

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a **Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Cruce Mathematicorum with Mathematical Mayhem*.

The interim Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School, Orleans, ON). The Assistant Mayhem Editor is Lynn Miller (Cairine Wilson Secondary School, Orleans, ON). The other staff members are Ann Arden (Osgoode Township District High School, Osgoode, ON), Nicole Diotte (Windsor, ON), Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON) and Daphne Shani (Bell High School, Nepean, ON).

Mayhem Problems

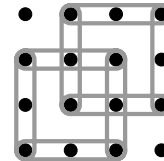
Veillez nous transmettre vos solutions aux problèmes du présent numéro avant le 1 septembre 2012. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

M501. *Proposé par l'Équipe de Mayhem.*

Un grillage **4 par 4** est formé de chevilles amovibles se situant à un centimètre l'une de l'autre, tel qu'illustré dans le schéma. Des élastiques sont attachés aux chevilles de façon à former des carrés; deux carrés différents de taille **2 par 2** sont illustrés dans le schéma. Combien de différents carrés sont possibles?



M502. *Proposé par l'Équipe de Mayhem.*

Lors de leur dernier match de basket, Ariane, Bernard et Claudine ont compté un total de **23** points entre eux. Chaque joueur a compté au moins **1** point; Claudine a compté au moins **10** fois. De combien de façons les **23** points peuvent-ils être répartis, s'il faut satisfaire aux conditions ci-haut? Par exemple, **5** points par Ariane, **3** points par Bernard et **15** points par Claudine serait une possibilité et **3** points par Ariane, **5** par Bernard et **15** par Claudine en serait une autre.

M503. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

Écrire n'importe quel nombre entier et, à la suite, y adjoindre son renversement. Par exemple, si on a écrit **13**, on obtiendra **1331**. Montrer que le résultat de cette opération est toujours divisible par **11**.

M504. *Proposé par Bruce Sawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

À l'intérieur d'un triangle rectangle de côtés **3**, **4** et **5**, deux cercles égaux sont tracés de façon à ce qu'ils soient tangents l'un à l'autre et à un des côtés. De plus, un des cercles est tangent à l'hypoténuse; l'autre cercle de la paire est tangent à l'autre côté. Déterminer les rayons des cercles dans les deux cas.

M505. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Démontrer que pour tout entier positif n , les nombres $A = 5n + 7$ et $B = 6n^2 + 17n + 12$ sont relativement premiers (i.e. n'ont aucun facteur commun sauf 1).

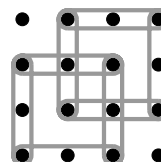
M506. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB; et Bruce Sawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Nous tentons de créer un ensemble de nombres, tels que chacun est obtenu en prenant seulement ses propres chiffres et leur appliquant les quelconques opérations arithmétiques et/ou symboles qui vous sont familiers. Chaque expression doit inclure au moins un symbole/opération; le nombre de fois qu'un chiffre apparaît est le même que dans le nombre lui-même. Par exemple, $1 = \sqrt{1}$, $36 = 6 \times 3!$ et $121 = 11^2$. Toute contribution valide sera reconnue.

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M501. *Proposed by the Mayhem Staff.*

A **4** by **4** square grid is formed by removable pegs that are one centimetre apart as shown in the diagram. Elastic bands may be attached to pegs to form squares, two different **2** by **2** squares are shown in the diagram. How many different squares are possible?



M502. *Proposed by the Mayhem Staff.*

At their last basketball game Alice, Bob and Cindy scored a total of **23** points between them. Each player got at least **1** point, and Cindy scored at least **10**. How many different ways could the **23** points been awarded to satisfy the conditions? For example: **5** points for Alice, **3** points for Bob, **15** for Cindy; and **3** points for Alice, **5** points for Bob, **15** for Cindy; are two different possibilities.

M503. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Write any number and then follow that number by adjoining its reversal. For example, if you write **13** then you would get **1331**. Show that the resulting number is always divisible by **11**.

M504. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Inside a right triangle with sides **3**, **4**, **5**, two equal circles are drawn that are tangent to one another and to one leg. One circle of the pair is tangent to the hypotenuse. The other circle of the pair is tangent to the other leg. Determine the radii of the circles in both cases.

