

THE CONTEST CORNER

No. 16

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Nous préférons les réponses électroniques et demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_CCNuméro du problème (exemple : Tremblay_Julie_CC1234.tex). De préférence, les lecteurs enverront un fichier au format \LaTeX et un fichier pdf pour chaque solution, bien que les autres formats (Word, etc.) soient aussi acceptés. Nous invitons les lecteurs à envoyer leurs solutions et réponses aux concours au rédacteur à l'adresse crux-contest@smc.math.ca. Nous acceptons aussi les contributions par la poste, envoyées à l'adresse figurant en troisième de couverture. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays; chaque solution doit également commencer sur une nouvelle page.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er octobre 2014**; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

CC76. Le point $P(a, b)$ est situé dans le premier quadrant. Une droite passant par P coupe les axes aux points Q et R de manière que le triangle OQR ait une aire de $2ab$, O étant l'origine. Démontrer qu'il y a trois droites possibles qui satisfont à cette condition.

CC77. Trois cercles sont tangents l'un à l'autre. Le premier cercle a pour rayon a , le deuxième a pour rayon b et le troisième a pour rayon $a + b$, a et b étant des nombres réels et $a, b > 0$. Déterminer le rayon d'un quatrième cercle tangent à chacun de ces trois cercles.

CC78. Soit $g(x) = x^3 + px^2 + qx + r$, p , q et r étant des entiers. Démontrer que si $g(0)$ et $g(1)$ sont tous deux impairs, alors l'équation $g(x) = 0$ ne peut admettre trois racines entières.

CC79. Démontrer que si n est un entier supérieur à 1, alors $n^4 + 4$ n'est pas un nombre premier.

CC80. Alphonse et Bérénice s'amuse avec n coffres-forts. Chaque coffre-fort peut être ouvert à l'aide d'une clé unique et chaque clé peut ouvrir un seul coffre-fort. Bérénice mêle les n clés au hasard, place une clé à l'intérieur de chaque coffre-fort, puis elle ferme chaque coffre-fort à l'aide de sa passe-partout. Alphonse choisit ensuite m coffres-forts ($m < n$) et Bérénice ouvre ces m coffres-forts à l'aide de la passe-partout. Alphonse prend les clés à l'intérieur de ces m coffres-forts et tente d'ouvrir les $n - m$ autres coffres-forts à l'aide de ces clés. Chaque fois qu'il réussit à ouvrir un coffre-fort, il peut utiliser la clé à l'intérieur de celui-ci pour tenter d'en ouvrir un autre. Il continue jusqu'à ce qu'il ait ouvert tous les coffres-forts ou qu'il ne puisse plus en ouvrir un autre. Soit $P_m(n)$ la probabilité pour qu'Alphonse puisse ouvrir tous les n coffres-forts à partir des m clés disponibles lors de son choix de m coffres-forts. Déterminer une formule pour $P_2(n)$.

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CC76. The point $P(a, b)$ lies in the first quadrant. A line, drawn through P , cuts the axes at Q and R such that the area of triangle OQR is $2ab$, where O is the origin. Prove that there are three such lines that satisfy these criteria.

CC77. The three following circles are tangent to each other: the first has radius a , the second has radius b , and the third has radius $a + b$ for some $a, b \in \mathbb{R}$ with $a, b > 0$. Find the radius of a fourth circle tangent to each of these three circles.

CC78. Let $g(x) = x^3 + px^2 + qx + r$, where p, q and r are integers. Prove that if $g(0)$ and $g(1)$ are both odd, then the equation $g(x) = 0$ cannot have three integer roots.

CC79. Show that if n is an integer greater than 1, then $n^4 + 4$ is not prime.

