

A Quadrangle's Centroid of Perimeter

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Introduction

In [3] we discussed relations between the vertex centroid and the centroid of area of a quadrangle. Here we compare the centroid of perimeter with the other two named centroids.

Setup

For the reader's convenience we adapt the setup described in [3] to our present purposes. We consider plane quadrangles $ABCD$ with vertices A, B, C, D — no three of them on a line — edges $[AB], [BC], [CD], [DA]$ and diagonals AC, BD ; the diagonals may be considered — depending on the situation — as segments or lines. The quadrangles may be convex, concave or crossed. The lengths of the edges are denoted in the usual manner by a, b, c, d while the position vectors of the vertices with respect to a suitably chosen origin are written as $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ (in computations using vector algebra).

We consider the centroid of vertices $S_{\mathcal{E}}$, the centroid of perimeter $S_{\mathcal{K}}$, and the centroid of area $S_{\mathcal{F}}$ ¹. The centroid of area $S_{\mathcal{F}}$ is not defined for crossed quadrangles.

The centroid of perimeter of a quadrangle

In order to get the centroid of perimeter $S_{\mathcal{K}}$ of the quadrangle one replaces the homogeneous system of mass by a system of mass points [6]. The masses of the sides are proportional to their lengths a, b, c, d and are concentrated in the midpoints of sides. So one takes these four midpoints and provides them with the masses a, b, c, d . If the quadrangle is a parallelogram then the centroid of perimeter is the intersection point of the diagonals and coincides with the vertex centroid. In general the centroid of the perimeter has the vector representation

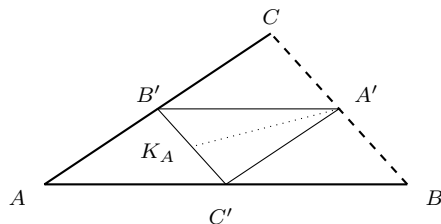
$$\vec{s}_{\mathcal{K}} = \frac{1}{2(a+b+c+d)} \left(a(\vec{a} + \vec{b}) + b(\vec{b} + \vec{c}) + c(\vec{c} + \vec{d}) + d(\vec{d} + \vec{a}) \right). \quad (1)$$

The synthetic determination of the centroid of the perimeter however has another basis. To this we recall first, how to find the centroid of a two-leg (with homogeneous mass on the sides).

We consider a two-leg consisting of the segments $[AB]$ and $[AC]$ with the lengths c and b , A, B, C not on a line.

* This note is an English adaption of the second part of a more comprehensive paper written in German [2]. The authors thank Chris Fisher for his helpful comments.

¹The symbols \mathcal{E} , \mathcal{K} , and \mathcal{F} suggest the German words for vertex (Ecke), edge (Kante), and area (Fläche).



Its centroid K_A belongs to the segment that runs from the midpoint C' of $[AB]$ to the midpoint B' of $[AC]$, that is, the middle parallel line of triangle ABC parallel to BC , and divides this segment in the ratio $b : c$.

Therefore the point K_A is the intersection point of the middle parallel line and the internal angular bisector of the triangle $A'B'C'$ running through the vertex A' , the midpoint of the segment $[BC]$. This construction is based on the angle bisector theorem: *An internal angular bisector of a triangle divides the opposite side in the ratio of the attached sides.*

In case of a quadrangle $ABCD$ one has four two-legs with the centroids K_A, K_B, K_C, K_D . The centroid of the perimeter S_K is then the intersection point of the diagonals of the quadrangle $K_A K_B K_C K_D$.

