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EDITORIAL

Dear ***Crux Mathematicorum*** Readers:

This issue is my last issue as Editor-in-Chief. I will continue to work on the journal until the end of June collecting material, and setting things up for the new editor. At this point in time, nobody has stepped forward, so there may be a delay before the September and October issues are produced. I will post any news to the ***Crux Mathematicorum*** Facebook page.

I count myself very lucky to have had the opportunity to work on such a fine journal twice. I am grateful that I have been able to work with all the wonderful members of the editorial board past and present. I must take a few moments to thank the people who have made it all possible.

I am fortunate to have associate editor JEFF HOOPER and editor-at-large BILL SANDS on the editorial board. I have trusted these fine gentlemen to spot all my spelling and grammatical errors and to point me in the correct direction. Their very thorough proofreading consistently picks up things that I have missed and I am indebted to them for their work.

We are privileged to be able to offer all the problems in ***Crux*** in both English and French. All of our translations are handled by three individuals. I must thank ANDRÉ LADOUCEUR, ROLLAND GAUDET and JEAN-MARC TERRIER for their quick, high quality translations as well as their suggestions for the English wordings of some problems.

Next I would like to thank those who provide regular features in *Crux*. A big thank you goes out to JOHN MCLOUGHLIN and former editor AMAR SODHI for always providing interesting book reviews that send me to the book store time and again. Thank you to ROBERT DAWSON for his fine work selecting interesting articles featured in the journal. MICHEL BATAILLE deserves thanks for his column, FOCUS ON..., which consistently receives praise from the readers. I have a tremendous amount of gratitude for NICOLAE STRUNGARU for his work on the OLYMPIAD CORNER. It is a huge job selecting the problems as well as doing all the editing of the solutions and I am very thankful for the great job he does.

We have added to our editorial board over the last little while. I want to thank CHIP CURTIS, LINO DEMASI, EDNA JAMES, MOHAMED OMAR, ALLEN O'HARA and PETER O'HARA. These six individuals stepped up and have joined the board. They have contributed to the editing of solutions and selection of the problems. Thank you to all of you, I hope you will continue in your position for years to come.

I must next thank my problems editors: EDWARD BARBEAU, CHRIS FISHER, ANNA KUCZYNSKA, LYNN MILLER and EDWARD WANG. These are my daily "go to" people whose main duties are: editing solutions to the numbered problems, selecting problems to be featured, and proofreading. On top of their regular responsibilities there are many other things that they do, some of which are acknowledged (such as writing articles, columns, and book reviews) and others which are not (like filling in missing details to solutions, finding references, rewording problem proposals, and adding comments to problems that will be of interest to readers). Thank you for your tireless work and always being there to say "what can I do next?".

Finally, I must thank you, the reader. I appreciate all the problems, solutions, articles, comments, references, and kind words you have sent over the last few years. A journal like ***Crux*** is really a function of the readership, and it is such a superb journal because we have such magnificent readers contributing. Thank you, I will miss you all.

Shawn Godin

THE CONTEST CORNER

No. 16

Shawn Godin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Nous préférons les réponses électroniques et demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_CCNuméro du problème *(exemple : Tremblay_Julie_CC1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats (Word, etc.) soient aussi acceptés. Nous invitons les lecteurs à envoyer leurs solutions et réponses aux concours au rédacteur à l'adresse* `crux-contest@smc.math.ca`. *Nous acceptons aussi les contributions par la poste, envoyées à l'adresse figurant en troisième de couverture. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er octobre 2014** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le franais précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

CC76. Le point $P(a, b)$ est situé dans le premier quadrant. Une droite passant par P coupe les axes aux points Q et R de manière que le triangle OQR ait une aire de $2ab$, O étant l'origine. Démontrer qu'il y a trois droites possibles qui satisfont à cette condition.

CC77. Trois cercles sont tangents l'un à l'autre. Le premier cercle a pour rayon a, le deuxième a pour rayon b et le troisième a pour rayon $a + b$, a et b étant des nombres réels et $a, b > 0$. Déterminer le rayon d'un quatrième cercle tangent à chacun de ces trois cercles.

CC78. Soit $g(x) = x^3 + px^2 + qx + r$, p, q et r étant des entiers. Démontrer que si $g(0)$ et $g(1)$ sont tous deux impairs, alors l'équation $g(x) = 0$ ne peut admettre trois racines entières.

CC79. Démontrer que si n est un entier supérieur à 1, alors $n^4 + 4$ n'est pas un nombre premier.

CC80. Alphonse et Bérénice s'amusent avec n coffres-forts. Chaque coffre-fort peut être ouvert à l'aide d'une clé unique et chaque clé peut ouvrir un seul coffre-fort. Bérénice mêle les n clés au hasard, place une clé à l'intérieur de chaque coffre-fort, puis elle ferme chaque coffre-fort à l'aide de sa passe-partout. Alphonse choisit ensuite m coffres-forts ($m < n$) et Bérénice ouvre ces m coffres-forts à l'aide de la passe-partout. Alphonse prend les clés à l'intérieur de ces m coffres-forts et tente d'ouvrir les $n - m$ autres coffres-forts à l'aide de ces clés. Chaque fois qu'il réussit à ouvrir un coffre-fort, il peut utiliser la clé à l'intérieur de celui-ci pour tenter d'en ouvrir un autre. Il continue jusqu'à ce qu'il ait ouvert tous les coffres-forts ou qu'il ne puisse plus en ouvrir un autre. Soit $P_m(n)$ la probabilité pour qu'Alphonse puisse ouvrir tous les n coffres-forts à partir des m clés disponibles lors de son choix de m coffres-forts. Déterminer une formule pour $P_2(n)$.

. . . .

CC76. The point $P(a, b)$ lies in the first quadrant. A line, drawn through P, cuts the axes at Q and R such that the area of triangle OQR is $2ab$, where O is the origin. Prove that there are three such lines that satisfy these criteria.

CC77. The three following circles are tangent to each other: the first has radius a, the second has radius b, and the third has radius $a + b$ for some $a, b \in \mathbb{R}$ with $a, b > 0$. Find the radius of a fourth circle tangent to each of these three circles.

CC78. Let $g(x) = x^3 + px^2 + qx + r$, where p, q and r are integers. Prove that if $g(0)$ and $g(1)$ are both odd, then the equation $g(x) = 0$ cannot have three integer roots.

CC79. Show that if n is an integer greater than 1, then $n^4 + 4$ is not prime.

CC80. Alphonse and Beryl play a game involving n safes. Each safe can be opened by a unique key and each key opens a unique safe. Beryl randomly shuffles the n keys, and after placing one key inside each safe, she locks all of the safes with her master key. Alphonse then selects m of the safes (where $m < n$), and Beryl uses her master key to open just the safes that Alphonse selected. Alphonse collects all of the keys inside these m safes and tries to use these keys to open up the other $n - m$ safes. If he can open a safe with one of the m keys, he can then use the key in that safe to try to open any of the remaining safes, repeating the process until Alphonse successfully opens all of the safes, or cannot open any more. Let $P_m(n)$ be the probability that Alphonse can eventually open all n safes starting from his initial selection of m keys. Determine a formula for $P_2(n)$.

CONTEST CORNER SOLUTIONS

CC26. A function f is defined in such a way that $f(1) = 2$, and for each positive integer $n > 1$,

$$f(1) + f(2) + f(3) + \cdots + f(n) = n^2 f(n).$$

Determine the value of $f(2013)$.
(*Inspired by question 5 from the 1994 CMC Invitational Mathematics Challenge, Grade 11.*)

Solved by George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain; Greg Cook, Angelo State University, San Angelo, TX, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA;
Marian Dincă, Bucharest, Romania; Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany; Leonard Giugiuc, Romania; Richard I. Hess, Rancho Palos Verdes, CA, USA; John G. Heuver, Grande Prairie, AB; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Norvald Midttun, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; Mihaï-Ioan Stoënescu, Bischwiller, France; Daniel Văcaru, Piteşti, Romania; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania.

We present Heuver's solution below.

For $n \geq 2$, we can write

$$f(1) + f(2) + f(3) + \cdots + f(n) = n^2 f(n) \tag{1}$$

and

$$f(1) + f(2) + f(3) + \cdots f(n-1) = (n-1)^2 f(n-1). \tag{2}$$

Subtracting (2) from (1) and rearranging for $f(n)$ we get

$$f(n) = \frac{(n-1)^2}{n^2-1} f(n-1) = \frac{n-1}{n+1} f(n-1). \tag{3}$$

By considering the recurrence relation in (3), we obtain

$$f(n) = \frac{2(n-1)!}{(n+1)!} f(1) = \frac{4}{n(n+1)}. \tag{4}$$

It follows from (4) that $f(2013) = \frac{4}{2013 \cdot 2014} = \frac{2}{2027091}$.

CC27. A $n \times n \times n$ cube has its faces ruled into n^2 unit squares. A path is to be traced on the surface of the cube starting at $(0,0,0)$ and ending at (n,n,n) moving only in a positive sense along the ruled lines. Determine the number of distinct paths.
(*Originally question 10 b) from the 1986 Descartes Contest.*)

No solutions were received.

CC28. The quartic polynomial $P(x)$ satisfies $P(1) = 0$ and attains its maximum value of 3 at both $x = 2$ and $x = 3$. Compute $P(5)$.
(*Originally question 5 from the 2012 Stanford Math Tournament, Algebra Problems.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Leonard Giugiuc, Romania; Richard I. Hess, Rancho Palos Verdes, CA, USA; David Jonathan, SMA Xaverius 1, Palembang, Indonesia; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. Two incorrect solutions were received.

We present two solutions. The first solution is a composite of solutions by Hess, Jonathan, Manes, Wang and Zvonaru.

Let $P(x) = ax^4 + bx^3 + cx^2 + dx + e$. We know, from the conditions of the problem, that $P(1) = 0$, $P(2) = P(3) = 3$ and $P'(2) = P'(3) = 0$. Substituting into $P(x)$ and $P'(x)$ we obtain the system of equations

$$\begin{aligned} a + b + c + d + e &= 0 \\ 16a + 8b + 4c + 2d + e &= 3 \\ 81a + 27b + 9c + 3d + e &= 3 \\ 32a + 12b + 4c + d &= 0 \\ 108a + 27b + 6c + d &= 0. \end{aligned}$$

Solving the system yields $a = -\dfrac{3}{4}$, $b = \dfrac{15}{2}$, $c = -\dfrac{111}{4}$, $d = 45$, and $e = -24$, so

$$P(x) = \frac{3}{4}x^4 + \frac{15}{2}x^3 - \frac{111}{4}x^2 + 45x - 24$$

and hence $P(5) = -24$.

Next we present a composite of the solutions from Curtis and Giugiuc.

Let $Q(x) = P(x) - 3$, then Q is a quartic polynomial that satisfies

$$Q(2) = Q(3) = 0, \quad \text{and} \quad Q'(2) = Q'(3) = 0.$$

Hence, 2 and 3 are double roots of Q, so

$$Q(x) = k(x-2)^2(x-3)^2$$

for some constant k, and thus

$$P(x) = k(x-2)^2(x-3)^2 + 3.$$

Using $P(1) = 0$ we obtain $k = -\frac{3}{4}$, so that

$$P(x) = -\frac{3}{4}(x-2)^2(x-3)^2 + 3,$$

whence

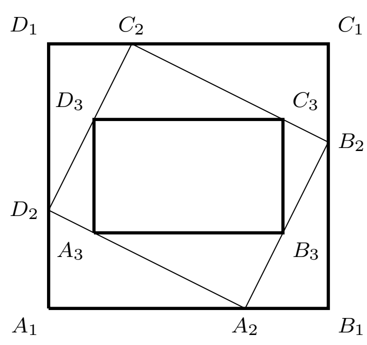
$$P(5) = -24.$$

CC29. Consider three parallelograms P_1, P_2, P_3. Parallelogram P_3 is inside parallelogram P_2, and the vertices of P_3 are on the edges of P_2. Parallelogram P_2 is inside parallelogram P_1, and the vertices of P_2 are on the edges of P_1. The sides of P_3 are parallel to the sides of P_1. Prove that one side of P_3 has length at least half the length of the parallel side of P_1.
(*Originally question 8 from the 2010 Sun Life Financial Repêchage Competition.*)

Solved by J. Chris Fisher, University of Regina, Regina, SK; and Titu Zvonaru, Cománeşti, Romania. No other solutions were received. We use the solution of Zvonaru, modified by the editor.