CC80. Alphonse and Beryl play a game involving n safes. Each safe can be opened by a unique key and each key opens a unique safe. Beryl randomly shuffles the n keys, and after placing one key inside each safe, she locks all of the safes with her master key. Alphonse then selects m of the safes (where $m < n$), and Beryl uses her master key to open just the safes that Alphonse selected. Alphonse collects all of the keys inside these m safes and tries to use these keys to open up the other $n - m$ safes. If he can open a safe with one of the m keys, he can then use the key in that safe to try to open any of the remaining safes, repeating the process until Alphonse successfully opens all of the safes, or cannot open any more. Let $P_m(n)$ be the probability that Alphonse can eventually open all n safes starting from his initial selection of m keys. Determine a formula for $P_2(n)$.



CONTEST CORNER SOLUTIONS

CC26. A function f is defined in such a way that $f(1) = 2$, and for each positive integer $n > 1$,

$$f(1) + f(2) + f(3) + \cdots + f(n) = n^2 f(n).$$

Determine the value of $f(2013)$.

(Inspired by question 5 from the 1994 CMC Invitational Mathematics Challenge, Grade 11.)

Solved by George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain; Greg Cook, Angelo State University, San Angelo, TX, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA;

Marian Dincă, Bucharest, Romania; Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany; Leonard Giugiuc, Romania; Richard I. Hess, Rancho Palos Verdes, CA, USA; John G. Heuver, Grande Prairie, AB; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Norvald Midttun, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; Mihaï-Ioan Stoënescu, Bischwiller, France; Daniel Văcaru, Pitești, Romania; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania.

We present Heuver's solution below.

For $n \geq 2$, we can write

$$f(1) + f(2) + f(3) + \cdots + f(n) = n^2 f(n) \tag{1}$$

and

$$f(1) + f(2) + f(3) + \cdots + f(n-1) = (n-1)^2 f(n-1). \tag{2}$$

Subtracting (2) from (1) and rearranging for $f(n)$ we get

$$f(n) = \frac{(n-1)^2}{n^2-1} f(n-1) = \frac{n-1}{n+1} f(n-1). \tag{3}$$

By considering the recurrence relation in (3), we obtain

$$f(n) = \frac{2(n-1)!}{(n+1)!} f(1) = \frac{4}{n(n+1)}. \tag{4}$$

It follows from (4) that $f(2013) = \frac{4}{2013 \cdot 2014} = \frac{2}{2027091}$.

CC27. A $n \times n \times n$ cube has its faces ruled into n^2 unit squares. A path is to be traced on the surface of the cube starting at $(0, 0, 0)$ and ending at (n, n, n) moving only in a positive sense along the ruled lines. Determine the number of distinct paths.

(Originally question 10 b) from the 1986 Descartes Contest.)

No solutions were received.

CC28. The quartic polynomial $P(x)$ satisfies $P(1) = 0$ and attains its maximum value of 3 at both $x = 2$ and $x = 3$. Compute $P(5)$.

(Originally question 5 from the 2012 Stanford Math Tournament, Algebra Problems.)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Leonard Giugiuc, Romania; Richard I. Hess, Rancho Palos Verdes, CA, USA; David Jonathan, SMA Xaverius 1, Palembang, Indonesia; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. Two incorrect solutions were received.

We present two solutions. The first solution is a composite of solutions by Hess, Jonathan, Manes, Wang and Zvonaru.

Let $P(x) = ax^4 + bx^3 + cx^2 + dx + e$. We know, from the conditions of the problem, that $P(1) = 0$, $P(2) = P(3) = 3$ and $P'(2) = P'(3) = 0$. Substituting into $P(x)$ and $P'(x)$ we obtain the system of equations

$$\begin{aligned} a + b + c + d + e &= 0 \\ 16a + 8b + 4c + 2d + e &= 3 \\ 81a + 27b + 9c + 3d + e &= 3 \\ 32a + 12b + 4c + d &= 0 \\ 108a + 27b + 6c + d &= 0. \end{aligned}$$

Solving the system yields $a = -\frac{3}{4}$, $b = \frac{15}{2}$, $c = -\frac{111}{4}$, $d = 45$, and $e = -24$, so

$$P(x) = \frac{3}{4}x^4 + \frac{15}{2}x^3 - \frac{111}{4}x^2 + 45x - 24$$

and hence $P(5) = -24$.

