OC96. Let $a, b > 1$ be two relatively prime integers. We define $x_1 = a$, $x_2 = b$ and

$$x_n = \frac{{x_{n-1}}^2 + {x_{n-2}}^2}{x_{n-1} + x_{n-2}}$$

for all $n \geq 3$. Prove that x_n is not an integer for all $n \geq 3$.

OC97. Let A be a set with 225 elements. Suppose that there are eleven subsets $A_1, \ldots, A_{11}$ of A such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup A_2 \cup \ldots \cup A_{11}| \geq 165$, and give an example for which equality holds.

OC98. Let ABC be a triangle with $\angle BAC = 60^\circ$. Let B_1 and C_1 be the feet of the bisectors from B and C. Let A_1 be the symmetrical of A with respect to the line B_1C_1. Prove that A_1, B and C are collinear.

OC99. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ so that, for all $x \in \mathbb{Q}^+$ we have

$$f\left(\frac{x}{x+1}\right) = \frac{f(x)}{x+1}$$

and

$$f\left(\frac{1}{x}\right) = \frac{f(x)}{x^3}\,.$$

OC100. Let a_n be the sequence defined by $a_0 = 1$, $a_1 = -1$, and

$$a_n = 6a_{n-1} + 5a_{n-2}$$

for all $n \geq 2$. Prove that $a_{2012} - 2010$ is divisible by 2011.

OLYMPIAD SOLUTIONS

OC36. The obtuse-angled triangle ABC has sides of length a, b, and c opposite the angles $\angle A$, $\angle B$ and $\angle C$ respectively. Prove that

$$a^3 \cos A + b^3 \cos B + c^3 \cos C < abc\,.$$

(*Originally question #6 from the 2008/9 British Mathematical Olympiad, Round 1.*)

Similar solutions by Michel Bataille, Rouen, France and Titu Zvonaru, Comăneşti, Romania. No other solution was received .

By the law of cosines we have

$$\cos(A) = \frac{b^2+c^2-a^2}{2bc}\,;\ \cos(B) = \frac{a^2+c^2-b^2}{2ac}\,;\ \cos(C) = \frac{a^2+b^2-c^2}{2ab}\,.$$

Then the inequality reduces to

$$a^4b^2 + a^2b^4 + b^4c^2 + b^2c^4 + c^4a^2 + c^2a^4 - (a^6 + b^6 + c^6 + 2a^2b^2c^2) < 0$$

or equivalently

$$(b^2+c^2-a^2)(a^2+c^2-b^2)(a^2+b^2-c^2) < 0$$

which is true since ABC is obtuse angled, and thus exactly one of $\cos(A)$, $\cos(B)$, $\cos(C)$ is negative.

Zvonaru observed that using the same argument we can prove that if ABC is acute angled then

$$a^3\cos(A) + b^3\cos(B) + c^3\cos(C) > abc$$

and if ABC is right triangle then

$$a^3\cos(A) + b^3\cos(B) + c^3\cos(C) = abc.$$

OC37. Find all integers n such that we can colour all the edges and diagonals of a convex n-gon by n given colours satisfying the following conditions:

(i) Every one of the edges or diagonals is coloured by only one colour;

(ii) For any three distinct colours, there exists a triangle whose vertices are vertices of the n-gon and the three edges are coloured by the three colours, respectively.

(*Originally question #5 from the 2009 Chinese Mathematical Olympiad.*)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

We prove that an integer $n \geq 3$ has the desired property if and only if n is odd.

Suppose that n has the required property. We show that n is odd. There are $\binom{n}{3}$ choices of three distinct colours out of n colours and also $\binom{n}{3}$ triangles out of n vertices. Hence, any two distinct such triangles have distinct sets of edge colours.

As there are exactly $\binom{n-1}{2}$ choices of three colours containing a fixed given color, each colour occurs in exactly $\binom{n-1}{2}$ triangles. On the other hand, every line segment belongs to $n-2$ triangles. Therefore, the number of segments of each colour is $\frac{1}{n-2}\binom{n-1}{2} = \frac{n-1}{2}$. We conclude that n is odd.