M505. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Prove that, for all positive integers n , the quantities $A = 5n + 7$ and $B = 6n^2 + 17n + 12$ are coprime (i.e. have no common factors other than 1).

M506. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB; and Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

We are trying to create a set of positive integers, that each can be formed using their own digits only, along with any mathematical operations and/or symbols that are familiar to you. Each expression must include at least one symbol/operation; the number of times a digit appears is the same as in the number itself. For example, $1 = \sqrt{1}$, $36 = 6 \times 3!$ and $121 = 11^2$. All valid contributions will be acknowledged.

Mayhem Solutions

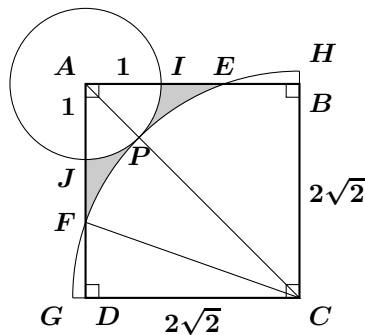
M463. *Proposed by the Mayhem Staff.*

The square $ABCD$ has side length $2\sqrt{2}$. A circle with centre A and radius 1 is drawn. A second circle with centre C is drawn so that it just touches the first circle at point P on AC . Determine the total area of the regions inside the square but outside the two circles.

Solution by Gloria Fang, student, University of Toronto Schools, Toronto, ON.

From the Pythagorean Theorem we get $AC = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{8+8} = \sqrt{16} = 4$. Since CP is a radius of the larger circle, and $AP = 1$ we get $CP = CA - PA = 4 - 1 = 3$, therefore the radius of the large circle is 3 .

Using $[A]$ to represent the area of figure A we get



$$\begin{aligned}
[GDF] &= [FCG] - [FCD] \\
&= \frac{\alpha}{2\pi} \cdot \pi(3)^2 - \frac{\sqrt{3^2 - (2\sqrt{2})^2} \cdot 2\sqrt{2}}{2} \\
&= \frac{9}{2}\alpha - \sqrt{2}
\end{aligned}$$

where $\alpha = \angle FCG = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$. Next, we have $[PCDF] = [PCG] - [GDF] = \frac{9\pi}{8} - \frac{9}{2}\alpha + \sqrt{2}$. Thus $[JPF] = [ACD] - [APJ] - [PCDF] = 4 - \frac{\pi}{8} - \left(\frac{9\pi}{8} - \frac{9}{2}\alpha + \sqrt{2}\right) = 4 - \sqrt{2} - \frac{5\pi}{4} + \frac{9}{2}\alpha$.

By symmetry $[IPE] = [JPE]$ thus the area of the regions inside the square but not inside a circle is $2[JPE]$, or

$$8 - 2\sqrt{2} - \frac{5\pi}{2} + 9\alpha = 8 - 2\sqrt{2} - \frac{5\pi}{2} + 9 \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right) \doteq 0.38 \text{ units.}$$

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain. Four incorrect solutions were received.

M464. *Proposed by the Mayhem Staff.*

Let $\lfloor x \rfloor$ be the greatest integer not exceeding x . For example, $\lfloor 3.1 \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$. Find all real numbers x for which $\lfloor \sqrt{x^2 + 1} - 1 \rfloor = 2$.

Solution by Florencio Cano Vargas, Inca, Spain.

The given equation is equivalent to the inequalities:

$$\begin{aligned}
2 \leq \sqrt{x^2 + 1} - 1 < 3 &\Leftrightarrow 3 \leq \sqrt{x^2 + 1} < 4 \\
&\Leftrightarrow 9 \leq x^2 + 1 < 16 \Leftrightarrow 8 \leq x^2 < 15
\end{aligned}$$

and then, the result is the interval $x \in (-\sqrt{15}, -2\sqrt{2}] \cup [2\sqrt{2}, \sqrt{15})$.

Also solved by GLORIA FANG, student, University of Toronto Schools, Toronto, ON; MUHAMMAD HAFIZ FARIZI, student, SMPN 8, Yogyakarta, Indonesia; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. Two incorrect solutions were received.