Let $P_i = A_iB_iC_iD_i$ for $i = 1, 2, 3$. Since affine transformations preserve the ratios of segment lengths along parallel lines, we may suppose that P_1, P_2, and P_3 are rectangles. [First map the outer parallelogram P_1 to a rectangle, in which case P_3 would also become a rectangle; then adjust the height A_1D_1 so that the diagonals A_2C_2 and B_2D_2 become equal, which forces P_2 to become a rectangle. This can always be accomplished since A_2C_2 grows from small to large and B_2D_2 remains fixed as A_1D_1 increases from 0 to infinity; thus, at some point in between they will be equal.]

Let $A_iB_iC_iD_i$ be the vertices of the rectangle P_i for $i = 1, 2, 3$.



We choose a system of coordinates such that $A_3(0,0)$, $B_3(a,0)$, $C_3(a,b)$, $D_3(0,b)$, where we assume that, without loss of generality, $a \geq b$.

Let $m > 0$ be the slope of the line A_2B_2, then the slope of the line A_2D_2 is $-\dfrac{1}{m}$. In order to find the coordinates of A_2, B_2, D_2, we have to solve the following systems:

$$A_2 : \begin{cases} y &= -\dfrac{1}{m}x \\ y &= m(x-a) \end{cases} ; \qquad B_2 : \begin{cases} y-b &= -\dfrac{1}{m}(x-a) \\ y &= m(x-a) \end{cases} ;$$

$$D_2 : \begin{cases} y-b &= mx \\ y &= -\dfrac{1}{m}x \end{cases} .$$

We obtain

$$A_2\left(\frac{am^2}{m^2+1}, -\frac{am}{m^2+1}\right),\ B_2\left(\frac{am^2+bm+a}{m^2+1}, \frac{bm^2}{m^2+1}\right),\ D_2\left(-\frac{bm}{m^2+1}, \frac{b}{m^2+1}\right).$$

It follows that

$$A_1\left(-\frac{bm}{m^2+1}, -\frac{am}{m^2+1}\right), \qquad B_1\left(\frac{am^2+bm+a}{m^2+1}, -\frac{am}{m^2+1}\right)$$

Since $A_3B_3 = a$ and $A_1B_1 = \dfrac{am^2+bm+a}{m^2+1} + \dfrac{bm}{m^2+1}$, it remains to prove that

$$\frac{am^2+2bm+a}{m^2+1} \le 2a \Leftrightarrow 2mb \le a(m^2+1),$$

which is true because $b \le a$ and $2m \le m^2+1$.

CC30. Two polite but vindictive children play a game as follows. They start with a bowl containing N candies, the number known to both contestants. In turn, each child takes (if possible) one or more candies, subject to the rule that no child may take, on any one turn, more than half of what is left. The winner is not the child who gets most candy, but the last child who gets to take some.
Thus, if there are 3 candies, the first player may only take one, as two would be more than half. The second player may take one of the remaining candies; and the first player cannot move and loses.
(a) Show that if the game begins with 2000 candies the first player wins.
(b) Show that if the game begins with $999\cdots999$ (2000 9's) candies, the first player wins.
(*Originally question 3 from the 2000 APICS contest.*)

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

We claim that if $N = 2^k - 1$ for some positive integer k, then the first player loses, and otherwise the first player wins. We prove this by induction on k. Our induction hypothesis is that for each positive integer k, a player whose turn starts with $2^k - 1$ candies loses, while a player wins if their turn starts with N candies, for $2^k - 1 < N < 2^{k+1} - 1$.

When $N = 1$ a player has no legal moves, and so that player loses. When $N = 2$, the player can take 1 candy, leaving the other player with 1, so the first player wins. This shows that our claim is true for $k = 1$.

Suppose that the claim is true for $k = m$, and consider $k = m+1$. If a player begins the turn with $N = 2^{m+1} - 1$ candies, then the allowable moves consist of taking r candies, where

$$r \in \{1, 2, 3, \dots, 2^m - 1\}.$$

The number of candies remaining after this move is an element of

$$\{2^m, 2^m + 1, \dots, 2^{m+1} - 2\}.$$

By the induction hypothesis, the second player is able to win in any of these games, so the first player loses when $N = 2^{m+1} - 1$.

When a player begins a turn with N candies, for $2^{m+1} - 1 < N < 2^{m+2} - 1$, they are able to take $N - (2^{m+1} - 1)$ candies, leaving $2^{m+1} - 1$ candies. This is a losing position for the second player, so the first player wins.

Since neither 2000 nor $10^{2000} - 1$ is of the form $2^k - 1$ for a positive integer k, they are both winning positions for the first player.

THE OLYMPIAD CORNER

No. 314

Nicolae Strungaru

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Nous préférons les réponses électroniques et demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_OCNuméro du problème *(exemple : Tremblay_Julie_OC1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats (Word, etc.) soient aussi acceptés. Nous invitons les lecteurs à envoyer leurs solutions et réponses aux concours au rédacteur à l'adresse* `crux-olympiad@smc.math.ca`. *Nous acceptons aussi les contributions par la poste, envoyées à l'adresse figurant en troisième de couverture. Le nom de la personne qui propose une solution doit figurer avec* chaque *solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

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Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, d'avoir traduit les problèmes.

OC136. Le quadrilatère $ABCD$ est inscrit dans un cercle avec centre O. Si $AB = \sqrt{2+\sqrt{2}}$ et $\angle AOB = 135^\circ$, déterminer l'aire maximum possible pour $ABCD$.

OC137. On dénote par $S(k)$ la somme des chiffres dans la représentation décimale de k. Démontre qu'il y a infiniment d'entiers positifs n tels que

$$S(2^n + n) < S(2^n)\,.$$

OC138. Déterminer tous les entiers positifs $a, b, c, p \geq 1$ tels que p est premier et

$$a^p + b^p = p^c.$$

OC139. Les nombres $1, 2, \ldots, 50$ sont écrits au tableau. À chaque minute, deux de ces nombres sont effacés et sont remplacés par leur différence positive. À la fin, un seul nombre demeure. Déterminer toutes les valeurs possibles pour ce nombre.

OC140. Soit ABC un triangle obtus avec $\angle A > 90°$ et avec cercle circonscrit Γ. Le point D se trouve sur le segment AB de façon à ce que $AD = AC$. Soit AK le diamètre de Γ et soit L le point d'intersection de AK et CD. Un cercle passant par D, K et L intersècte Γ à $P \neq K$. Étant donné que $AK = 2$, $\angle BCD = \angle BAP = 10°$, démontrer que

$$DP = \sin\left(\frac{\angle A}{2}\right).$$

. . . .

OC136. Quadrilateral $ABCD$ is inscribed in a circle with centre O. If $AB = \sqrt{2+\sqrt{2}}$ and $\angle AOB = 135°$, find the maximum possible area of $ABCD$.

OC137. We denote by $S(k)$ the sum of the digits in the decimal representation of k. Prove that there are infinitely many positive integers n for which

$$S(2^n + n) < S(2^n).$$

OC138. Find all positive integers $a, b, c, p \geq 1$ such that p is a prime and

$$a^p + b^p = p^c.$$

OC139. The numbers $1, 2, \ldots, 50$ are written on a blackboard. Each minute any two numbers are erased and their positive difference is written instead. At the end one number remains. Find all the values this number can take.

OC140. Let ABC be an obtuse triangle with $\angle A > 90°$ and circumcircle Γ. Point D is on the segment AB such that $AD = AC$. Let AK be a diameter of Γ, and let L be the point of intersection of AK and CD. A circle passing through D, K, L intersects Γ at $P \neq K$. Given that $AK = 2$, $\angle BCD = \angle BAP = 10°$, prove that

$$DP = \sin\left(\frac{\angle A}{2}\right).$$

OLYMPIAD SOLUTIONS

OC76. For any positive integer n, let a_n be the exponent of the largest power of 2 which occurs as a factor of $5^n - 3^n$. Also, let b_n be the exponent of the largest power of 2 which divides n. Show that

$$a_n \leq b_n + 3$$

for all n.
(*Originally question 1 from the 2011 British IMO selection, Day 2.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Norvald Midttun, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; Daniel Văcaru, Piteşti, Romania; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Midttun.

If n is odd, then

$$5^n - 3^n \equiv 1 - 3 \equiv 2 \pmod 4 .$$

Therefore $a_n = 1$ and $b_n = 0$, thus the inequality holds.

Now, let $n = 2^m \cdot q$ with q odd and $m \geq 1$. Then

$$\begin{aligned} 5^{2^m \cdot q} - 3^{2^m \cdot q} &= \left(5^{2^{m-1}\cdot q} + 3^{2^{m-1}\cdot q}\right)\left(5^{2^{m-1}\cdot q} - 3^{2^{m-1}\cdot q}\right) \\ &= \left(5^{2^{m-1}\cdot q} + 3^{2^{m-1}\cdot q}\right)\left(5^{2^{m-2}\cdot q} + 3^{2^{m-2}\cdot q}\right)\left(5^{2^{m-2}\cdot q} - 3^{2^{m-2}\cdot q}\right) \\ &\vdots \\ &= \left(5^{2^{m-1}\cdot q} + 3^{2^{m-1}\cdot q}\right)\left(5^{2^{m-2}\cdot q} + 3^{2^{m-2}\cdot q}\right)\left(5^{2^{m-3}\cdot q} + 3^{2^{m-3}\cdot q}\right) \times \\ &\qquad \cdots \left(5^{2\cdot q} + 3^{2\cdot q}\right)\left(5^q + 3^q\right)\left(5^q - 3^q\right) \quad (1) \end{aligned}$$

As all odd perfect squares are congruent to 1 (mod 4) we have

$$\begin{aligned} 5^{2^{m-1}\cdot q} + 3^{2^{m-1}\cdot q} &\equiv 5^{2^{m-2}\cdot q} + 3^{2^{m-2}\cdot q} \\ &\equiv 5^{2^{m-3}\cdot q} + 3^{2^{m-3}\cdot q} \equiv \cdots \equiv 5^{2\cdot q} + 3^{2\cdot q} \equiv 2 \pmod 4 \end{aligned}$$

Therefore, the exponent of the largest power of 2 that divides

$$\left(5^{2^{m-1}\cdot q} + 3^{2^{m-1}\cdot q}\right)\left(5^{2^{m-2}\cdot q} + 3^{2^{m-2}\cdot q}\right)\left(5^{2^{m-3}\cdot q} + 3^{2^{m-3}\cdot q}\right)\cdots\left(5^{2\cdot q} + 3^{2\cdot q}\right)$$

is $m - 2$.

Since q is odd, we have exactly as in the first part of the proof

$$5^q - 3^q \equiv 2 \pmod 4 .$$

We claim that
$$5^q + 3^q \equiv 8 \pmod{16}\ .$$
Indeed, let $q = 2k+1$. Then
$$5^{2k+1} + 3^{2k+1} \equiv 5 \cdot 25^k + 3 \cdot 9^k \equiv 5 \cdot 9^k + 3 \cdot 9^k \equiv 8 \cdot 9^k \equiv 8 \pmod{16}\ .$$
This shows that in this case $a_n = b_n + 3$.

