PROBLEM SOLVER'S TOOLKIT

No. 2

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The Problem Solver's Toolkit is a new feature in ***Crux Mathematicorum****. It will contain short articles on topics of interest to problem solvers at all levels. Occasionally, these pieces will span several issues.*

Counting With Care

Counting problems usually involve quite elementary mathematics and techniques, yet can prove to be quite complex. In many cases the correct way of looking at the problem can simplify or greatly illuminate the situation.

As an example, I was recently visiting my family and we decided to play a game. There were six of us and we had to break into three groups of two. Someone asked me off hand "how many different ways could the groups be formed?" Caught off guard (it was my holidays) I gave a wrong answer: 90. I knew as soon as it was out of my mouth that it was wrong, but they would never know so I let it be and decided to think about it later.

The solution that I did in my head was: first we need to form the first team from the six, then another team from the four and the final group from the two. As such, I calculated

$$\binom{6}{2} \times \binom{4}{2} \times \binom{2}{2} = 15 \times 6 \times 1 = 90.$$

The result seemed a little big, but it impressed them and I was sure they wouldn't think about it again ..., but I would. My mistake, a typical one with counting problems, was that my method counts the groups many times. That is, if I label our players A, B, C, D, E and F, my method considers choosing A and B as team #1, C and D as team #2 and E and F as team #3 as different from choosing C and D as team #1 , and E and F as team #2, and A and B as team #3. As a result, each of my groupings of teams has been counted 6 times, thus the real (and less impressive) answer to the question is

$$\frac{\binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}}{3!} = 15.$$

Now we could leave it at that and just take away the message "be careful when you are counting", but, like many counting problems, we can attack this from another point of view.

Consider one of the players, A for example, and consider him "fixed". That is, he needs a partner one way or the other, and there are 5 ways to pick one for him. Next, pick one of the remaining players to consider as "fixed" and she needs a partner as well, which we can do in 3 ways. All that remains is 2 people who become the last team. Thus, the number of ways to make the teams is just

$$5 \times 3 = 15$$

which is much easier to see.

The next problem is a favourite of mine.

> **Problem** (question S3, 2001 Interprovincial Mathematics Olympiad, senior team portion, [2002 : 42, 392]) *Eight boxes, each a unit cube, are packed in a $2 \times 2 \times 2$ crate, open at the top. The boxes are taken out one by one. In how many ways can this be done? (Remember that a box in the bottom layer can only be removed after the box above it has been removed.)*

If it suits you, take a few minutes and try this problem before you read on.

This problem was from back when I was doing the Skoliad Corner, before we started printing solutions from readers. I remember quite clearly receiving a solution from a reader that, although correct, was quite long and complicated. The reader had basically broken the problem down into all the possible cases. Their solution started like this:

> We only have 4 choices for the first box, as it must come from the top. For the second box, we have four choices, but we must break them up into two cases:
>
> **Case 1:** We choose another box from the top (3 possible choices), then our next choice we will have four boxes to choose from, two from the top and two from the bottom, which leads to two more cases.
>
> **Case 2:** We chose the box from the bottom (1 possible choice), so for our next choice we only have 3 boxes to choose from (all from the top).
>
> $\vdots$

Since choosing a box from the top or choosing a box from the bottom lead to different scenarios for the next choice, the problem broke down into a large number of cases. As such, the solution was very long and hard to follow (to make sure nothing was double counted, my sin from the first problem, or nothing was left out). In the end it was correct.

I love this problem because it is so simple to state and understand, yet it can be tricky to solve. I have given this problem to high school teachers in a number of workshops and it always causes a stir.

The key to solving the problem, is choosing the correct point of view. I know that a bottom box cannot be removed until the one above it has been removed. Start by numbering the top boxes 1, 2, 3, 4 and numbering each bottom box with the same number as the box above it. Now, when I pull the boxes out, I will just record the number on them. When you see something like 2, 3, 1, 2, 4, 1, 3, 4 there is no ambiguity, the first 2 has to be the top box, and the second 2 has to be the bottom box. As a result, our problem is equivalent to finding the number of arrangements of the eight numbers 1, 1, 2, 2, 3, 3, 4, 4 which is

$$\frac{8!}{2!2!2!2!} = 2520.$$

The 8! in the solution is there since we are arranging 8 objects, but this includes all possible arrangements, including when two identical numbers are "switched". For example the 8! would count both 1**1**223344 and **1**1223344 (where I have bolded and enlarged one of the ones so that you can see the "switch"). For each distinct digit, there are 2! ways that the two copies can be interchanged, hence we divide by all of these factors to eliminate duplicate arrangements from the original total. This solution is much more illuminating and satisfying than two pages of cases.

Here are a couple of related problems to try:

1. Twenty-four boxes, each a unit cube, are packed in a crate 2 units long, 3 units wide and 4 units deep, open at the top. The boxes are taken out one by one. In how many ways can this be done?

2. A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe? (problem 16, 2001 AMC12 contest)

So, when you tackle a counting problem, take some time and think of different ways it can be interpreted. A good practice would be to try to find multiple solutions to any given problem, it will help to count with care.