Centroid of perimeter = centroid of vertices

Now we assume that the centroid of the perimeter S_K of our quadrangle coincides with its vertex centroid S_E which has the vector representation

$$\vec{s}_E = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d}).$$

As in [3] we take the vertex centroid as origin which gives

$$\vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{o}. \tag{2}$$

From (1) we get

$$\vec{o} = a(\vec{a} + \vec{b}) + b(\vec{b} + \vec{c}) + c(\vec{c} + \vec{d}) + d(\vec{d} + \vec{a}). \tag{3}$$

Equation (2) provides the substitution $\vec{d} = -\vec{a} - \vec{b} - \vec{c}$ which yields

$$\vec{o} = (a - c)(\vec{a} + \vec{b}) + (b - d)(\vec{b} + \vec{c}) = (a - c)\vec{a} + (a - c + b - d)\vec{b} + (b - d)\vec{c},$$

that is

$$-(a + b - c - d)\vec{b} = (a - c)\vec{a} + (b - d)\vec{c}. \tag{4}$$

The corresponding substitution $\vec{b} = -\vec{a} - \vec{c} - \vec{d}$ yields analogously

$$-(a + b - c - d)\vec{d} = (b - d)\vec{a} + (a - c)\vec{c}. \tag{5}$$

We see that the argument depends on whether or not $a + b - c - d = 0$. If $a + b - c - d \neq 0$, then

$$-\vec{b} = \frac{a-c}{a+b-c-d}\vec{a} + \frac{b-d}{a+b-c-d}\vec{c},$$

$$-\vec{d} = \frac{b-d}{a+b-c-d}\vec{a} + \frac{a-c}{a+b-c-d}\vec{c}.$$

These equations show: Reflecting the vertices B and D in the origin yields points on the diagonal AC , that is, the diagonals BD and AC are parallel and we have a crossed quadrangle.

It will now be convenient to treat separately the situation where the diagonals of the quadrangle are not parallel. From the previous paragraph we necessarily have

$$a + b - c - d = 0. \quad (6)$$

and the equations (4), (5) simplify to

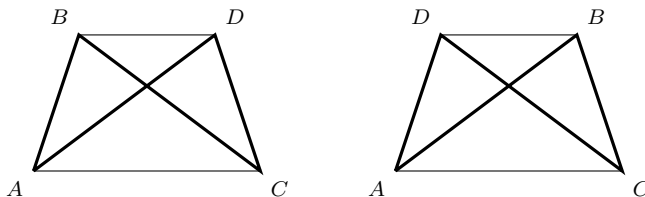
$$\vec{o} = (a-c)\vec{a} + (b-d)\vec{c},$$

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From equation (6) we know $(b-d) = -(a-c)$ which leads by substitution in the first of these equations to $\vec{o} = (a-c) \cdot (\vec{a} - \vec{c})$. Since $\vec{a} \neq \vec{c}$ we conclude $a = c$ and consequently $b = d$. Starting with the triangle ABD the vertex C is an intersection point of the circles centered at the points B and D with radii b, c respectively. Thus when (6) holds, our quadrangle is either (a) a parallelogram (which is a centrally symmetric quadrangle) or (b) a mirror symmetric crossed quadrangle whose diagonals are perpendicular to the mirror. In case (b) the diagonals are parallel, which we momentarily set aside to state

Theorem (part 1.) *The centroid of vertices and the centroid of perimeter of a quadrangle with nonparallel diagonals coincide if and only if the quadrangle is a parallelogram.*

Now we consider quadrangles with parallel diagonals.



If (6) holds, then we have as stated at the end of the previous paragraph two types of mirror symmetric crossed quadrangles whose vertex centroids coincide with the centroids of perimeter shown in the diagram.

Finally, we return to quadrangles with $a + b - c - d \neq 0$ (and therefore with parallel diagonals). We introduce coordinates such that the x -axis is the center

line of the diagonals; without loss of generality we may assume that the diagonals AC , BD satisfy equations $y = -1$, $y = 1$ respectively. We take, using equation (2),

$$A(a_1, -1), B(b_1, 1), C(c_1, -1), D(-a_1 - b_1 - c_1, 1).$$

The midpoints of the sides of the quadrangle all have second coordinate 0, so equation (3) reduces to a condition only on the first coordinate and can be written in the form