OC77. Find all functions $f : (0, \infty) \to (0, \infty)$ so that for all $x, y \in (0, \infty)$ we have
$$f(x)f(y) = f(y)f\left(xf(y)\right) + \frac{1}{xy}\,.$$
(*Originally question 6 from the 2011 Czech Republic Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France.

It is easy to check that $f(x) = x + \frac{1}{x}$ is a solution. We show that there is no other solution.

Let $f : (0, \infty) \to (0, \infty)$ be any solution. Replacing x by $\frac{x}{f(y)}$ we get
$$f\left(\frac{x}{f(y)}\right) = f(x) + \frac{1}{xy} \tag{2}$$
therefore
$$f\left(\frac{1}{f(y)}\right) = a + \frac{1}{y} \tag{3}$$
where $a = f(1)$. It follows from (3) that f is injective.

Setting $y = 1$ in (3) we get
$$f\left(\frac{1}{a}\right) = a + 1 \tag{4}$$
while setting $y = 1, x = \frac{1}{a}$ in (2) yields
$$f\left(\frac{1}{a^2}\right) = f\left(\frac{1}{a}\right) + a \tag{5}$$
therefore
$$f\left(\frac{1}{a^2}\right) = 2a + 1\,.$$
On another hand, setting $y = \frac{1}{a+1}$ in (3) yields
$$f\left(\frac{1}{f(\frac{1}{a+1})}\right) = 2a + 1\,.$$

As f is injective, we get

$$f\left(\frac{1}{a+1}\right) = a^2$$

therefore

$$f\left(\frac{1}{f(a)}\right) = a^2\,.$$

Setting $y = \dfrac{1}{a}$ in (3) yields

$$f\left(\frac{1}{f(\frac{1}{a})}\right) = 2a\,. \tag{6}$$

This shows that $a^2 = 2a$ and hence, as $a = f(1) > 0$ we get $a = 2$.

Now, replacing x by $\dfrac{1}{f(x)}$ and y by 1 in the original relation we get

$$f\left(\frac{1}{f(x)}\right) = f\left(\frac{2}{f(x)}\right) + \frac{f(x)}{2}\,.$$

Now, combining (3) with $a = 2$ we have

$$f\left(\frac{1}{f(x)}\right) = 2 + \frac{1}{x}\,.$$

Moreover, from (2) we get

$$f\left(\frac{2}{f(x)}\right) = f(2) + \frac{1}{2x}\,.$$

Therefore

$$2 + \frac{1}{x} = f(2) + \frac{1}{2x} + \frac{f(x)}{2}$$

or

$$f(x) = 4 - 2f(2) + \frac{1}{x}\,.$$

Now, setting in the given relation $x = y = 1$ we get

$$2^2 = 2f(2) + 1 \Rightarrow 2f(2) = 3\,,$$

which shows that

$$f(x) = 1 + \frac{1}{x}\,.$$

This completes the proof.

OC78. Let $a_1 = 1, a_2 = 5, a_3 = 14, a_4 = 19, \ldots$ be the sequence of positive integers starting with 1, followed by all integers with the sum of the digits divisible by 5. Prove that for all n we have

$$a_n \leq 5n\,.$$

(*Originally question 4 from 2011 Kazahstan National Olympiad, Grade 9.*)

Solved by Oliver Geupel, Brühl, NRW, Germany; and Daniel Văcaru, Piteşti, Romania. There was one incomplete solution. We give the writeup from Geupel.

For any positive integer n, define the set

$$A_n = \{5n-5,\ 5n-4,\ 5n-3,\ 5n-2,\ 5n-1\}.$$

We have $a_1 \in A_1$.

Moreover, for each $n \geq 2$, the elements in A_n only differ in the last digit, thus the sums of digits of the members of A_n are in distinct residue classes modulo 5.

Therefore, exactly one member of each A_n has a sum of digits that is divisible by 5. Consequently, $a_n \in A_n$ for $n = 1, 2, \ldots$, which implies

$$a_n \leq \max A_n = 5n - 1.$$

This completes the proof.

OC79. Let D be a point different from the vertices on the side BC of a ΔABC. Let I, I_1 and I_2 be the incenters of $\Delta ABC, \Delta ABD$ respectively ΔADC. Let E be the second intersection point of the circumcircles of ΔAI_1I and ΔADI_2, and let F be the second intersection point of the circumcircles of ΔAII_2 and ΔAI_1D. If $AI_1 = AI_2$, prove that

$$\frac{EI}{FI} \cdot \frac{ED}{FD} = \frac{{EI_1}^2}{{FI_2}^2}\,.$$

(*Originally question 1 from the 2011 Turkey Team Selection Test, Day 2.*)

No solution to this problem was received.

OC80. Let G be a simple graph with $3n^2$ vertices ($n \geq 2$), such that the degree of each vertex of G is not greater than $4n$, there exists at least one vertex of degree one, and between any two vertices, there is a path of length ≤ 3. Prove that the minimum number of edges that G might have is equal to $\dfrac{7n^2 - 3n}{2}$. (*Originally question 3 from 2011 China Team Selection Test, Quiz 3, Day 1.*)

No solution to this problem was received.

BOOK REVIEWS

John McLoughlin

The Joy of x: A Guided Tour of Math, from One to Infinity by Steven Strogatz
Eamon Dolan/Houghton Mifflin Harcourt, 2013
ISBN: 978-0-547-51765-0, Paperback/e-book, 336 pages, US$15.95 (print/electronic)

Reviewed by ***S. Swaminathan****, Dalhousie University, Halifax, NS*

Type the letter 'x' in Google and you learn that in mathematics 'x' is commonly used as the name for an independent variable or unknown value and that the modern tradition of using x to represent an unknown was started by René Descartes in '*La Géométrie*' (1637). Many high school students find it difficult to understand its use for an unknown quantity in a simple arithmetical problem; the ensuing frustration is often the cause of their subsequent apathy towards mathematics.

The Cornell University mathematician Steven Strogatz is well known for his popular articles concerning the beauty and fun of mathematical topics. Each chapter of the present book offers an "Aha!" moment, starting with why numbers are so helpful and progressing through the wondrous truths implicit in topics concerning π, the Pythagorean theorem, irrational numbers, fat tails, even the rigours and surprising charms of calculus. For example, he explains how Michael Jordan's high jump dunks can help explain the fundamentals of calculus. He discusses topics which concern some of life's mysteries such as: Did O.J. do it?; How can one flip a mattress to get the maximum wear out of it and thereby learn some group theory?; How many people should one date before settling down?; and Why are some infinities bigger than others? The only prerequisites that are needed to read this book are curiosity and common sense. The 'joy' of the title is found through his clear, ingenious and often funny explanations of the most vital and exciting principles of mathematical topics.

The thirty chapters of the book are divided into six parts: Numbers, Relationships, Shapes, Change, Data and Frontiers. Part 1, *Numbers* deals with kindergarten and grade-school arithmetic, stressing how helpful numbers can be and how uncannily effective they are in describing the world. Part 2, *Relationships* generalizes from working with numbers to working with relationships between numbers. Part 3, *Shapes* deals with geometry and trigonometry and introduces new levels of rigour through logic and proof. Part 4, *Change* explains the role of infinity in Calculus. Part 5, *Data* is concerned with probability, statistics, networks and data mining. With the right kinds of math and the rights kinds of data, it is shown how to pull meaning from maelstrom. Finally, Part 6, *Frontiers* discusses the edge of mathematical knowledge, the borderland between what's known and what remains elusive, under the headings: The Loneliest Numbers; Group Think; Twist & Shout; Think Globally; Analyze This! (infinite series); and The Hilbert Hotel.

If anyone tells you "I hate math" ask that person to read this book; that opinion will get revised!

PROBLEM SOLVER'S TOOLKIT

No. 6

J. Chris Fisher

The Problem Solver's Toolkit is a new feature in ***Crux Mathematicorum****. It will contain short articles on topics of interest to problem solvers at all levels. Occasionally, these pieces will span several issues.*

Harmonic Sets Part 3: The Harmonic Mean File

Murray Klamkin, problem solver extraordinaire, was closely associated with *Crux Mathematicorum* during its first, and his last, 30 years. Well into his 70s he claimed that he could remember every result he ever proved, plus its proof. For those youngsters who are reading this, let me warn you that Klamkin was an exception. I realized the need for a good filing system when still in my 30s. The result that convinced me of my vulnerability was one that at first sight I found hard to believe:

> *If AA' and BB' are two line segments that are on the same side of the line AB and perpendicular to it, then the distance d to the line from the point where AB' intersects $A'B$ is independent of the distance AB.*

I easily found a quick argument to show that not only is the result "obvious", but $d = \dfrac{AA' \cdot BB'}{AA' + BB'}$. Unfortunately, a few years later I could not remember my neat proof and had to resort to a proof by algebra. Afterwards, after much effort, I finally recovered my original argument and filed it away for safe keeping. Before turning to that proof, here are a couple other lessons about filing that I learned the hard way: when an item fits in more than one place, put a note in each relevant file indicating the file where that item is located; also, list the contents of each file on its cover. In these final two installments of the four-part series we will look at the items in my first file, the harmonic-mean file.

Now for the proof that d is independent of the distance between A and B. Figure 1, on the next page, almost says it all.

Note that the line A^*B^* parallel to $A'B$ through B' determines a triangle A^*AB^* that is similar to triangle $A'AB$; the dilatation that shrinks the larger to the smaller takes $B'B$ to $D'D$ and A^*A to $A'A$, whence $\dfrac{d}{b} = \dfrac{a}{a+b}$, or

$$d = \frac{ab}{a+b},$$

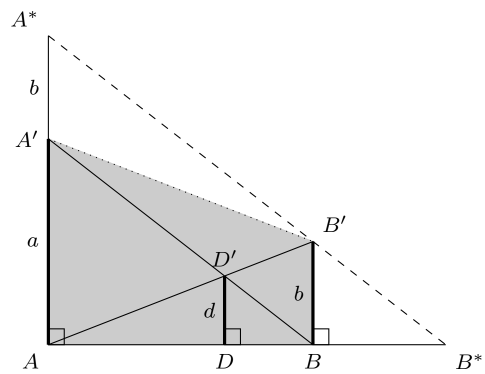

Figure 1: $d = \dfrac{ab}{a+b}$.

which is half the harmonic mean of a and b and is independent of the distance AB, as claimed.

Dictionaries tell us that the *harmonic mean* h of the numbers a and b equals the reciprocal of the arithmetic mean of the reciprocals of a and b. What a mouthful! It is perhaps less formidable in symbols:

$$\frac{1}{h} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \qquad \text{or} \qquad h = \frac{2ab}{a+b}.$$

The harmonic mean arises naturally when finding average rates. For example, if one pedals a bicycle up the hill at 8 km/h and back down at 24 km/h, then the average speed for the return trip is 12 km/h, the harmonic mean of 8 and 24. Almost the same problem, except here we want the combined rate rather than the average: if it takes 8 minutes for person A to peel the potatoes, and 24 for person B, then how long would it take them if they worked together? No, not 32 minutes, nor 16, but 6 minutes, which is *half* the harmonic mean. As we go through my harmonic-mean file, we shall see that in geometry the harmonic mean pops up all over the place.

Looking carefully at the argument based on Figure 1, we see that we never used perpendicularity — we require only that AA', BB', and DD' be parallel. Moreover, if you complete the trapezoid $A'ABB'$ of Figure 1 you get the theorem that

> *the line that is parallel to the bases of a trapezoid and that passes through the intersection of its diagonals is intercepted by the nonparallel sides in a segment whose length is the harmonic mean of the bases.*

The independence of d (in the result discussed at the start) can also be easily seen dynamically — without computing its value — with the help of a strain whose axis is BB' and whose centre is the point at infinity of the line AB. Recall (from the first installment) that a strain is the perspective collineation that fixes all points of the axis and slides points along lines through the centre. As in Figure 2, the points A, A', D, D' slide along horizontal lines to $\hat{A}, \hat{A}', \hat{D}, \hat{D}'$ while the lengths along the segments parallel to BB' remain constant: BB' is fixed while $AA' = \hat{A}\hat{A}'$ and $DD' = \hat{D}\hat{D}'$.