Next suppose that $n \geq 3$ is odd. We prove that n has the desired property. Suppose that there are n colours $C_0,\ C_1,\ \ldots,\ C_{n-1}$ and n vertices $P_0,\ P_1,\ \ldots,\ P_{n-1}$. Consider indices of colours and indices of vertices modulo n. Let us colour the edge with end points P_i and P_j with colour C_{i+j}. Let C_i, C_j, and C_k be any three distinct colours. It is now straightforward to verify that the triangle $P_{(i+j-k)/2}P_{(i+k-j)/2}P_{(j+k-i)/2}$ has edges with colours C_i, C_j, and C_k. (Here the calculation of indices is within the additive ring $\mathbb{Z}_n$. As n is odd, 2 is invertible in $\mathbb{Z}_n$.) This completes the proof.

OC38. Let a, b, c be positive real numbers such that $ab+bc+ca=\frac{1}{3}$. Prove the inequality:

$$\frac{a}{a^2-bc+1}+\frac{b}{b^2-ca+1}+\frac{c}{c^2-ab+1}\geq\frac{1}{a+b+c}\,.$$

(*Originally question #4 from the 16*$^{\text{th}}$ *Macedonian Mathematical Olympiad.*)

Solved by John Asmanis, Chalkida, Greece ; Michel Bataille, Rouen, France ; Marian Dincă, Bucharest, Romania ; Oliver Geupel, Brühl, NRW, Germany ; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA ; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy ; Henry Ricardo, Tappan, NY, USA and Titu Zvonaru, Cománeşti, Romania. We give the solution of Bataille.

First, we observe that $a^2-bc+1=a^2+ab+ac-(ab+ac+bc)+1=a(a+b+c)+\frac{2}{3}$.

We denote by $s:=a+b+c$. Then, the inequality becomes equivalent to

$$\frac{1}{s^2}\leq\frac{a}{s}\cdot\frac{1}{as+\frac{2}{3}}+\frac{b}{s}\cdot\frac{1}{bs+\frac{2}{3}}+\frac{c}{s}\cdot\frac{1}{cs+\frac{2}{3}}\,. \qquad (1)$$

Since $f(x)=\frac{1}{x}$ is convex on $(0,\infty)$ and $\frac{a}{s}+\frac{b}{s}+\frac{c}{s}=1$ by the Jensen inequality we get

$$\begin{aligned}\frac{a}{s}\cdot\frac{1}{as+\frac{2}{3}}+\frac{b}{s}\cdot\frac{1}{bs+\frac{2}{3}}+\frac{c}{s}\cdot\frac{1}{cs+\frac{2}{3}}&\geq\frac{1}{\frac{a}{s}(as+\frac{2}{3})+\frac{b}{s}(bs+\frac{2}{3})+\frac{c}{s}(cs+\frac{2}{3})}\\&=\frac{1}{a^2+b^2+c^2+\frac{2}{3}}\\&=\frac{1}{a^2+b^2+c^2+2(ab+ac+bc)}=\frac{1}{s^2}\,,\end{aligned}$$

which proves (1).

OC39. Given a positive integer n, let $b(n)$ denote the number of positive integers whose binary representations occur as blocks of consecutive integers in the binary expansion of n. For example $b(13)=6$ because $13=1101_2$, which contains as consecutive blocks the binary representations of $13=1101_2$, $6=110_2$, $5=101_2$, $3=11_2$, $2=10_2$ and $1=1_2$.

Show that if $n \leq 2500$, then $b(n) \leq 39$, and determine the values of n for which equality holds.
(*Originally question #4 from the 2008/9 British Mathematical Olympiad, Round 2.*)

Solved by Oliver Geupel, Brühl, NRW, Germany and Titu Zvonaru, Cománeşti, Romania. We give a combination of their similar solutions.