$$(a_1 + b_1) \cdot (a - c) = (b_1 + c_1) \cdot (d - b).$$

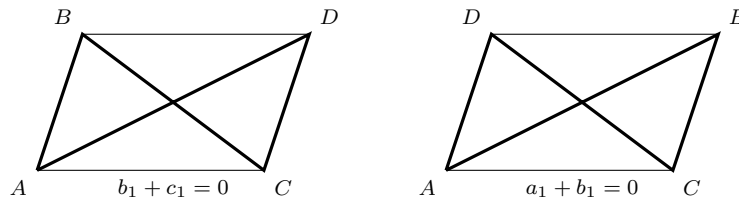
This is a (square-)root equation since

$$\begin{aligned} a &= \sqrt{(b_1 - a_1)^2 + 4a_2^2}, & c &= \sqrt{(b_1 - c_1)^2 + 4a_2^2}, \\ b &= \sqrt{(a_1 + b_1 + 2c_1)^2 + 4a_2^2}, & d &= \sqrt{(2a_1 + b_1 + c_1)^2 + 4a_2^2}. \end{aligned}$$

It can be transformed into a polynomial equation by the usual procedure of multiple squaring. Factoring at each step, if possible, (most conveniently done by means of a computer algebra system) yields the following polynomial factors

$$a_1 - c_1, a_1 + 2b_1 + c_1, b_1 + c_1, a_1 + c_1, a_1 + b_1.$$

At least one of these terms must vanish. The first two can't since this would imply either $A = C$ or $B = D$, contrary to our definition of a quadrangle. If $b_1 + c_1 = 0$ we have $\vec{b} = -\vec{c}$ and $\vec{a} = -\vec{d}$, whence the quadrangle is centrally symmetric. The same holds if $a_1 + b_1 = 0$ implying $\vec{a} = -\vec{b}$, $\vec{c} = -\vec{d}$. In both cases the origin is the vertex centroid as well as the centroid of the perimeter.



Finally, if $a_1 + c_1 = 0$, then $a = c$, $b = d$ and therefore $a + b - c - d = 0$ which we already discussed.

In summary,

Theorem (part 2) *The centroid of vertices and the centroid of perimeter of a quadrangle with parallel diagonals coincide if and only if the quadrangle is either mirror symmetric with the diagonals perpendicular to the axis or crossed centrally symmetric.*

Centroid of perimeter = centroid of area

For parallelograms all three sorts of centroids coincide. Conversely we have seen that a non-crossed quadrangle whose vertex centroid coincides either with the centroid of area or with the centroid of perimeter must be a parallelogram. The

question remains whether there are non-crossed quadrangles other than parallelograms whose centroids of area and perimeter coincide. We will restrict our attention to kites that have this property.

In order to describe them, we choose orthogonal coordinates such that the vertices of the kites can be presented as $A(0, 1)$, $B(-p, 0)$, $C(0, -1)$, $D(q, 0)$ with $p > 0$ and $p \neq q$. For $q > 0$ the kite is convex, for $q < 0$ concave. A kite is a mirror symmetric quadrangle; all its centroids belong to the x -axis of our coordinate system, which is the kite's axis of symmetry. We show:

- If $0 < p < \sqrt{3}$ there is just one positive and just one negative value for q such that the centroids under consideration of the corresponding kite coincide, whence one of the corresponding kites will be convex, the other concave.
- If $p = \sqrt{3}$ there is a suitable negative value for q yielding exactly one concave kite of the desired kind.
- If $\sqrt{3} < p$ there are exactly two negative values for q yielding two concave kites of the desired kind.