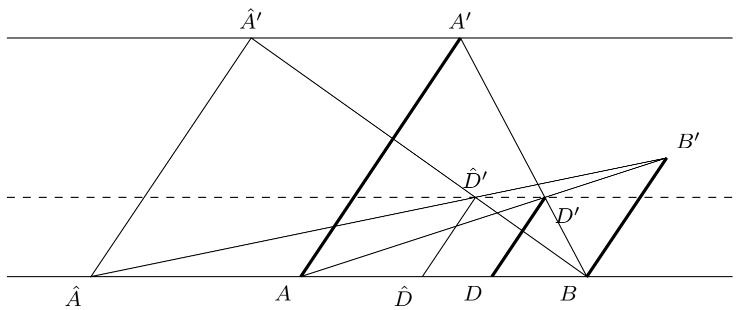

Figure 2: Fix B and B' while letting A and A' move along their horizontal lines to positions $\hat{A}$ and $\hat{A}'$. The distance of the intersection point $\hat{D}'$ from the line AB is independent of the choice of $\hat{A}$.

Next, consider four collinear points A, B, C, and D. We saw in the second installment that B and D are harmonic conjugates with respect to A and C (and the four points form a harmonic set) if and only if the cross ratio $\dfrac{AB \cdot CD}{AD \cdot CB}$ equals -1. A simple calculation shows that for points in the Euclidean plane, the length of AC is the harmonic mean of the segments AB and AD if and only if B and D are harmonic conjugates with respect to A and C:

$$\begin{aligned} AC &= \frac{2AB \cdot AD}{AB + AD} \\ 2AB \cdot AD &= AC \cdot (AB + AD) \\ AB \cdot (AD - AC) &= AD \cdot (AC - AB) \\ AB \cdot CD &= AD \cdot BC = -AD \cdot CB \end{aligned}$$

We conclude this month's installment with a further look at Figure 1. It can be recognized as the initial step of the affine version of Figure 3.5A in [1, p. 32], which indicates how to construct a *harmonic sequence* $A_1, A_2, A_3 \ldots$ from three collinear points O, A_1, and A_2. For $j > 1$ in the sequence, A_j is the harmonic conjugate of O with respect to A_{j-1} and A_{j+1}; this means that

$$OA_{j+1} \cdot A_j A_{j-1} = A_{j+1} A_j \cdot OA_{j-1},$$

or, if you prefer, OA_j is the harmonic mean of OA_{j+1} and OA_{j-1}. Figure 3 shows the start of the infinite harmonic sequence $A_1, A_2, A_3 \ldots$ along the base OA_1 of the parallelogram $OA_1A_1'P$. The harmonic relationship follows directly from the definition of harmonic sets (as given in the previous installment): For all $j > 1$, points O and A_j are diagonal points of the quadrangle whose vertices are P, A'_{j+1}, A'_j, and the common point at infinity of the lines OP and $A_jA'_j$; the remaining diagonals meet OA_j at A_{j-1} and A_{j+1}.

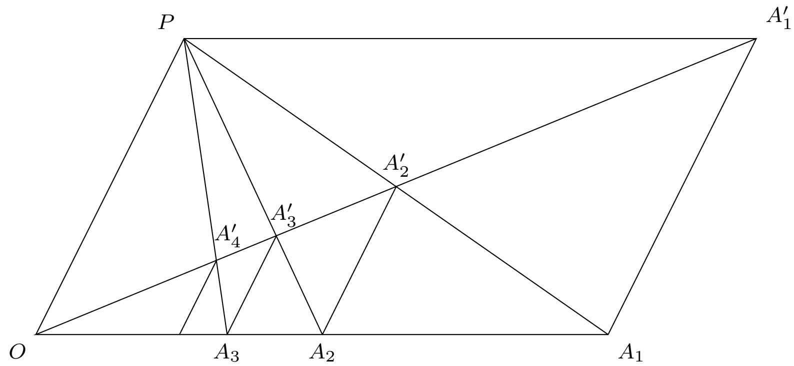

Figure 3: Construction of the harmonic sequence $A_1, A_2, A_3 \ldots$.

Returning to algebra, we see that if segment OA_1 has unit length, then the lengths OA_j form the familiar harmonic sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$. Coxeter [1, p. 23] explains the source of the word *harmonic* by labeling $C = O, E = A_5$, and $G = A_3$, and observing that "if the segment CA_1 represents a stretched string, tuned to the note C, the same string stopped at E or G will play the other notes of the major triad." Of course, $\frac{1}{3}$ is the harmonic mean of $\frac{1}{5}$ and 1.

In my file along with this example of harmonic sets is an article [3] by two 14-year-olds who discovered the construction of Figure 3 using *The Geometer's Sketchpad*; the article was brought to my attention by the media frenzy purporting that the students' construction had slipped by the notice of mathematicians for millennia [2]. To the contrary, of course, the construction has been widely known for centuries.

References

[1] H.S.M. Coxeter, *Projective Geometry*, 2nd ed. Springer-Verlag, 1987.

[2] Leslie Chess Feller, The Eternal Challenge of Euclid's Geometry, *The New York Times*, March 7, 1999.

[3] Dan Litchfield, Dave Goldenheim, Euclid, Fibonacci, Sketchpad, *Math. Teacher*, **90**:1 (Jan. 1997) 8-12.

A Quadrangle's Centroid of Perimeter

Rudolf Fritsch and Günter Pickert

Introduction

In [3] we discussed relations between the vertex centroid and the centroid of area of a quadrangle. Here we compare the centroid of perimeter with the other two named centroids.

Setup

For the reader's convenience we adapt the setup described in [3] to our present purposes. We consider plane quadrangles $ABCD$ with vertices A, B, C, D — no three of them on a line — edges $[AB]$, $[BC]$ $[CD]$, $[DA]$ and diagonals AC, BD; the diagonals may be considered — depending on the situation — as segments or lines. The quadrangles may be convex, concave or crossed. The lengths of the edges are denoted in the usual manner by a, b, c, d while the position vectors of the vertices with respect to a suitably chosen origin are written as $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ (in computations using vector algebra).

We consider the centroid of vertices $S_{\mathcal{E}}$, the centroid of perimeter $S_{\mathcal{K}}$, and the centroid of area $S_{\mathcal{F}}$[1]. The centroid of area $S_{\mathcal{F}}$ is not defined for crossed quadrangles.

The centroid of perimeter of a quadrangle

In order to get the centroid of perimeter $S_{\mathcal{K}}$ of the quadrangle one replaces the homogeneous system of mass by a system of mass points [6]. The masses of the sides are proportional to their lengths a, b, c, d and are concentrated in the midpoints of sides. So one takes these four midpoints and provides them with the masses a, b, c, d. If the quadrangle is a parallelogram then the centroid of perimeter is the intersection point of the diagonals and coincides with the vertex centroid. In general the centroid of the perimeter has the vector representation

$$\overrightarrow{s_{\mathcal{K}}} = \frac{1}{2(a+b+c+d)}\left(a(\vec{a}+\vec{b})+b(\vec{b}+\vec{c})+c(\vec{c}+\vec{d})+d(\vec{d}+\vec{a})\right). \qquad (1)$$

The synthetic determination of the centroid of the perimeter however has another basis. To this we recall first, how to find the centroid of a two-leg (with homogeneous mass on the sides).

We consider a two-leg consisting of the segments $[AB]$ and $[AC]$ with the lengths c and b, A, B, C not on a line.

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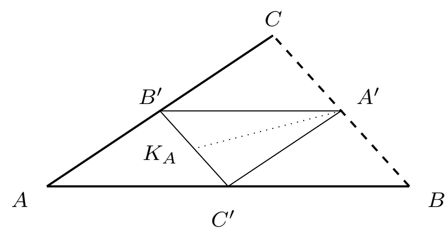
* This note is an English adaption of the second part of a more comprehensive paper written in German [2]. The authors thank Chris Fisher for his helpful comments.

[1] The symbols $\mathcal{E}$, $\mathcal{K}$, and $\mathcal{F}$ suggest the German words for vertex (Ecke), edge (Kante), and area (Fläche).



Its centroid K_A belongs to the segment that runs from the midpoint C' of $[AB]$ to the midpoint B' of $[AC]$, that is, the middle parallel line of triangle ABC parallel to BC, and divides this segment in the ratio $b : c$.

Therefore the point K_A is the intersection point of the middle parallel line and the internal angular bisector of the triangle $A'B'C'$ running through the vertex A', the midpoint of the segment $[BC]$. This construction is based on the the angle bisector theorem: *An internal angular bisector of a triangle divides the opposite side in the ratio of the attached sides.*

In case of a quadrangle $ABCD$ one has four two-legs with the centroids K_A, K_B, K_C, K_D. The centroid of the perimeter $S_{\mathcal{K}}$ is then the intersection point of the diagonals of the quadrangle $K_AK_BK_CK_D$.

Centroid of perimeter = centroid of vertices

Now we assume that the centroid of the perimeter $S_{\mathcal{K}}$ of our quadrangle coincides with its vertex centroid $S_{\mathcal{E}}$ which has the vector representation

$$\overrightarrow{s_{\mathcal{E}}} = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d}).$$

As in [3] we take the vertex centroid as origin which gives

$$\vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{o}. \tag{2}$$

From (1) we get

$$\vec{o} = a(\vec{a} + \vec{b}) + b(\vec{b} + \vec{c}) + c(\vec{c} + \vec{d}) + d(\vec{d} + \vec{a}). \tag{3}$$

Equation (2) provides the substitution $\vec{d} = -\vec{a} - \vec{b} - \vec{c}$ which yields

$$\vec{o} = (a - c)(\vec{a} + \vec{b}) + (b - d)(\vec{b} + \vec{c}) = (a - c)\vec{a} + (a - c + b - d)\vec{b} + (b - d)\vec{c},$$

that is

$$-(a + b - c - d)\vec{b} = (a - c)\vec{a} + (b - d)\vec{c}. \tag{4}$$

The corresponding substitution $\vec{b} = -\vec{a} - \vec{c} - \vec{d}$ yields analogously

$$-(a + b - c - d)\vec{d} = (b - d)\vec{a} + (a - c)\vec{c}. \tag{5}$$

We see that the argument depends on whether or not $a + b - c - d = 0$. If $a + b - c - d \neq 0$, then

$$-\vec{b} = \frac{a-c}{a+b-c-d}\vec{a} + \frac{b-d}{a+b-c-d}\vec{c},$$

$$-\vec{d} = \frac{b-d}{a+b-c-d}\vec{a} + \frac{a-c}{a+b-c-d}\vec{c}.$$

These equations show: Reflecting the vertices B and D in the origin yields points on the diagonal AC, that is, the diagonals BD and AC are parallel and we have a crossed quadrangle.