For any positive integers ℓ and m and any bit string $x \in \{0,1\}^m$ (i.e., a word of length m over the alphabet $\{0,1\}$), let $B(x,\ell)$ denote the number of distinct bit strings of length ℓ and with leading bit 1 that occur as consecutive blocks in x. Let $B(x) = \sum_{\ell=1}^{\infty} B(x,\ell)$. Moreover, let $\beta(n)$ denote the binary (bit string) representation of the integer n. The identity $B(\beta(n)) = b(n)$ clearly holds.

Claim 1: If $2^{11} \leq n \leq 2050$ then $b(n) \leq 37$.

Indeed, $x = \beta(n)$ has the form $100x_9$ with $x_9 \in \{0,1\}^9$. By counting the possible bit strings, we obtain $B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4$. We have $B(x,4) \leq 7$, because there are not more than 6 blocks inside x_9 and one additional block starting at the leftmost bit of x. Similarly, we obtain $B(x,5) \leq 6$, $B(x,6) \leq 5$, $B(x,7) \leq 4$, $B(x,8) \leq 3$, $B(x,9) \leq 2$, and $B(x,10) = B(x,11) = B(x,12) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 37,$$

which completes the proof of Claim 1.

Now we assume that $n < 2048$.

Claim 2: If $n \leq 2^{10}$ then $b(n) \leq 36$.

By a similar counting to the one in Claim 1, we obtain $B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4, B(x,4) \leq 8$.

As a block of length 5 in $x = x_1x_2...x_{10}$ can only start at x_1, x_2, x_3, x_4, x_5 or x_6 we get $B(x,5) \leq 6$ and similarly $B(x,6) \leq 5$, $B(x,7) \leq 4$, $B(x,8) \leq 3$, $B(x,9) \leq 2$, and $B(x,10) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 36,$$

which completes the proof of Claim 2.

Claim 3: If $n \leq 2^{11}$ and $x = \beta(n)$, it holds $B(x,1) \leq 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4$, $B(x,4) \leq 8$, $B(x,5) \leq 7$, $B(x,6) \leq 6$, $B(x,7) \leq 5$, $B(x,8) \leq 4$, $B(x,9) \leq 3$, $B(x,10) \leq 2$, and $B(x,11) \leq 1$.

The first four relations follow immediately from the observation that, by counting all the binary strings of length k starting with 1 we have

$$B(x,k) \leq 2^{k-1}.$$

For the others it suffices to observe that there are exactly $11 - k + 1$ possible starting positions for any block of length k in x. Thus

$$B(X, k) \leq 12 - k\,.$$

This proves the claim.

Now we assume that $1024 \leq n < 2048$. Then $x := \beta(n) = 1x_2x_3...x_{11}$.

We split the problem in four cases:

Case 1: $x_2 = 0$. Then, $x := \beta(n) = 10x_3...x_{11}$, and exactly as in Claim 1 we get $B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4$, $B(x,4) \leq 7$, $B(x,5) \leq 6$, $B(x,6) \leq 5$, $B(x,7) \leq 4$, $B(x,8) \leq 3$, $B(x,9) \leq 2$, and $B(x,10) = B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 36\,.$$

Case 2: $x_2 = 1, x_3 = 0$. Then $x := \beta(n) = 110x_4...x_{11}$, and exactly as in Claim 1 we get $B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4$, $B(x,4) \leq 7$, $B(x,5) \leq 6$, $B(x,6) \leq 5$, $B(x,7) \leq 4$, $B(x,8) \leq 3$, $B(x,9) \leq 2$, $B(x,10) \leq 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 37\,.$$

Case 3: $x_2 = 1, x_3 = 1, x_4 = 0$. Then $x := \beta(n) = 1110x_5...x_{11}$, and exactly as in Claim 1 we get $B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4$, $B(x,4) \leq 7$, $B(x,5) \leq 6$, $B(x,6) \leq 5$, $B(x,7) \leq 4$, $B(x,8) \leq 3$, $B(x,9) \leq 3$, $B(x,10) \leq 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 38\,.$$

Case 4: $x_2 = 1, x_3 = 1, x_4 = 1$. We will also split this case in 7 subcases:

SubCase 4a: $x_5 = x_6 = 0$. Then $x := \beta(n) = 111100x_7...x_{11}$.