Next we present a composite of the solutions from Curtis and Giugiuc.

Let $Q(x) = P(x) - 3$, then Q is a quartic polynomial that satisfies

$$Q(2) = Q(3) = 0, \quad \text{and} \quad Q'(2) = Q'(3) = 0.$$

Hence, 2 and 3 are double roots of Q , so

$$Q(x) = k(x - 2)^2(x - 3)^2$$

for some constant k , and thus

$$P(x) = k(x-2)^2(x-3)^2 + 3.$$

Using $P(1) = 0$ we obtain $k = -\frac{3}{4}$, so that

$$P(x) = -\frac{3}{4}(x-2)^2(x-3)^2 + 3,$$

whence

$$P(5) = -24.$$

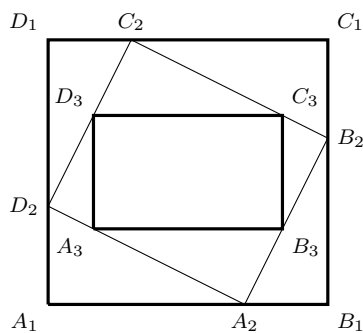
CC29. Consider three parallelograms P_1, P_2, P_3 . Parallelogram P_3 is inside parallelogram P_2 , and the vertices of P_3 are on the edges of P_2 . Parallelogram P_2 is inside parallelogram P_1 , and the vertices of P_2 are on the edges of P_1 . The sides of P_3 are parallel to the sides of P_1 . Prove that one side of P_3 has length at least half the length of the parallel side of P_1 .

(Originally question 8 from the 2010 Sun Life Financial Repêchage Competition.)

Solved by J. Chris Fisher, University of Regina, Regina, SK; and Titu Zvonaru, Comănești, Romania. No other solutions were received. We use the solution of Zvonaru, modified by the editor.

Let $P_i = A_iB_iC_iD_i$ for $i = 1, 2, 3$. Since affine transformations preserve the ratios of segment lengths along parallel lines, we may suppose that P_1, P_2 , and P_3 are rectangles. [First map the outer parallelogram P_1 to a rectangle, in which case P_3 would also become a rectangle; then adjust the height A_1D_1 so that the diagonals A_2C_2 and B_2D_2 become equal, which forces P_2 to become a rectangle. This can always be accomplished since A_2C_2 grows from small to large and B_2D_2 remains fixed as A_1D_1 increases from 0 to infinity; thus, at some point in between they will be equal.]

Let $A_iB_iC_iD_i$ be the vertices of the rectangle P_i for $i = 1, 2, 3$.



We choose a system of coordinates such that $A_3(0, 0)$, $B_3(a, 0)$, $C_3(a, b)$, $D_3(0, b)$, where we assume that, without loss of generality, $a \geq b$.

Let $m > 0$ be the slope of the line A_2B_2 , then the slope of the line A_2D_2 is $-\frac{1}{m}$. In order to find the coordinates of A_2 , B_2 , D_2 , we have to solve the following systems:

$$A_2 : \begin{cases} y = -\frac{1}{m}x \\ y = m(x-a) \end{cases} ; \quad B_2 : \begin{cases} y-b = -\frac{1}{m}(x-a) \\ y = m(x-a) \end{cases} ;$$

$$D_2 : \begin{cases} y-b = mx \\ y = -\frac{1}{m}x \end{cases} .$$

We obtain

$$A_2 \left(\frac{am^2}{m^2+1}, -\frac{am}{m^2+1} \right), \quad B_2 \left(\frac{am^2+bm+a}{m^2+1}, \frac{bm^2}{m^2+1} \right), \quad D_2 \left(-\frac{bm}{m^2+1}, \frac{b}{m^2+1} \right).$$