It will now be convenient to treat separately the situation where the diagonals of the quadrangle are not parallel. From the previous paragraph we necessarily have

$$a + b - c - d = 0. \tag{6}$$

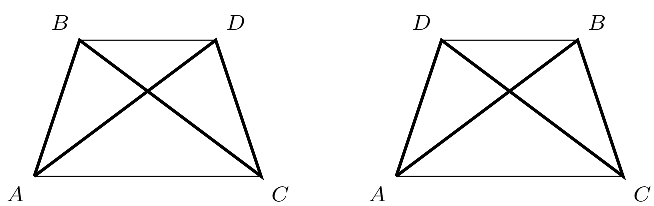
and the equations (4), (5) simplify to

$$\begin{aligned} \vec{o} &= (a-c)\vec{a} + (b-d)\vec{c}, \\ \vec{o} &= (b-d)\vec{a} + (a-c)\vec{c}. \end{aligned}$$

From equation (6) we know $(b - d) = -(a - c)$ which leads by substitution in the first of these equations to $\vec{o} = (a - c) \cdot (\vec{a} - \vec{c})$. Since $\vec{a} \neq \vec{c}$ we conclude $a = c$ and consequently $b = d$. Starting with the triangle ABD the vertex C is an intersection point of the circles centered at the points B and D with radii b, c respectively. Thus when (6) holds, our quadrangle is either (a) a parallelogram (which is a centrally symmetric quadrangle) or (b) a mirror symmetric crossed quadrangle whose diagonals are perpendicular to the mirror. In case (b) the diagonals are parallel, which we momentarily set aside to state

Theorem (part 1.) *The centroid of vertices and the centroid of perimeter of a quadrangle with nonparallel diagonals coincide if and only if the quadrangle is a parallelogram.*

Now we consider quadrangles with parallel diagonals.



If (6) holds, then we have as stated at the end of the previous paragraph two types of mirror symmetric crossed quadrangles whose vertex centroids coincide with the centroids of perimeter shown in the diagram.

Finally, we return to quadrangles with $a + b - c - d \neq 0$ (and therefore with parallel diagonals). We introduce coordinates such that the x-axis is the center line of the diagonals; without loss of generality we may assume that the diagonals AC, BD satisfy equations $y=-1$, $y=1$ respectively. We take, using equation (2),

$$A(a_1,-1),\ \ B(b_1,1),\ \ C(c_1,-1),\ \ D(-a_1-b_1-c_1,1).$$

The midpoints of the sides of the quadrangle all have second coordinate 0, so equation (3) reduces to a condition only on the first coordinate and can be written in the form

$$(a_1+b_1)\cdot(a-c)=(b_1+c_1)\cdot(d-b).$$

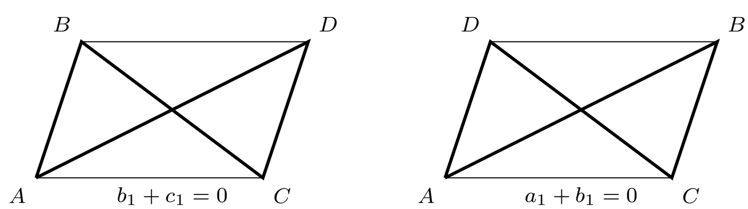
This is a (square-)root equation since

$$\begin{aligned} a &= \sqrt{(b_1-a_1)^2+4a_2^2}, & c &= \sqrt{(b_1-c_1)^2+4a_2^2}\,,\\ b &= \sqrt{(a_1+b_1+2c_1)^2+4a_2^2}, & d &= \sqrt{(2a_1+b_1+c_1)^2+4a_2^2}\,. \end{aligned}$$

It can be transformed into a polynomial equation by the usual procedure of multiple squaring. Factoring at each step, if possible, (most conveniently done by means of a computer algebra system) yields the following polynomial factors

$$a_1-c_1,\ a_1+2b_1+c_1,\ b_1+c_1,\ a_1+c_1,\ a_1+b_1.$$

At least one of these terms must vanish. The first two can't since this would imply either $A=C$ or $B=D$, contrary to our definition of a quadrangle. If $b_1+c_1=0$ we have $\vec{b}=-\vec{c}$ and $\vec{a}=-\vec{d}$, whence the quadrangle is centrally symmetric. The same holds if $a_1+b_1=0$ implying $\vec{a}=-\vec{b}$, $\vec{c}=-\vec{d}$. In both cases the origin is the vertex centroid as well as the centroid of the perimeter.



Finally, if $a_1+c_1=0$, then $a=c$, $b=d$ and therefore $a+b-c-d=0$ which we already discussed.

In summary,

Theorem (part 2) *The centroid of vertices and the centroid of perimeter of a quadrangle with parallel diagonals coincide if and only if the quadrangle is either mirror symmetric with the diagonals perpendicular to the axis or crossed centrally symmetric.*

Centroid of perimeter = centroid of area

For parallelograms all three sorts of centroids coincide. Conversely we have seen that a non-crossed quadrangle whose vertex centroid coincides either with the centroid of area or with the centroid of perimeter must be a parallelogram. The question remains whether there are non-crossed quadrangles other than parallelograms whose centroids of area and perimeter coincide. We will restrict our attention to kites that have this property.

In order to describe them, we choose orthogonal coordinates such that the vertices of the kites can be presented as $A(0,1)$, $B(-p,0)$, $C(0,-1)$, $D(q,0)$ with $p>0$ and $p \neq q$. For $q>0$ the kite is convex, for $q<0$ concave. A kite is a mirror symmetric quadrangle; all its centroids belong to the x-axis of our coordinate system, which is the kite's axis of symmetry. We show:

- If $0<p<\sqrt{3}$ there is just one positive and just one negative value for q such that the centroids under consideration of the corresponding kite coincide, whence one of the corresponding kites will be convex, the other concave.
- If $p=\sqrt{3}$ there is a suitable negative value for q yielding exactly one concave kite of the desired kind.
- If $\sqrt{3}<p$ there are exactly two negative values for q yielding two concave kites of the desired kind.

We compute the centroid of area of the triangle BCD

$$S_A\left(\frac{q-p}{3},\frac{1}{3}\right)$$

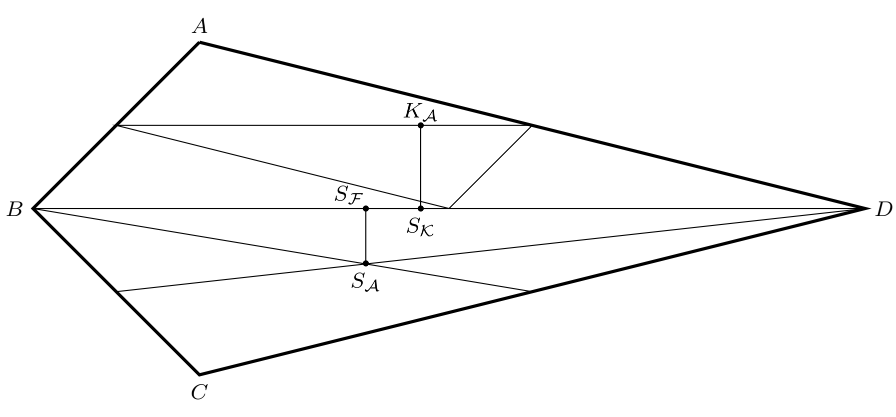
and obtain the centroid of area of the kite

$$S_{\mathcal{F}}\left(\frac{q-p}{3},0\right).$$

We obtain the centroid of perimeter of the kite from (1)

$$S_{\mathcal{K}}\left(\frac{dq-ap}{2(a+d)},0\right).$$

How these centers can be constructed is shown in the following diagram



The condition for $S_{\mathcal{F}} = S_{\mathcal{K}}$ is:

$$\frac{q-p}{3} = \frac{dq-ap}{2(a+d)},$$

or more simply

$$2(a+d)(q-p) = 3(dq-ap).$$

This is a root equation since $a = b = \sqrt{1+p^2}$, $c = d = \sqrt{1+q^2}$. We solve for q in terms of p. To this end we transform it first to

$$(2q+p)a = (q+2p)d.$$

Squaring yields

$$(2q+p)^2(1+p^2) = (q+2p)^2(1+q^2).$$

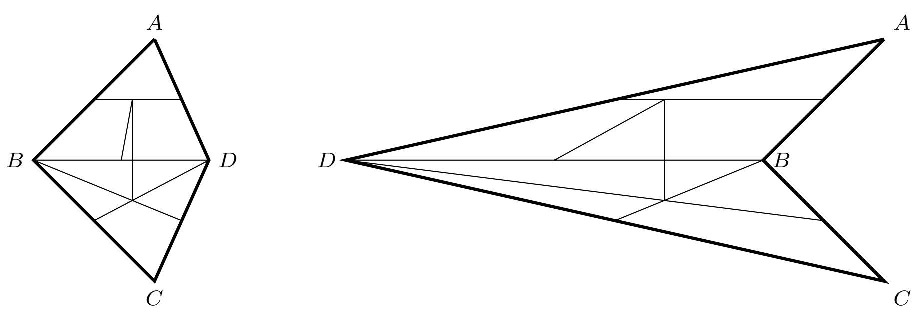
So we have the polynomial equation for q:

$$q^4 + 4q^3p - 3q^2 + 3p^2 - 4qp^3 - p^4 = 0.$$

Evidently it has roots $\pm p$. For $+p$ we have a rhombus, a special parallelogram, which is not relevant. The value $-p$ is extraneous. Splitting those roots we obtain the quadratic equation

$$q^2 + 4pq + p^2 - 3 = 0$$

with the roots $-2p \pm \sqrt{3(1+p^2)} = -2p \pm a\sqrt{3}$. Both values satisfy the given equation. Only when $p < \sqrt{3}$ can one of the roots be positive, in which case the resulting kite would be convex. When $p = \sqrt{3}$, one of the roots is $q = 0$ and the points A, C, D would be collinear, which is excluded by hypothesis; thus there is just one kite when $p = 3$ and it is concave. Since p is always positive, all other values of p yield two negative values for q and, consequently, a pair of concave kites. The diagram shows the two kites with $p = 1$ for which the centroids of perimeter and of area coincide.



Our treatment leaves open a further question: Do there exist non-crossed quadrangles beside parallelograms and kites whose centroid of perimeter and centroid of area coincide?

Preview

We shall conclude these thoughts in a further note. In [4] we connect the centroid of vertices to van Aubel's Square Theorem [1].

References

[1] H. H. van Aubel, *Note concernant les centres de carrés construits sur les côtés d'un polygon quelconque,* Nouvelle Correspondance Mathématique, **4** 1878, pp. 40-44.

[2] R. Fritsch, G. Pickert, *Vierecksschwerpunkte I, Die Wurzel* **48**(2) (2014); parts II and III are to appear in issues (3/4) and (5).

[3] R. Fritsch, G. Pickert, *The Seebach-Walser Line of a Quadrangle,* CRUX **39**(4), 2013, pp. 178-184.

[4] R. Fritsch, G. Pickert, *A Quadrangle's Centroid of Vertices and van Aubel's Square Theorem,* to appear in CRUX.

[5] C. Kimberling, *Encyclopedia of Triangle Centers – ETC,* `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[6] K. Seebach, *Über Schwerpunkte von Dreiecken, Vierecken und Tetraedern, Teil 1,* Didaktik der Mathematik, **11** 1983, pp. 270-282.

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PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. Nous préférons les réponses électroniques et demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX* et *un fichier pdf pour chaque solution, bien que les autres formats (Word, etc.) soient aussi acceptés. Nous invitons les lecteurs à envoyer leurs solutions au rédacteur à l'adresse* `crux-redacteurs@smc.math.ca`*. Nous acceptons aussi les contributions par la poste, envoyées à l'adresse figurant en troisième de couverture. Le nom de la personne qui propose une solution doit figurer avec* chaque *solution, de même que l'tablissement qu'elle fréquente, sa ville et son pays; chaque solution doit également commencer sur une nouvelle page. Un astérisque ($\star$) signale un problème proposé sans solution.*

Nous sommes surtout à la recherche de problèmes originaux, mais d'autres problèmes intéressants peuvent aussi être acceptables pourvu qu'ils ne soient pas trop connus et que leur provenance soit indiquée. Normalement, si l'on connaît l'auteur d'un problème, on ne doit pas le proposer sans lui en demander la permission. Les solutions connues doivent accompagner les problèmes proposés. Si la solution n'est pas connue, la personne qui propose le problème doit tenter de justifier l'existence d'une solution. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_Proposition_Année_numéro *(exemple : Tremblay_Julie_Proposition_2014_4.tex, s'il s'agit du 4e problème proposé par Julie en 2014).*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er octobre 2014***; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal d'avoir traduit les problèmes.

