with 2000 candies the first player wins.
(b) Show that if the game begins with $999\cdots999$ (2000 9's) candies, the first player wins.
(*Originally question 3 from the 2000 APICS contest.*)

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

We claim that if $N = 2^k - 1$ for some positive integer k, then the first player loses, and otherwise the first player wins. We prove this by induction on k. Our induction hypothesis is that for each positive integer k, a player whose turn starts with $2^k - 1$ candies loses, while a player wins if their turn starts with N candies, for $2^k - 1 < N < 2^{k+1} - 1$.

When $N = 1$ a player has no legal moves, and so that player loses. When $N = 2$, the player can take 1 candy, leaving the other player with 1, so the first player wins. This shows that our claim is true for $k = 1$.

Suppose that the claim is true for $k = m$, and consider $k = m+1$. If a player begins the turn with $N = 2^{m+1} - 1$ candies, then the allowable moves consist of taking r candies, where

$$r \in \{1, 2, 3, \dots, 2^m - 1\}.$$

The number of candies remaining after this move is an element of

$$\{2^m, 2^m + 1, \dots, 2^{m+1} - 2\}.$$

By the induction hypothesis, the second player is able to win in any of these games, so the first player loses when $N = 2^{m+1} - 1$.

When a player begins a turn with N candies, for $2^{m+1} - 1 < N < 2^{m+2} - 1$, they are able to take $N - (2^{m+1} - 1)$ candies, leaving $2^{m+1} - 1$ candies. This is a losing position for the second player, so the first player wins.

Since neither 2000 nor $10^{2000} - 1$ is of the form $2^k - 1$ for a positive integer k, they are both winning positions for the first player.