We compute the centroid of area of the triangle BCD

$$S_A \left(\frac{q-p}{3}, \frac{1}{3} \right)$$

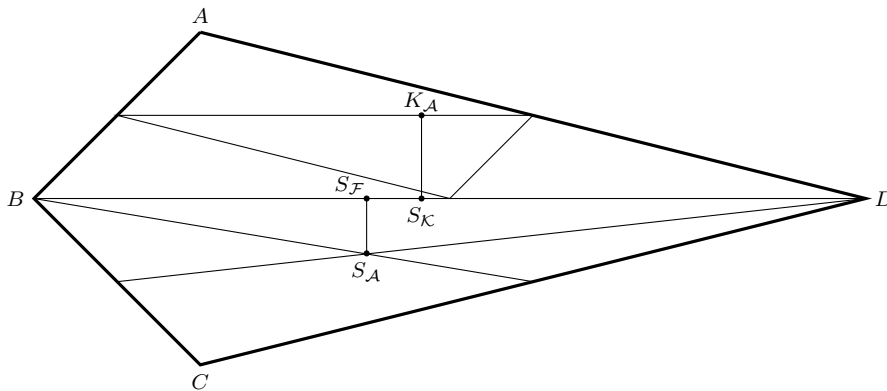
and obtain the centroid of area of the kite

$$S_{\mathcal{F}} \left(\frac{q-p}{3}, 0 \right).$$

We obtain the centroid of perimeter of the kite from (1)

$$S_{\mathcal{K}} \left(\frac{dq - ap}{2(a+d)}, 0 \right).$$

How these centers can be constructed is shown in the following diagram



The condition for $S_{\mathcal{F}} = S_{\mathcal{K}}$ is:

$$\frac{q-p}{3} = \frac{dq-ap}{2(a+d)},$$

or more simply

$$2(a+d)(q-p) = 3(dq-ap).$$

This is a root equation since $a = b = \sqrt{1+p^2}$, $c = d = \sqrt{1+q^2}$. We solve for q in terms of p . To this end we transform it first to

$$(2q+p)a = (q+2p)d.$$

Squaring yields

$$(2q+p)^2(1+p^2) = (q+2p)^2(1+q^2).$$

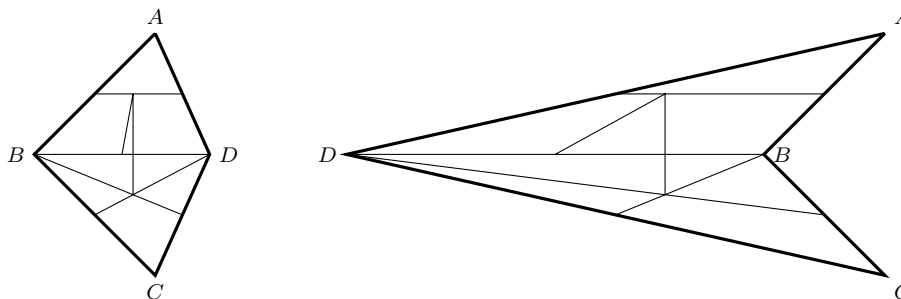
So we have the polynomial equation for q :

$$q^4 + 4q^3p - 3q^2 + 3p^2 - 4qp^3 - p^4 = 0.$$

Evidently it has roots $\pm p$. For $+p$ we have a rhombus, a special parallelogram, which is not relevant. The value $-p$ is extraneous. Splitting those roots we obtain the quadratic equation

$$q^2 + 4pq + p^2 - 3 = 0$$

with the roots $-2p \pm \sqrt{3(1+p^2)} = -2p \pm a\sqrt{3}$. Both values satisfy the given equation. Only when $p < \sqrt{3}$ can one of the roots be positive, in which case the resulting kite would be convex. When $p = \sqrt{3}$, one of the roots is $q = 0$ and the points A, C, D would be collinear, which is excluded by hypothesis; thus there is just one kite when $p = 3$ and it is concave. Since p is always positive, all other values of p yield two negative values for q and, consequently, a pair of concave kites. The diagram shows the two kites with $p = 1$ for which the centroids of perimeter and of area coincide.



Our treatment leaves open a further question: Do there exist non-crossed quadrangles beside parallelograms and kites whose centroid of perimeter and centroid of area coincide?

Preview

We shall conclude these thoughts in a further note. In [4] we connect the centroid of vertices to van Aubel's Square Theorem [1].

References

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- [2] R. Fritsch, G. Pickert, *Vierecksschwerpunkte I, Die Wurzel* **48**(2) (2014); parts II and III are to appear in issues (3/4) and (5).
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