3851. *Proposé par Billy Jin, Waterloo Collegiate Institute, Waterloo, ON; et Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Soit $U = \{1, 2, 3, \ldots, n\}$ où $n \in \mathbb{N}$, et soit $S \subseteq U$ avec $|S| = k$ où $0 < k \leq n$.Trouver le nombre de paires non ordonnées (X, Y) telles que $S = X\Delta Y$ où X et Y sont des sous-ensembles de U, et $X\Delta Y = (X-Y)\cup(Y-X) = (X\cup Y)-(X\cap Y)$ est la différence symétrique de X and Y.

3852. *Proposé par Václav Konečný, Big Rapids, MI, É-U.*

On donne les graphes de deux fonctions f et g, positives, continues et croissantes, satisfaisant à $0 < f(x) < g(x)$ pour tout $x \geq 0$. On considère le système d'équations

$$\begin{aligned} f(x_1) + f(x_2) &= K \\ g(x_1) + g(x_2) &= L. \end{aligned}$$

Si $x_1 > 0$ et $K > 0$ sont donnés, trouver $x_2 > 0$ et $L > 0$ via la construction classique grecque (avec la règle et le compas) de sorte que le système d'équation soit satisfait.

3853. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit a, b, c trois nombres réels positifs tels que $a + b + c = 3$. Montrer que

$$\frac{a}{b(2c+a)} + \frac{b}{c(2a+b)} + \frac{c}{a(2b+c)} \geq 1.$$

3854. *Proposé par Paul Yiu, Florida Atlantic University, Boca Raton, FL, É-U.*

Montrer que la parabole tangente aux bissectrices interne et externe des angles B et C du triangle ABC a comme foyer le sommet A et comme directrice la droite BC.

3855. *Proposé par Leonard Giugiuc, Roumanie.*

Soit a et b deux nombres réels avec $0 < a < b$ et $\dfrac{1+ab}{b-a} \leq \sqrt{3}$. Montrer que

$$(1+a^2)(1+b^2) \geq 4a(a+b).$$

Quand y a-t-il égalité ?

3856. *Proposé par Nguyen Ngoc Giag, Vietnam Institute of Educational Sciences, Ha Noi, Vietnam.*

On donne un triangle ABC avec les bissectrices internes AA', BB', CC'. Les bissectrices CC' et BB' coupent respectivement $A'B'$ en F et $C'A'$ en E. Montrer que si $BE = CF$ alors le triangle ABC est isocèle.

3857. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit a, b, c trois nombres réels positifs tels que $abc = 1$. Montrer que

$$\frac{a^{n+2}}{a^n + (n-1)b^n} + \frac{b^{n+2}}{b^n + (n-1)c^n} + \frac{c^{n+2}}{c^n + (n-1)a^n} \geq \frac{3}{n}$$

pour chaque entier positif n.

3858. *Proposé par Michel Bataille, Rouen, France.*

Soit a, b deux nombres réels positifs avec $a \neq b$. Résoudre le système

$$\begin{aligned} a^2x^2 - 2abxy + b^2y^2 - 2a^2bx - 2ab^2y + a^2b^2 &= 0 \\ abx^2 + (a^2 - b^2)xy - aby^2 + ab^2x - a^2by &= 0 \end{aligned}$$

pour $(x, y) \in \mathbb{R}^2$.

3859. *Proposé par Jung In Lee, École Secondaire Scientifique de Séoul, Séoul, République de Corée.*

On définit la suite $\{F_n\}$ par $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ pour $n \geq 1$. Pour tout nombre naturel m, on définit $v_2(m)$ comme $v_2(m) = n$ if $2^n \mid m$ et $2^{n+1} \nmid m$. Trouver tous les nombres entiers positifs n satisfaisant l'équation

$$v_2(n!) = v_2(F_1F_2 \cdots F_n).$$

3860. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $n \geq 3$ un entier impair et soit $A \in M_n(\mathbb{Z})$. Montrer que le déterminant de $3A + 4A^T$ est divisible par 7. Le résultat reste-t-il vrai si n est un entier pair ?

. . . .

3851. *Proposed by Billy Jin, Waterloo Collegiate Institute, Waterloo, ON; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let $U = \{1, 2, 3, \ldots, n\}$ where $n \in \mathbb{N}$, and let $S \subseteq U$ with $|S| = k$ where $0 < k \leq n$. Determine the number of unordered pairs (X, Y) such that $S = X\Delta Y$ where X and Y are subsets of U, and $X\Delta Y = (X-Y)\cup(Y-X) = (X\cup Y)-(X\cap Y)$ is the symmetric difference of X and Y.

3852. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Given the graphs of two positive, continuous, increasing functions f, g, satisfying $0 < f(x) < g(x)$ for all $x \geq 0$. Consider the following system of equations

$$\begin{aligned} f(x_1) + f(x_2) &= K \\ g(x_1) + g(x_2) &= L. \end{aligned}$$

If $x_1 > 0$ and $K > 0$ are given, find $x_2 > 0$ and $L > 0$ by the Classical Greek construction (compass and straightedge) such that the system of equations is satisfied.

3853. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{a}{b(2c+a)} + \frac{b}{c(2a+b)} + \frac{c}{a(2b+c)} \geq 1.$$

3854. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Show that the parabola tangent to the internal and external bisectors of angles B and C of triangle ABC has focus at vertex A and directrix the line BC.

3855. *Proposed by Leonard Giugiuc, Romania.*

Let a and b be real numbers with $0 < a < b$ and $\dfrac{1+ab}{b-a} \le \sqrt{3}$. Prove that

$$(1+a^2)(1+b^2) \ge 4a(a+b).$$

When does equality hold?

3856. *Proposed by Nguyen Ngoc Giag, Vietnam Institute of Educational Sciences, Ha Noi, Vietnam.*

Given a triangle ABC with internal angle bisectors AA', BB', CC'. Bisector CC' meets $A'B'$ at F, and bisector BB' meets $C'A'$ at E. Prove that if $BE = CF$ then triangle ABC is isosceles.

3857. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a^{n+2}}{a^n + (n-1)b^n} + \frac{b^{n+2}}{b^n + (n-1)c^n} + \frac{c^{n+2}}{c^n + (n-1)a^n} \ge \frac{3}{n}$$

for each positive integer n.

3858. *Proposed by Michel Bataille, Rouen, France.*

Let a, b be positive real numbers with $a \ne b$. Solve the system

$$\begin{aligned} a^2x^2 - 2abxy + b^2y^2 - 2a^2bx - 2ab^2y + a^2b^2 &= 0 \\ abx^2 + (a^2-b^2)xy - aby^2 + ab^2x - a^2by &= 0 \end{aligned}$$

for $(x, y) \in \mathbb{R}^2$.

3859. *Proposed by Jung In Lee, Seoul Science High School, Seoul, Republic of Korea.*

The sequence $\{F_n\}$ is defined by $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n \ge 1$. For any natural number m, define $v_2(m)$ as $v_2(m) = n$ if $2^n \mid m$ and $2^{n+1} \nmid m$. Find all positive integer n that satisfy the equation

$$v_2(n!) = v_2(F_1F_2\cdots F_n).$$

3860. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $n \ge 3$ be an odd integer and let $A \in M_n(\mathbb{Z})$. Prove that the determinant of $3A + 4A^T$ is divisible by 7. Does the result hold when n is an even integer?

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3751. [2012 : 241, 243] *Proposed by Richard K. Guy, University of Calgary, Calgary, AB.*

The edge lengths of a quadrilateral are $AB = 5$, $BC = 10$, $CD = 11$, $DA = 14$.

(a) If the quadrilateral is cyclic, what is the diameter of its circumcircle?

(b) If we alter the order of the edges, does it affect the answer to (a)?

I. Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

For part (a) note that

$$AB^2 + DA^2 = 5^2 + 14^2 = 221 = 10^2 + 11^2 = BC^2 + CD^2.$$

Hence, if we take $BD = \sqrt{221}$, then ABD and BCD will form two right triangles that share their hypotenuse BD, which implies that the resulting quadrilateral $ABCD$ has a circumcircle whose diameter is $BD = \sqrt{221}$.

For part (b) the answer is *no*, altering the order will generally produce a new quadrilateral, but the circumcircle of the new quadrilateral will have the same diameter. To see this, we denote the centre of the circle of part (a) by O. Altering the order of the edges is the same as interchanging the triangles OAB, OBC, OCD, ODA. No matter how these triangles might be permuted, the four angles at O will still sum to 360°, and the sides opposite O would form the sides of a new quadrilateral that is still inscribed in the circle whose radius is $OA = OB = OC = OD = \frac{\sqrt{221}}{2}$.

II. Solution by John Hawkins and David R. Stone, Georgia Southern University, Statesboro, GA, USA.

We answer both parts together. In the cyclic quadrilateral $ABCD$, let $a = AB = 5$, $b = BC = 10$, $c = CD = 11$, and $d = DA = 14$, and let $s = \frac{1}{2}(a + b + c + d) = 20$ be the semiperimeter. Brahmagupta's formula gives us the area of the cyclic quadrilateral:

$$A = \sqrt{(s-a)(s-b)(s-c))(s-d)} = \sqrt{(20-5)(20-10)(20-11)(20-14)} = 90.$$

Note that it is a symmetric polynomial in the four variables, so that A is invariant with respect to altering the order of the side lengths. Also the product

$$P = (ac + bd)(ad + bc)(ab + cd) = (55 + 140)(70 + 110)(50 + 154) = 195 \cdot 180 \cdot 204$$

is a symmetric polynomial and therefore invariant under reordering of the side lengths. On the *MathWorld* web page for cyclic quadrilaterals [or any other standard reference] we find the formula $4RA = \sqrt{P}$ involving the circumradius R; consequently, the diameter equals

$$2R = \frac{\sqrt{P}}{2A} = \frac{\sqrt{195 \cdot 180 \cdot 204}}{2 \cdot 90} = \sqrt{13 \cdot 17} = \sqrt{221},$$

and it will not change when the order of the edges is altered.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; MARIAN DINCĂ, Bucharest, Romania; JOHN G. HEUVER, Grande Prairie, AB; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; ERCOLE SUPPA, Teramo, Italy; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; TITU ZVONARU, Cománeşti, Romania; and the proposer.

We commonly accept convexity to be part of the definition of a cyclic quadrilateral. Both solutions show further that the area of a cyclic quadrilateral will not change when the order of the edges is altered. More precisely, given the four line segments that form a cyclic quadrilateral, they will, in general, in their six possible orders form three convex quadrilaterals that are not congruent, yet they will have the same circumradius and the same area. Only the proposer addressed the corresponding results for crossed quadrilaterals that are inscribed in a circle. The first solution shows that when the edges are not just rearranged, but are allowed to form a crossed quadrilateral, those quadrilaterals will still have the same circumcircle. (The formula for $2R$ in the second solution requires convexity; it should not be used for crossed quadrilaterals. Indeed, the circumradius of a crossed quadrilateral that is inscribed in a circle is generally different from the common circumradius of its convex mates.)

Most of the submissions were similar to one of the featured solutions, although many provided more background details. Such details were discussed recently in the solution of the related problem 2724, which appeared in the March issue [2013 : 148-149].

3752. [2012 : 241, 243] *Proposed by Péter Ivády, Budapest, Hungary.*

Show that if $n \geq 2$ is a positive integer then

$$\frac{1}{2}\left[1 + \frac{1}{n}\left(1 - \frac{1}{n}\right)\right]^2 < \left(1 - \frac{1}{2^3}\right)\left(1 - \frac{1}{3^3}\right)\cdots\left(1 - \frac{1}{n^3}\right)$$

holds.

