THE OLYMPIAD CORNER

No. 300

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Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 août 2013**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

OC66. Soit $n \geq 2$ un entier positif. Trouver toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ de sorte que

$$f(x - f(y)) = f(x + y^n) + f(f(y) + y^n), \qquad \forall x, y \in \mathbb{R}.$$

OC67. Un 2011-gone convexe est dessiné au tableau. Pierre est occupé à dessiner ses diagonales de sorte que chaque nouvelle diagonale ne coupe pas plus qu'une seule des diagonales déjà dessinées. Quel est le plus grand nombre de diagonales que Pierre peut dessiner ?

OC68. Trouver tous les entiers x, y de sorte que

$$x^3 + x^2 + x = y^2 + y\,.$$

OC69. Soit n un entier positif et soit $P(x, y) = x^n + xy + y^n$. Montrer qu'on ne peut pas trouver deux polynômes $G(x, y)$ et $H(x, y)$ à coefficients réels tels que

$$P(x, y) = G(x, y) \cdot H(x, y)\,.$$

OC70. $\triangle ABC$ est un triangle tel que $\angle C$ and $\angle B$ sont acutangles. Soit D un point variable sur BC tel que $D \neq B, C$ and AD n'est pas perpendiculaire à BC. Soit d la droite passant par D et perpendiculaire à BC. Supposons que $d \cap AB = E, d \cap AC = F$. Soit M, N, P les centres des cercles inscrits de $\triangle AEF, \triangle BDE, \triangle CDF$. Montrer que A, M, N, P sont cocycliques si et seulement d passe par le centre du cercle inscrit de $\triangle ABC$.

OC66. Let $n \geq 2$ be a positive integer. Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that

$$f(x - f(y)) = f(x + y^n) + f(f(y) + y^n), \qquad \forall x, y \in \mathbb{R}.$$

OC67. A convex 2011-gon is drawn on the board. Peter keeps drawing its diagonals in such a way that each newly drawn diagonal intersects no more than one of the already drawn diagonals. What is the greatest number of diagonals that Peter can draw?

OC68. Find all integers x, y so that

$$x^3 + x^2 + x = y^2 + y\,.$$

OC69. Let n be a positive integer and let $P(x,y) = x^n + xy + y^n$. Prove that we cannot find two non-constant polynomials $G(x,y)$ and $H(x,y)$ with real coefficients such that

$$P(x,y) = G(x,y)\cdot H(x,y)\,.$$

OC70. $\triangle ABC$ is a triangle such that $\angle C$ and $\angle B$ are acute. Let D be a variable point on BC such that $D \neq B, C$ and AD is not perpendicular to BC. Let d be the line passing through D and perpendicular to BC. Assume $d \cap AB = E$, $d \cap AC = F$. Let M, N, P be the incentres of $\triangle AEF$, $\triangle BDE$, $\triangle CDF$. Prove that A, M, N, P are concyclic if and only if d passes through the incentre of $\triangle ABC$.

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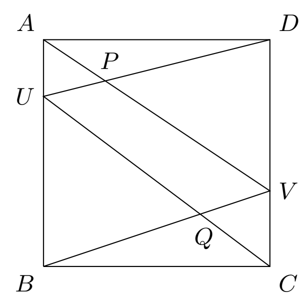
OC6. In the diagram, $ABCD$ is a square, with U and V interior points of the sides AB and CD respectively. Determine all the possible ways of selecting U and V so as to maximize the area of the quadrilateral $PUQV$. (*Originally question # 3 from the 1992 Canadian Mathematical Olympiad.*)



Solved by Michel Bataille, Rouen, France; Florencio Cano Vargas, Inca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Bataille.

We will use $[\cdot]$ to denote the area. Let a be the length of the sides of the square. Since

$$[AVB] + [DUC] = \frac{a^2}{2} + \frac{a^2}{2} = [ABCD]\,, \tag{1}$$

we get

$$[PUQV] = [APD] + [QBC]\,.$$

Let $u = AU$ and $v = CV$, and let P', Q' be the orthogonal projections of P onto AD, respectively Q onto BC. Then

$$\frac{AP'}{AD} = \frac{PP'}{DV} \Rightarrow AP' = \frac{AD}{DV}PP' = \frac{a}{a-v}PP'\,.$$

Similarly we get $DP' = \frac{a}{u}PP'$.