M465. *Proposed by Antonio Ledesma López, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain.*

The integer **20114022** is divisible by **2011**. Determine if there exists a positive integer that is divisible by **2011** and whose digits add to **2011**.

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

The answer is yes. The positive integer $201120112011 \cdots 201110055$, which contains 500 groups with digits 2011 and $10055 = 5 \times 2011$ as its final digits, is divisible by 2011 because a direct division would show that it is equal to $2011 \times 100010001 \cdots 1000100005$, where the last number of this product contains 499 groups with digits 1000 and the digits 100005 as its final digits. Another calculation shows that the digits add to

$$\begin{aligned} & (2 + 0 + 1 + 1) + (2 + 0 + 1 + 1) + \cdots + (2 + 0 + 1 + 1) + (1 + 0 + 0 + 5 + 5) \\ &= \underbrace{4 + 4 + 4 + \cdots + 4}_{500 \text{ times}} + 11 = 2011. \end{aligned}$$

Other solutions can be obtained in a similar way, for example $20112011 \cdots 201112066$, made up of 499 groups with digits 2011 and $12066 = 3 \times 2011$ as its final digits.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; GLORIA FANG, student, University of Toronto Schools, Toronto, ON; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

M466. Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.

Determine all pairs (m, n) of positive integers such that $2^m - 2 = n!$.

I. Solution by Sally Li, student, Marc Garneau Collegiate Institute, Toronto, ON.

The condition is equivalent to $2^{m-1} - 1 = \frac{n!}{2}$. Since $\frac{n!}{2}$ is non-zero, it must be odd, thus n can only be 1, 2 or 3. Trying each case we find the only solutions are $m = n = 2$ or $m = n = 3$.

II. Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

We have $2^m - 2 = n!$ or $2^m = 2 + n!$. If $n \geq 4$, then $2 + n! \geq 26$, thus $m > 4$. Hence, $2^m \equiv 0 \pmod{4}$ and $n! + 2 \equiv 2 \pmod{4}$, a contradiction.

Therefore, $n \leq 3$. By direct calculation we get $(m, n) = (3, 3)$ or $(m, n) = (2, 2)$.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; GLORIA FANG, student, University of Toronto Schools, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer. One incorrect solution was received.

M467. Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

Determine all real numbers x for which

$$(x - 2010)^3 + (2x - 2010)^3 + (4020 - 3x)^3 = 0.$$

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

First, we observe that the given equation can be written as

$$(x - 2010)^3 + (2x - 2010)^3 = (3x - 4020)^3.$$

Putting $a = x - 2010$, $b = 2x - 2010$, and $c = 3x - 4020$, we have $a + b = c$ and $a^3 + b^3 = c^3$ from which immediately follows $(a + b)^3 = a^3 + b^3$ or $3ab(a + b) = 0$. To get the solutions we consider the following two cases:

- (i) $a + b = 3x - 4020 = 0$ from which we obtain the solution $x = 1340$;
- (ii) $ab = (x - 2010)(2x - 2010) = 0$ from which we obtain the solutions $x = 2010$ and $x = 1005$.

Since the polynomial has degree three and we have found three solutions, on account of the Fundamental Theorem of Algebra, the given equation does not have more roots and we are done.

Also solved by MIHÁLY BENCZE, Brasov, Romania; FLORENCIO CANO VARGAS, Inca, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; GLORIA FANG, student, University of Toronto Schools, Toronto, ON; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

Bencze generalized the problem by writing it as $(ax - b)^3 + (cx - d)^3 + (b + d - (a + c)x)^3 = 0$, with $ac(a + c) \neq 0$, which has solutions $x_1 = \frac{b}{a}$, $x_2 = \frac{d}{c}$ and $x_3 = \frac{b+d}{a+c}$. Thus setting $a = 1$, $b = 2010$, $c = 2$ and $d = 2010$ yields the given problem and its solutions.

M468. *Proposed by Gheorghe Ghiță, M. Eminescu National College, Buzău, Romania.*

Determine all pairs (p, q) of prime numbers for which each of $p + q$, $p + q^2$, $p + q^3$, and $p + q^4$ is a prime number.