Solution by Haohao Wang and Jerzy Wojdylo, Southeast Missouri State University, Cape Girardeau, Missouri, USA.

We will prove the claim by induction on n.

First, if $n \geq 2$, then the claim holds since

$$\frac{1}{2}\left[1+\frac{1}{2}\left(1-\frac{1}{2}\right)\right]^2 = \frac{25}{32} < \frac{7}{8} = \left(1-\frac{1}{2^3}\right).$$

Assume the claim is true for $n = k$. So we have

$$\frac{1}{2}\left[1+\frac{1}{k}\left(1-\frac{1}{k}\right)\right]^2 < \left(1-\frac{1}{2^3}\right)\left(1-\frac{1}{3^3}\right)\dots\left(1-\frac{1}{k^3}\right) \tag{1}$$

and we need to show that the claim is true for $n = k+1$. Multiplying inequality (1) by $\left(1-\frac{1}{(k+1)^3}\right)$, we obtain

$$\begin{aligned}\frac{1}{2}\left[1+\frac{1}{k}\left(1-\frac{1}{k}\right)\right]^2 &\left(1-\frac{1}{(k+1)^3}\right)\\ &< \left(1-\frac{1}{2^3}\right)\left(1-\frac{1}{3^3}\right)\dots\left(1-\frac{1}{k^3}\right)\left(1-\frac{1}{(k+1)^3}\right)\end{aligned} \tag{2}$$

Now, we notice that

$$\begin{aligned}&\frac{1}{2}\left[1+\frac{1}{k}\left(1-\frac{1}{k}\right)\right]^2\left(1-\frac{1}{(k+1)^3}\right) - \frac{1}{2}\left[1+\frac{1}{k+1}\left(1-\frac{1}{k+1}\right)\right]^2\\ &= \frac{3-11k^2-8k^3+3k^4+2k^5}{2k^3(1+k)^4} = \frac{(2k^2-8)k^3+(3k^2-11)k^2+3}{2k^3(1+k)^4} > 0,\end{aligned}$$

since both $2k^2-8 \geq 0$ and $3k^2-11 > 0$ as $k \geq 2$. Therefore, we have

$$\frac{1}{2}\left[1+\frac{1}{k+1}\left(1-\frac{1}{k+1}\right)\right]^2 < \frac{1}{2}\left[1+\frac{1}{k}\left(1-\frac{1}{k}\right)\right]^2\left(1-\frac{1}{(k+1)^3}\right). \tag{3}$$

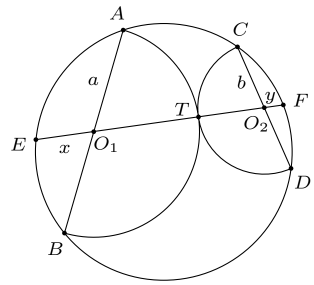
Thus, from (2) and (3) the claim is true for $n = k+1$.

This completes the proof of the original inequality for all $n \geq 2$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; MIHAÏ-IOAN STOËNESCU, Bischwiller, France; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3753. [2012 : 241, 243] *Proposed by Abdilkadir Altintaş, mathematics teacher, Emirdağ, Turkey.*

Semi-circles with centres O_1 and O_2 are drawn on chords AB and CD of a circle Γ such that they are tangent at T. The line through O_1 and O_2 intersects Γ at E and F. If $O_1A = a$, $O_2C = b$, $O_1E = x$ and $O_2F = y$, show that $a-b = x-y$.



Solution by several respondents.

By the Intersecting Chords Theorem, we find that $a^2 = x(a+b+y)$ and $b^2 = y(a+b+x)$. Therefore

$$a^2 - b^2 = (x-y)(a+b)$$

from which the result follows.

Solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; DAO THANH OAI, Kien Xuong, Thai Binh, Viet Nam; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW, Germany; LEONARD GIUGIUC, Romania; JOHN G. HEUVER, Grande Prairie, AB; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; PANAGIOTE LIGOURAS, Leonardo da Vinci High School, Noci, Italy; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; MIHAÏ-IOAN STOËNESCU, Bischwiller, France; ERCOLE SUPPA, Teramo, Italy; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; JACQUES VERNIN, Marseille, France; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Cománeşti, Romania; and the proposer.

Dao proposed the following generalization for which he enclosed a proof by Lui González: Suppose that the circles with diameters AB and CD do not necessarily intersect and that their radical axis meets EF at T. Then $TO_1 - TO_2 = EO_1 - EO_2$.

3754. [2012 : 242, 243] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove that in all scalene triangles ΔABC the inequality

$$576\sqrt{3}r^3 < \frac{w_a^2 - w_b^2}{b-a} + \frac{w_b^2 - w_c^2}{c-b} + \frac{w_c^2 - w_a^2}{a-c} < 72\sqrt{3}R^3$$

holds, where w_a, w_b and w_c are the lengths of the angle bisectors; R is the radius of the circumcircle; and r is the inradius of ΔABC.

Solution by Oliver Geupel, Brühl, NRW, Germany.

The two inequalities do not generally hold. Consider a right triangle with sides $a = 5t$, $b = 4t$, and $c = 3t$ where t is a positive real parameter. Its semiperimeter is $s = 6t$. Straightforward computations yield

$$w_a^2 = \frac{4bcs(s-a)}{(b+c)^2} = \frac{288}{49}t^2, \qquad w_b^2 = \frac{45}{4}t^2, \qquad w_c^2 = \frac{160}{9}t^2,$$

$$R = \frac{5}{2}t, \qquad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = t.$$

The left inequality thus specializes to $576\sqrt{3}t^3 < \dfrac{2624}{147}t$, which is false when

$$t^2 \geq \frac{2624}{147 \cdot 576\sqrt{3}}.$$

The right inequality rewrites as $\dfrac{2624}{147}t < 1125\sqrt{3}t^3$. But this fails when

$$t^2 \leq \frac{2624}{147 \cdot 1125\sqrt{3}}.$$

Consequently, both inequalities are not generally valid.

One incorrect solution was received.

Upon closer inspection, the proposer lost a factor part way through his solution (as did the person who sent in the incorrect solution). As a result, the original inequality should have read

$$576\sqrt{3}r^3 < \sum_{cyclic} \frac{(w_a^2 - w_b^2)(b+c)^2(a+c)^2}{c(b-a)(a+b+c)} < 72\sqrt{3}R^3$$

which is less appealing to look at than the original. Using Geupel's right triangle example in the inequality above yields

$$576\sqrt{3}t^3 < 1524t^3 < 1125\sqrt{3}t^3$$

which is true.

3755. [2012 : 242, 244] *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Find all real numbers $a \leq b \leq c \leq d$ which form an arithmetic progression which satisfy the two equations $a + b + c + d = 1$ and $a^2 + b^2 + c^2 + d^2 = d$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA; and Titu Zvonaru, Cománeşti, Romania(independently).

Let the four numbers be $a = m-3h$, $b = m-h$, $c = m+h$, $d = m+3h$ where $h \geq 0$. The two equations are equivalent to $4m = 1$ and $4m^2 + 20h^2 = m + 3h$. This leads to $20h^2 = 3h$ which implies that $h = 0$ or $h = \frac{3}{20}$. The two solutions are

$$(a,b,c,d) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(-\frac{1}{5}, \frac{1}{10}, \frac{2}{5}, \frac{7}{10}\right).$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE

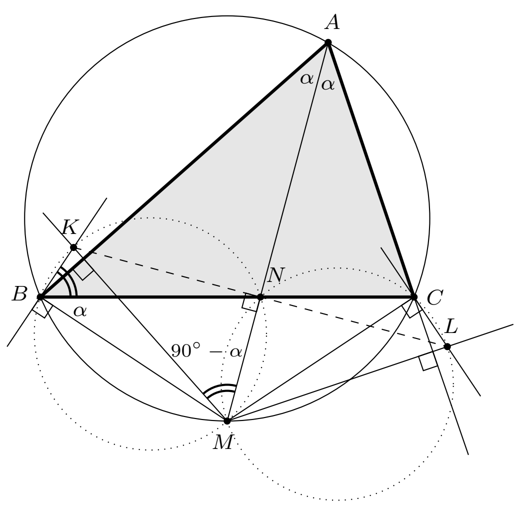
BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MATEI COICULESCU, East Lyme High School, East Lyme, CT, USA; GREG COOK, student, Angelo State University, San Angelo, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; LEONARD GIUGIUC, Romania; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ANGEL PLAZA, University of Las Palmas de Gran Canaria, Spain; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; MIHAÏ-IOAN STOËNESCU, Bischwiller, France; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; STAN WAGON, Macalester College, St. Paul, MN, USA; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer.

The proposer points out that if we merely require that $a+b+c+d$ be an integer, then we get exactly two more solutions $(a,b,c,d) = (0,0,0,0), (-9/10, -3/10, 3/10, 9/10)$.

3756. [2012 : 242, 244] *Proposed by Michel Bataille, Rouen, France.*

Let triangle ABC be inscribed in circle Γ and let M be the midpoint of the arc BC of Γ not containing A. The perpendiculars to AB through M and to MB through B intersect at K and the perpendiculars to AC through M and to MC through C intersect at L. Prove that the lines BC, AM intersect at the midpoint of KL.

Solution by Ricardo Barroso Campos, University of Seville, Seville, Spain, modified by the editor.



Let $N = BC \cap AM$, and $\alpha = \frac{\angle BAC}{2}$. We can assume without loss of generality that $\angle CBA < \angle ACB$ as in the figure (or use directed angles). Because M is the midpoint of the arc BC,

$$\alpha = \angle BAM = \angle MAC = \angle MBC.$$

Using the right angles first at the intersection of KM and AB and then at B, we have

$$\angle NMK = \angle AMK = 90^\circ - \alpha \quad \text{and} \quad \angle NBK = \angle MBK - \angle MBC = 90^\circ - \alpha.$$

From $\angle NMK = \angle NBK$ and $\angle MBK = 90^\circ$ we deduce that $KBMN$ is inscribed in a circle whose diameter is MK, which makes $\angle MNK = 90^\circ$ also. Analogously, $LCNM$ is cyclic with diameter ML and $\angle MNL = 90^\circ$. Because MN is perpendicular to both NK and NL, N must lie on the line KL. Also, the right triangles KNM and LNM have corresponding angles of $90^\circ - \alpha$ at their common vertex M and they share the side MN; consequently, they are congruent, whence $NK = NL$. That is, N is the midpoint of KL.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; MIHAÏ-IOAN STOËNESCU, Bischwiller, France; ERCOLE SUPPA, Teramo, Italy; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; JACQUES VERNIN, Marseille, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer. There was one incomplete submission.