Then

$$a = AP' + DP' = \frac{a}{a-v}PP' + \frac{a}{u}PP'\,,$$

which yields

$$PP' = \frac{u(a-v)}{a+u-v}.$$

In a similar way, we obtain

$$QQ' = \frac{v(a-u)}{a+v-u}\,.$$

Since $[APD] = \frac{1}{2}a \cdot PP'$ and $[BQC] = \frac{1}{2}a \cdot QQ'$, (1) yields

$$[PUQV] = \frac{a}{2}\left(\frac{u(a-v)}{a+u-v} + \frac{v(a-u)}{a+v-u}\right).$$

Now, by AM-GM, $u(a-v) \le \frac{(a+u-v)^2}{4}$ and $v(a-u) \le \frac{(a+v-u)^2}{4}$, with equality if and only if $u+v=a$.

Thus

$$[PUQV] \le \frac{a}{8}(a+u-v+a+v-u) = \frac{a^2}{4}.$$

In conclusion, $[PUQV] \le \frac{a^2}{4}$ and the maximum value $\frac{a^2}{4}$ is attained exactly when $AU + VC = a$.

OC7. Let n be a natural number such that $n \ge 2$. Show that

$$\frac{1}{n+1}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right).$$

(*Originally question # 3 from the 1998 Canadian Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Manes.

Our inequality is equivalent to

$$n\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) > (n+1)\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right).$$

Using the well known Catalan's inequality

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n},$$

we get

$$1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} = \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right).$$

Thus the inequality to prove becomes

$$n\left(\frac{1}{2} + \frac{1}{4} + \ldots \frac{1}{2n}\right) + n\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) > (n+1)\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right),$$

or equivalently

$$n\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) > \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right). \qquad (1)$$

Since $n \ge 2$, for all $1 \le j \le n$ we have $n + j < 2nj$ and hence:

$$\frac{n}{n+j} > \frac{1}{2j}.$$

Adding these inequalities yields (1) and completes the proof.

OC8. For each real number r let T_r be the transformation of the plane that takes the point (x, y) into the point $(2^r x, r2^r x + 2^r y)$. Let F be the family of all such transformations *i.e.* $F = \{T_r : r \in \mathbb{R}\}$. Find all curves $y = f(x)$ whose graphs remain unchanged by every transformation in F.
(*Originally question # 2 from the 1983 Canadian Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Bataille.

It is readily checked that T_r is a bijective transformation with $T_r^{-1} = T_{-r}$. Thus, it suffices to find all curves $\mathcal{C}_f : y = f(x)$ such that $T_r(\mathcal{C}_f) \subset \mathcal{C}_f$ for all real r.
We show that the solutions are the curves $y = f_{n,p}(x)$ where $f_{n,p}$ is defined by

$$f_{n,p}(x) = \begin{cases} x(\log_2(x) + p) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ x(\log_2(|x|) + n) & \text{if } x < 0 \end{cases}$$

where n, p are real constants.

First, let $f_{n,p}$ be such a function, $M(x, f_{n,p}(x))$ be any point on $\mathcal{C}_{f_{n,p}}$ and r any real number. Then $T_r(M)$ has coordinates $x' = 2^r x, \quad y' = r2^r x + 2^r f_{n,p}(x)$.

If $x = 0$, then $y' = 2^r f_{n,p}(0) = 0 = f_{n,p}(x')$.

If $x < 0$, then

$$f_{n,p}(x') = 2^r x(\log_2(|2^r x|) + n) = 2^r x(r + \log_2(|x|) + n) = r2^r x + 2^r f_{n,p}(x) = y'$$

hence $y' = f_{n,p}(x')$ and $T_r(M)$ is on $\mathcal{C}_{f_{n,p}}$.