It follows that

$$A_1 \left(-\frac{bm}{m^2+1}, -\frac{am}{m^2+1} \right), \quad B_1 \left(\frac{am^2+bm+a}{m^2+1}, -\frac{am}{m^2+1} \right)$$

Since $A_3B_3 = a$ and $A_1B_1 = \frac{am^2+bm+a}{m^2+1} + \frac{bm}{m^2+1}$, it remains to prove that

$$\frac{am^2+2bm+a}{m^2+1} \leq 2a \Leftrightarrow 2mb \leq a(m^2+1),$$

which is true because $b \leq a$ and $2m \leq m^2+1$.

CC30. Two polite but vindictive children play a game as follows. They start with a bowl containing N candies, the number known to both contestants. In turn, each child takes (if possible) one or more candies, subject to the rule that no child may take, on any one turn, more than half of what is left. The winner is not the child who gets most candy, but the last child who gets to take some. Thus, if there are 3 candies, the first player may only take one, as two would be more than half. The second player may take one of the remaining candies; and the first player cannot move and loses.

(a) Show that if the game begins with 2000 candies the first player wins.

(b) Show that if the game begins with $999 \cdots 999$ (2000 9's) candies, the first player wins.

(Originally question 3 from the 2000 APICS contest.)

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

We claim that if $N = 2^k - 1$ for some positive integer k , then the first player loses, and otherwise the first player wins. We prove this by induction on k . Our induction hypothesis is that for each positive integer k , a player whose turn starts

with $2^k - 1$ candies loses, while a player wins if their turn starts with N candies, for $2^k - 1 < N < 2^{k+1} - 1$.

When $N = 1$ a player has no legal moves, and so that player loses. When $N = 2$, the player can take 1 candy, leaving the other player with 1, so the first player wins. This shows that our claim is true for $k = 1$.

Suppose that the claim is true for $k = m$, and consider $k = m + 1$. If a player begins the turn with $N = 2^{m+1} - 1$ candies, then the allowable moves consist of taking r candies, where

$$r \in \{1, 2, 3, \dots, 2^m - 1\}.$$

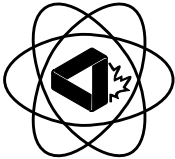
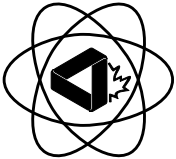
The number of candies remaining after this move is an element of

$$\{2^m, 2^m + 1, \dots, 2^{m+1} - 2\}.$$

By the induction hypothesis, the second player is able to win in any of these games, so the first player loses when $N = 2^{m+1} - 1$.

When a player begins a turn with N candies, for $2^{m+1} - 1 < N < 2^{m+2} - 1$, they are able to take $N - (2^{m+1} - 1)$ candies, leaving $2^{m+1} - 1$ candies. This is a losing position for the second player, so the first player wins.

Since neither 2000 nor $10^{2000} - 1$ is of the form $2^k - 1$ for a positive integer k , they are both winning positions for the first player.

 <div style="text-align: center;"> <p>A Taste Of Mathematics Aime-T-On les Mathématiques ATOM</p> </div> 
<p>ATOM Volume XI: Problems for Junior Mathematics Leagues by Bruce L.R. Shawyer & Bruce B. Watson (both of Memorial University of Newfoundland)</p> <p>The problems in this volume were originally designed for mathematics competitions aimed at students in the junior high school levels (grade 7 to 9) and including those students who may have the talent, ambition and mathematical expertise to represent Canada internationally. The problems herein function as a source of “out of classroom” mathematical enrichment that teachers and parents/guardians of appropriate students may assign to their charges. This volume is similar to previous publications on Problems for Mathematics Leagues in this series.</p> <p>There are currently 13 booklets in the series. For information on titles in this series and how to order, visit the ATOM page on the CMS website:</p> <p style="text-align: center;">http://cms.math.ca/Publications/Books/atom.</p>