3757. [Correction, 2012 : 284, 286] *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let A, B, C be the angles (measured in radians), R the circumradius and r the inradius of a triangle. Prove that

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \leq \frac{9}{2\pi} \cdot \frac{R}{r}.$$

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Consider the function $f(x) = \ln\left(\frac{\sin\frac{x}{2}}{x}\right) = \ln\left(\sin\frac{x}{2}\right) - \ln x$, $x \in (0, \pi)$. Straightforward computations show that $f'(x) = \frac{1}{2}\cot\frac{x}{2} - \frac{1}{x}$ and

$$f''(x) = -\frac{1}{4}\csc^2 x + \frac{1}{x^2} = \frac{\sin^2\left(\frac{x}{2}\right) - \frac{1}{4}x^2}{x^2\sin^2\left(\frac{x}{2}\right)} = \frac{\left(\sin\frac{x}{2} + \frac{x}{2}\right)\left(\sin\frac{x}{2} - \frac{x}{2}\right)}{x^2\sin^2 x}.$$

Since $0 < \sin\frac{x}{2} < \frac{x}{2}$ for $x \in (0, \pi)$ we have $f''(x) < 0$ so f is concave on $(0, \pi)$. Hence, by Jensen's Inequality we have

$$\frac{1}{3}f(A) + \frac{1}{3}f(B) + \frac{1}{3}f(C) \leq f\left(\frac{A+B+C}{3}\right) = f\left(\frac{\pi}{3}\right)$$

so

$$\frac{1}{3}\left(\ln\left(\frac{\sin\frac{A}{2}}{A}\right) + \ln\left(\frac{\sin\frac{B}{2}}{B}\right) + \ln\left(\frac{\sin\frac{C}{2}}{C}\right)\right) \leq \ln\left(\frac{\sin\frac{\pi}{6}}{\frac{\pi}{3}}\right) = \ln\left(\frac{3}{2\pi}\right)$$

or

$$\ln\left(\frac{\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{ABC}\right)\le\ln\left(\frac{3}{2\pi}\right)^3$$

so

$$\frac{1}{ABC}\le\frac{27}{8\pi^3}\cdot\frac{1}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}\,. \tag{1}$$

It is well known that

$$\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}=\frac{r}{4R}\,. \tag{2}$$

From (1) and (2) we then have

$$\frac{1}{ABC}\le\frac{27}{2\pi^3}\cdot\frac{R}{r}\,. \tag{3}$$

Finally, using (3) and the obvious inequality $AB+BC+CA\le\frac{1}{3}(A+B+C)^2$ we have

$$\frac{1}{A}+\frac{1}{B}+\frac{1}{C}=\frac{AB+BC+CA}{ABC}\le\frac{(A+B+C)^2}{3ABC}=\frac{9}{2\pi}\cdot\frac{R}{r}$$

and our proof is complete. Clearly equality holds if and only if $A=B=C$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MARIAN DINCĂ, Bucharest, Romania; NERMIN HODŽIĆ, Dobošnica, Bosnia and Herzegovina and SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

*Arslanagić gave a similar proof and pointed out that (1) is actually inequality 6.59 on p. 188 of the book "*Recent Advances in Geometric Inequalities*" (Kluwer Academic Publishers, Dordrecht/Boston/London, 1989) by D.S. Mitrinović, J.E. Pečarić and V. Volenec. We decided to give a proof for completeness. Dinca pointed out that since $\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\ge\frac{9}{A+B+C}=\frac{9}{\pi}$ by the AM-HM inequality, the result can be strengthened to a double inequality*

$$\frac{9}{\pi}\le\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\le\frac{9}{2\pi}\cdot\frac{R}{r}$$

or

$$2r\le\frac{2\pi r}{9}\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)\le R$$

which is a refinement of the famous Euler inequality $2r\le R$.

3758. [2012 : 242, 244] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a point X on the segment BC, construct a point A such that the incircle of triangle ABC touches BC at X, and that the line joining the Gergonne point and the Nagel points of the triangle is parallel to BC.

Composite of solutions by Daniel Văcaru, Piteşti, Romania; and by the proposer.

We assume the desired triangle ABC exists and let $a=BC$, $b=CA$, $c=AB$, and $s=\frac{1}{2}(a+b+c)$. If X and Z are the points where the incircle touches the sides BC and BA, then (by definition) the cevians AX and CZ intersect in the Gergonne point G_e. From the standard properties of incircles we know that

$BX = BZ = s - b$, $AZ = s - a$, and $CX = s - c$. Because we have been given the segment BC and a point X on it, we therefore know the lengths $s - b = BX$ and $s - c = CX$ and must find $s - a$. Applying Menelaus's theorem to triangle ABX with transversal CG_eZ we have

$$-1 = \frac{AG_e}{G_eX} \cdot \frac{XC}{CB} \cdot \frac{BZ}{ZA} = \frac{AG_e}{G_eX} \cdot \frac{s-c}{-a} \cdot \frac{s-b}{s-a},$$

whence,

$$\frac{AG_e}{G_eX} = \frac{a(s-a)}{(s-b)(s-c)}. \tag{1}$$

We next let X' and Y' be the points where the excircles touch the sides BC and AC, so that the cevians AX' and BY' intersect (by definition) in the Nagel point N_a. Again we know that $BX' = AY' = s-c$, $CX' = s-b$, and $CY' = s-a$. Applying Menelaus's theorem to triangle $AX'C$ with transversal BN_aY', we have

$$-1 = \frac{AN_a}{N_aX'} \cdot \frac{X'B}{BC} \cdot \frac{CY'}{Y'A} = \frac{AN_a}{N_aX'} \cdot \frac{-(s-c)}{a} \cdot \frac{s-a}{s-c},$$

whence,

$$\frac{AN_a}{N_aX'} = \frac{a}{s-a}. \tag{2}$$

Because the transversal G_eN_a of the triangle AXX' is parallel to XX' if and only if $\frac{AG_e}{G_eX} = \frac{AN_a}{N_aX'}$, we deduce from (1) and (2) that G_eN_a is parallel to BC (which contains the segment XX') if and only if

$$(s-a)^2 = (s-b)(s-c). \tag{3}$$

Consequently, we want $s - a$ to be the geometric mean of $s - b$ and $s - c$, a quantity whose construction was given by Euclid. From this we get the lengths $b = (s-a)+(s-c)$ and $c = (s-a)+(s-b)$. Because our argument is reversible, as long as the quantities a, b, c satisfy the triangle inequality, there will be a unique triangle ABC that satisfies (3), whose incircle touches BC at the given point X. Here, then, is its construction.

1. Construct the perpendicular to BC at X, and call P either point where it intersects the circle whose diameter is BC. (Then PX is the geometric mean of $BX = s - b$ and $CX = s - c$; that is, $PX = \sqrt{(s-b)(s-c)} = s - a$.)

2. Call B' and C' the points where the circle with centre X and radius XP intersects the line BC, labeled so that B and C' are on the same side of X. (Then $BB' = BX + XB' = s - b + s - a = c$ and $C'C = C'X + XC = s - a + s - c = b$.)

3. The desired third vertex A will be either point where the circle with centre B and radius $c = BB'$ intersects the circle with centre C and radius $b = CC'$.

It remains to verify that the circles in the third step of the construction will always intersect; that is, we must show that the constructed segments a, b, c satisfy the triangle inequality:

$$\begin{aligned} c+b &= BB'+CC' = (BX+XB')+(CX+XC') > BX+XC = a, \\ a+b &= (BX+XC)+(CX+XC') > BX+XC' = BX+XB' = c, \\ a+c &= (BX+XC)+(BX+XB') > CX+XB' = CX+XC' = b. \end{aligned}$$

This proves the existence of the constructed triangle ABC whose incircle touches BC at the given point X and whose sides satisfy equation (3), so that $G_eN_a||BC$ as required.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comănești, Romania.

3759. [2012 : 242, 244] *Proposed by Nguyen Minh Ha, Hanoi, Vietnam.*

Given a convex polygon $A_1A_2\cdots A_n$ with an interior point P. Let $a_i = \sum_{j=1}^{n} A_iA_j$. Prove that $\sum_{i=1}^{n} PA_i < \max_{1\le j\le n}\{a_j\}$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

The position vector $\vec{P}$ of a point P can be expressed as a convex linear combination of the position vectors of the A_i's, $i = 1, 2, \dots, n$

$$\vec{P} = \sum_{i=1}^{n} \lambda_i \vec{A}_i$$

where $\lambda_i > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. Hence,

$$\vec{A}_i - \vec{P} = \vec{A}_i - \sum_{j=1}^{n} \lambda_j \vec{A}_j = \sum_{j=1}^{n} \lambda_j(\vec{A}_i - \vec{A}_j), \quad i = 1, 2, \dots, n.$$

By the triangle inequality, we then have, for each $i = 1, 2, \dots, n$,

$$|PA_i| = \left|\sum_{j=1}^{n} \lambda_j(\vec{A}_i - \vec{A}_j)\right| < \sum_{j=1}^{n} \lambda_j|A_jA_i|. \tag{1}$$

[*Ed. : For clarity we use $|XY|$ to denote the distance between points X and Y, that is, the length of the vector $\vec{Y} - \vec{X}$.*]

Note that the inequality in (1) is strict since the vectors $\vec{A}_i - \vec{A}_j$ are not collinear.

Adding the inequality in (1) over $i = 1, 2, \ldots, n$ we then obtain

$$\sum_{i=1}^{n} |PA_i| < \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_j |A_j A_i| = \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} |A_j A_i|$$
$$= \sum_{j=1}^{n} \lambda_j a_j \le \max_{1 \le j \le n} \{a_j\}.$$

This completes the proof.

Also solved by the proposer.

Geupel remarked that the problem is a generalization of problem **2215** *[1997 : 109; 1998 : 121] which dealt with the case $n = 3$. This same special problem was also posed in the internet forum* Art of Problem Solving. *The solution by a solver nicknamed* gemath *straightforwardly generalizes to his proof featured above. He gave the following reference:*

`http://www.artofproblemsolving.com/forum/viewtopic.php?p=634149.`

3760. [2012 : 243, 244] *Proposed by Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $p \ge 2$ be an integer. Determine the limit

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{\sqrt[p]{n}}{\sum_{j=1}^{p} \sqrt[p]{k^j (n+k)^{p-j+1}}}.$$

Solution by Haohao Wang and Jerzy Wojdylo, Southeast Missouri State University, Cape Girardeau, Missouri, USA.

We claim that

$$\lim_{n\to\infty} \sum_{k\ge 1} \frac{\sqrt[p]{n}}{\displaystyle\sum_{j=1}^{p} \sqrt[p]{k^j \left(n+k\right)^{p-j+1}}} = \frac{p}{p-1}.$$

To prove our claim, we note that since $a + a^2 + \cdots + a^p = \dfrac{a\left(1-a^p\right)}{1-a}$, we have

$$\begin{aligned}
\sum_{j=1}^{p} \sqrt[p]{k^j \left(n+k\right)^{p-j+1}} &= \left(n+k\right)^{(p+1)/p} \left[\sum_{j=1}^{p} \left(\frac{k}{n+k}\right)^{j/p}\right] \\
&= \left(n+k\right)^{(p+1)/p} \cdot \frac{\left(\frac{k}{n+k}\right)^{1/p} \left(1 - \frac{k}{n+k}\right)}{1 - \left(\frac{k}{n+k}\right)^{1/p}} \\
&= \frac{\left(n+k\right)^{(p+1)/p} \left(\frac{k}{n+k}\right)^{1/p} \frac{n}{n+k}}{1 - \left(\frac{k}{n+k}\right)^{1/p}} \\
&= \frac{nk^{1/p}}{1 - \left(\frac{k}{n+k}\right)^{1/p}}.
\end{aligned}$$

Thus

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{\sqrt[p]{n}}{\sum\limits_{j=1}^{p}\sqrt[p]{k^j\,(n+k)^{p-j+1}}} = \lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{\sqrt[p]{n}\left[1-\left(\frac{k}{n+k}\right)^{1/p}\right]}{nk^{1/p}}$$

$$= \lim_{n\to\infty} n^{(1-p)/p}\cdot\sum_{k=1}^{\infty}\left[k^{-1/p}-(n+k)^{-1/p}\right]$$

$$= \lim_{n\to\infty} n^{(1-p)/p}\cdot\sum_{k=1}^{n}k^{-1/p}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{-1/p}$$

$$= \int_0^1 x^{-1/p}\,dx$$

$$= \frac{p}{p-1},$$

as claimed.

Also solved by JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; ANASTASIOS KOTRONONIS, Athens, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer. Three incorrect solutions were received.

Note that this problem is a generalization of problem Q1011 ***Math. Mag.*** *84(3), 2011, p. 230.*