Solution by Florencio Cano Vargas, Inca, Spain.

Let p, q be prime numbers. If we require that $p + q > 2$ has to be prime, either $p = 2$ or $q = 2$ to ensure that $p + q$ is odd. Let us consider two cases separately:

Case 1: $q = 2$

In this case, p must be an odd prime number $p \geq 3$. Clearly, $p = 3$ is a solution since $p + q = 5$, $p + q^2 = 7$, $p + q^3 = 11$ and $p + q^4 = 19$ are prime numbers. For $p > 3$ we will study the values of the expressions modulo 3. Let $p \equiv t \pmod{3}$ where $t = 1$ or $t = 2$ since $p > 3$ is a prime. Then

$$\begin{aligned} p + q &\equiv t + 2 \pmod{3} \\ p + q^2 &\equiv t + 4 \equiv t + 1 \pmod{3} \end{aligned}$$

and therefore, either $p + q$ or $p + q^2$ is divisible by three, thus they both cannot be prime.

Case 2: $p = 2$

In this case, q must be an odd prime number $q \geq 3$. Clearly, $q = 3$ is a solution since $p + q = 5$, $p + q^2 = 11$, $p + q^3 = 29$ and $p + q^4 = 83$ are prime

numbers. For $q > 3$ we will study the values of the expressions modulo 3. Let $q \equiv t \pmod{3}$ where $t = 1$ or $t = 2$ since $q > 3$ is a prime, then

$$\begin{aligned} p + q &\equiv 2 + t \pmod{3} \\ p + q^2 &\equiv 2 + t^2 \pmod{3}. \end{aligned}$$

If $t = 1$, $p + q$ is divisible by three and if $t = 2$, $p + q^2$ is divisible by three, therefore, either $p + q$ or $p + q^2$ is divisible by three, thus they both cannot be prime.

Summarizing, the only pairs (p, q) of prime numbers which satisfy the condition of the problem are $(2, 3)$ and $(3, 2)$.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; GLORIA FANG, student, University of Toronto Schools, Toronto, ON; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

M469. *Proposed by Antonio Ledesma López, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain.*

Prove that, for all real numbers x , we have $(2^{\sin x} + 2^{\cos x})^2 \geq 2^{2-\sqrt{2}}$.

Solution by Gloria Fang, student, University of Toronto Schools, Toronto, ON.

By the AM-GM inequality we have that $\frac{2^{\sin x} + 2^{\cos x}}{2} \geq \sqrt{2^{\sin x} \cdot 2^{\cos x}}$, so $(2^{\sin x} + 2^{\cos x})^2 \geq 4 \cdot 2^{\sin x} \cdot 2^{\cos x}$ thus $(2^{\sin x} + 2^{\cos x})^2 \geq 2^{2+\sin x+\cos x}$. Using well known trigonometric identities we get

$$\begin{aligned} \sin x + \cos x &= \frac{\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x}{\frac{1}{\sqrt{2}}} \\ &= \frac{\cos \frac{\pi}{4} \sin x + \sin \frac{\pi}{4} \cos x}{\frac{1}{\sqrt{2}}} \\ &= \sqrt{2} \sin \left(\frac{\pi}{4} + x \right). \end{aligned}$$

Since $-1 \leq \sin \left(\frac{\pi}{4} + x \right) \leq 1$, we must have $\sqrt{2} \sin \left(\frac{\pi}{4} + x \right) \geq -\sqrt{2}$. Thus

$$(2^{\sin x} + 2^{\cos x})^2 \geq 2^{2+\sin x+\cos x} = 2^{2+\sqrt{2} \sin \left(\frac{\pi}{4} + x \right)} \geq 2^{2-\sqrt{2}}.$$

Also solved by MIHÁLY BENCZE, Brasov, Romania; FLORENCIO CANO VARGAS, Inca, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

[Ed.: Note the solution could be shortened by noting that $\sin 2x = 2 \sin(x) \cos(x) \leq 1 \Rightarrow (\sin(x) + \cos(x))^2 \leq 2 \Rightarrow 2 - \sqrt{2} \leq 2 + \sin(x) + \cos(x)$, from which the result follows.]