Then, $B(x,5) \leq 5$, $B(x,6) \leq 5, B(x,7) \leq 4, B(x,8) \leq 4$, and hence from Claim 3 we get

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 38\,.$$

SubCase 4b: $x_5 = 0, x_6 = 1, x_7 = 0$. Then $x := \beta(n) = 1111010x_8x_9x_{10}x_{11}$.

$B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4, B(x,4) \leq 6, B(x,5) \leq 5$, $B(x,6) \leq 5$, $B(x,7) \leq 4, B(x,8) \leq 4, B(x,9) \leq 3, B(x,10) \leq 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 37\,.$$

SubCase 4c: $x_5 = 0, x_6 = 1, x_7 = 1$. Then $x := \beta(n) = 1111011x_8x_9x_{10}x_{11}$.

$B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4, B(x,4) \leq 7, B(x,5) \leq 6$, $B(x,6) \leq 5$, $B(x,7) \leq 4, B(x,8) \leq 4, B(x,9) \leq 3, B(x,10) \leq 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 39\,.$$

To get equality we must have $B(x,1) = 1$, $B(x,2) = 2$, $B(x,3) = 4, B(x,4) = 7, B(x,5) = 6$, $B(x,6) = 5$, $B(x,7) = 4, B(x,8) = 4, B(x,9) = 3, B(x,10) = 2$, and $B(x,11) = 1$.

If $x_8 = 0$, we get $B(x,4) \leq 6$, thus $b(n) \leq 38$.

Thus we can only get equality if $x_8 = 1$. Then $x = 11110111x_9x_{10}x_{11}$. Since $B(x,3) = 4$ we must have all 4 blocks $100, 101, 110, 111$ in the binary representation of n. Hence either $x_9 = x_{10} = 0$ or $x_{10} = x_{11} = 0$, which shows that equality can only occur for

$$x \in \{11110111000, 11110111001, 11110111100\} = X\,.$$

As 11110111100 doesn't contain 1000 and 1001, $B(11110111100, 4) \neq 7$, and hence $B(11110111100) \neq 39$.

As 11110111001 doesn't contain 1010 and 1000, $B(11110111100, 4) \neq 7$, and hence $B(11110111100) \neq 39$.

As 11110111100 doesn't contain 1010 and 1000, $B(11110111100, 4) \neq 7$, and hence $B(11110111100) \neq 39$.

Thus, there is no solution in this case.

SubCase 4d: $x_5 = 1, x_6 = 0, x_7 = 0$. Then $x := \beta(n) = 1111100x_8x_9x_{10}x_{11}$.

$B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4, B(x,4) \leq 5, B(x,5) \leq 5$, $B(x,6) \leq 5$, $B(x,7) \leq 4, B(x,8) \leq 4, B(x,9) \leq 3, B(x,10) \leq 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 37\,.$$

SubCase 4e: $x_5 = 1, x_6 = 0, x_7 = 1$. Then $x := \beta(n) = 1111101x_8x_9x_{10}x_{11}$.

$B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4, B(x,4) \leq 6, B(x,5) \leq 6$, $B(x,6) \leq 5$, $B(x,7) \leq 5, B(x,8) \leq 4, B(x,9) \leq 3, B(x,10) \leq 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 39\,.$$

To get equality we must have $B(x,1) = 1$, $B(x,2) = 2$, $B(x,3) = 4, B(x,4) = 6, B(x,5) = 6$, $B(x,6) = 5$, $B(x,7) = 5, B(x,8) = 4, B(x,9) = 3, B(x,10) = 2$, and $B(x,11) = 1$.

If $x_8 = 0$, then $x = 11111010x_9x_{10}x_{11}$ we get $B(x,4) \leq 5$, thus $b(n) \leq 38$.