The case $x > 0$ is easily treated in a similar way. Thus,

$$T_r(\mathcal{C}_{f_{n,p}}) \subset \mathcal{C}_{f_{n,p}} .$$

Conversely, let f be such that $T_r(\mathcal{C}_f) \subset \mathcal{C}_f$ for all r. This means that for all r, x, we have

$$f(2^r x) = r2^r x + 2^r f(x) . \tag{1}$$

Taking $r = 1, x = 0$ in (1) yields $f(0) = 2f(0)$, hence we must have $f(0) = 0$.

Let $\alpha > 0$. Setting $r = \log_2(\alpha)$ and $x = 1$ in (1), we get

$$f(\alpha) = \alpha \log_2(\alpha) + \alpha f(1) = \alpha(\log_2(\alpha) + p) ,$$

where $p = f(1)$.

Now setting $r = \log_2(\alpha)$ and $x = -1$ in (1), we get

$$f(-\alpha) = -\alpha \log_2(\alpha) + \alpha f(-1) = (-\alpha)(\log_2(|-\alpha|) + n) ,$$

where $n = -f(-1)$. It follows that $f = f_{n,p}$.

OC9. A deck of $2n + 1$ cards consists of a joker and, for each number between 1 and n inclusive, two cards marked with that number. The $2n + 1$ cards are placed in a row, with the joker in the middle. For each k with $1 \le k \le n$, the two cards numbered k have exactly $k - 1$ cards between them. Determine all the values of n not exceeding 10 for which this arrangement is possible. For which values of n is it impossible?
(*Originally question # 5 from the 1992 Canadian Mathematical Olympiad.*)

Solution by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

We prove that such an arrangement is possible if and only if

$$n \in \{3, 4, 7, 8\} . \qquad (1)$$

Let's number the places from left to right $1, 2, 3, ..., 2n+1$. The joker is placed on $n+1$. For each $1 \le k \le n$, let the two cards with number k be on position a_k and $a_k + k$. Then we have

$$\sum_{k=1}^{n} (a_k + a_k + k) = \left(\sum_{k=1}^{2n+1} k \right) - (n+1) .$$

Thus

$$\sum_{k=1}^{n} a_k = \frac{3n(n+1)}{4} .$$

Thus, either n or $n+1$ is divisible by 4, which implies (1).

To complete the proof, we need to show that these n work. Indeed we have the following:

$$\begin{array}{lcc} n = 3 & : & 2, 3, 2, J, 3, 1, 1 \\ n = 4 & : & 2, 4, 2, 3, J, 4, 3, 1, 1 \\ n = 7 & : & 5, 3, 4, 7, 3, 5, 4, J, 6, 7, 1, 1, 6 \\ n = 8 & : & 6, 8, 5, 7, 1, 1, 6, 5, J, 8, 7, 4, 2, 3, 2, 4, 3 \end{array}$$

OC10. The number 1987 can be written as a three digit number xyz in some base b. If $x + y + z = 1 + 9 + 8 + 7$, determine all possible values of x, y, z, b.
(*Originally question # 2 from the 1987 Canadian Mathematical Olympiad.*)

Solved by Florencio Cano Vargas, Inca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Cano Vargas.

Let $xyz_{(b)} = 1987$.

Thus

$$xb^2 + yb + z = 1987 . \qquad (1)$$

Since x, y, z are digits in base b it follows immediately that $b > 10$ and $b^2 < 1987$.

From the problem we also have

$$x + y + z = 25. \qquad (2)$$

Subtracting (2) from (1) we get

$$(b - 1)\,(xb + x + y) = 1962 .$$

And hence

$$(b - 1) | 2 \cdot 3^2 \cdot 109 .$$

Since $b^2 < 1987$ we have $109 \nmid (b - 1)$ and hence $(b - 1)$ is a divisor of 18. Hence, using $b > 10$, we get $b - 1 = 18$. Thus $b = 19$.

We showed that $b = 19$ is the only possible solution. We see now that this works. Indeed, changing 1987 into base 19 we get

$$1987_{(10)} = 59E_{(19)} ,$$

where E is the digit eleven in base 19, and this satisfies the requirements of the problem.