Thus we can only get equality if $x_8 = 1$. Then $x = 11111011x_9x_{10}x_{11}$. Since $B(x,3) = 4$ we must have all 4 blocks $100, 101, 110, 111$ in the binary representation of n. Hence either $x_9 = x_{10} = 0$ or $x_{10} = x_{11} = 0$, which shows that equality can only occur for

$$x \in \{11111011000, 11111011001, 11111011100\}\,.$$

It is easy to check that both $x = 11111011000$ and $x = 11111011001$ work.
Thus, in this case, we get equality for $n = 2008$ and $n = 2009$.

SubCase 4f: $x_5 = 1, x_6 = 1, x_7 = 0$. Then $x := \beta(n) = 1111110x_8x_9x_{10}x_{11}$.
$B(x,1) = 1$, $B(x,2) \le 2$, $B(x,3) \le 4, B(x,4) \le 5, B(x,5) \le 5$, $B(x,6) \le 6$, $B(x,7) \le 5, B(x,8) \le 4, B(x,9) \le 3, B(x,10) \le 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \le 38\,.$$

SubCase 4g: $x_5 = 1, x_6 = 1, x_7 = 1$. Then $x := \beta(n) = 1111111x_8x_9x_{10}x_{11}$.
$B(x,1) = 1$, $B(x,2) \le 2$, $B(x,3) \le 4, B(x,4) \le 5, B(x,5) \le 5$, $B(x,6) \le 6$, $B(x,7) \le 5, B(x,8) \le 4, B(x,9) \le 3, B(x,10) \le 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \le 38\,.$$

This proves that for all $n \le 2500$ we have $b(n) \le 39$ with equality if and only if

$$n \in \{2008, 2009\}\,.$$

OC40. Let M and N be the intersection of two circles, Γ_1 and Γ_2. Let AB be the line tangent to both circles closer to M, say $A \in \Gamma_1$ and $B \in \Gamma_2$. Let C be the point symmetrical to A with respect to M, and D the point symmetrical to B with respect to M. Let E and F be the intersections of the circle circumscribed around DCM and the circles Γ_1 and Γ_2, respectively.

Show that the circles circumscribed around the triangles MEF and NEF have radii of the same length.
(*Originally question #5 from the 2009 Italian Team Selection Test.*)

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany and Titu Zvonaru, Comǎneşti, Romania. We give the solution of Zvonaru.

Let P be the intersection point of AB and MN. Using the power of a point with respect to a circle we get

$$PA^2 = PM \cdot PN = PB^2\,.$$

Thus P is the midpoint of AB. As M is the midpoint of BD we get

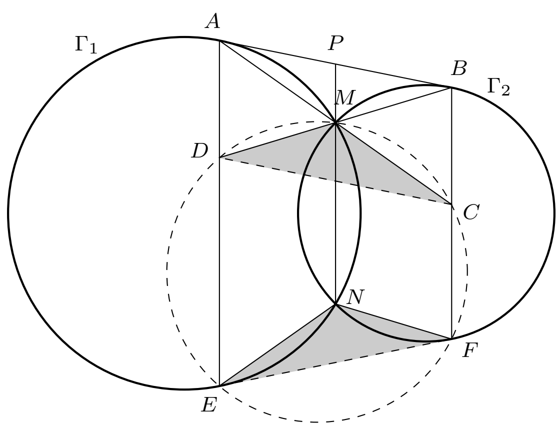
$$AD \parallel PM \parallel BC\,.$$

Combining the fact that M, D, E, F, C are on the same circle with the fact that $ABCD$ is a parallelogram and AB tangent to Γ_1, we get

$$\angle DEM = \angle DCM = \angle MAB = \angle AEM\,,$$

$$\angle CFM = \angle MDC = \angle MBA = \angle BFM\,.$$

Hence A, E, M are collinear and B, C, F are also collinear.



It results that the trapezoids $DEFC$, $AENM$, $BFNM$ are cyclic, hence isosceles. It follows that

$$DC = EF\,;\; EN = AM = MC\,;\; NF = BM = MD\,.$$

Thus the triangles NEF and MCD are congruent. Since the triangles MEF and MCD are inscribed in the same circle, the statement of the problem follows.