M504. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Inside a right triangle with sides 3, 4, 5, two equal circles are drawn that are tangent to one another and to one leg. One circle of the pair is tangent to the hypotenuse. The other circle of the pair is tangent to the other leg. Determine the radii of the circles in both cases.

M505. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Prove that, for all positive integers n, the quantities $A = 5n + 7$ and $B = 6n^2 + 17n + 12$ are coprime (i.e. have no common factors other than 1).

M506. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB; and Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

We are trying to create a set of positive integers, that each can be formed using their own digits only, along with any mathematical operations and/or symbols that are familiar to you. Each expression must include at least one symbol/operation; the number of times a digit appears is the same as in the number itself. For example, $1 = \sqrt{1}$, $36 = 6 \times 3!$ and $121 = 11^2$. All valid contributions will be acknowledged.

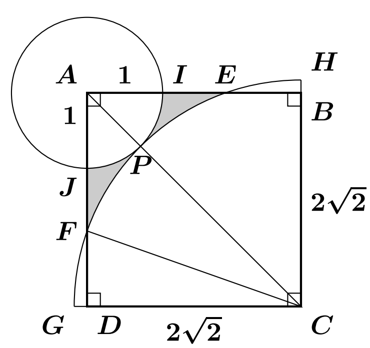
Mayhem Solutions

M463. *Proposed by the Mayhem Staff.*

The square $ABCD$ has side length $2\sqrt{2}$. A circle with centre A and radius 1 is drawn. A second circle with centre C is drawn so that it just touches the first circle at point P on AC. Determine the total area of the regions inside the square but outside the two circles.

Solution by Gloria Fang, student, University of Toronto Schools, Toronto, ON.

From the Pythagorean Theorem we get $AC = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{8+8} = \sqrt{16} = 4$. Since CP is a radius of the larger circle, and $AP = 1$ we get $CP = CA - PA = 4 - 1 = 3$, therefore the radius of the large circle is 3.



Using $[A]$ to represent the area of figure A we get

$$\begin{aligned}[GDF] &= [FCG] - [FCD] \\ &= \frac{\alpha}{2\pi} \cdot \pi(3)^2 - \frac{\sqrt{3^2 - (2\sqrt{2})^2} \cdot 2\sqrt{2}}{2} \\ &= \frac{9}{2}\alpha - \sqrt{2}\end{aligned}$$

where $\alpha = \angle FCG = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$. Next, we have $[PCDF] = [PCG] - [GDF] = \frac{9\pi}{8} - \frac{9}{2}\alpha + \sqrt{2}$. Thus $[JPF] = [ACD] - [APJ] - [PCDF] = 4 - \frac{\pi}{8} - \left(\frac{9\pi}{8} - \frac{9}{2}\alpha + \sqrt{2}\right) = 4 - \sqrt{2} - \frac{5\pi}{4} + \frac{9}{2}\alpha$.

By symmetry $[IPE] = [JPE]$ thus the area of the regions inside the square but not inside a circle is $2[JPE]$, or

$$8 - 2\sqrt{2} - \frac{5\pi}{2} + 9\alpha = 8 - 2\sqrt{2} - \frac{5\pi}{2} + 9\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right) \doteq 0.38 \text{ units.}$$

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain. Four incorrect solutions were received.

M464. *Proposed by the Mayhem Staff.*

Let $\lfloor x \rfloor$ be the greatest integer not exceeding x. For example, $\lfloor 3.1 \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$. Find all real numbers x for which $\lfloor \sqrt{x^2+1} - 1 \rfloor = 2$.

Solution by Florencio Cano Vargas, Inca, Spain.

The given equation is equivalent to the inequalities:

$$\begin{aligned}2 \le \sqrt{x^2+1} - 1 < 3 \Leftrightarrow 3 \le \sqrt{x^2+1} < 4 \\ \Leftrightarrow 9 \le x^2 + 1 < 16 \Leftrightarrow 8 \le x^2 < 15\end{aligned}$$

and then, the result is the interval $x \in (-\sqrt{15}, -2\sqrt{2}] \cup [2\sqrt{2}, \sqrt{15})$.

Also solved by GLORIA FANG, student, University of Toronto Schools, Toronto, ON; MUHAMMAD HAFIZ FARIZI, student, SMPN 8, Yogyakarta, Indonesia; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. Two incorrect solutions were received.

M465. *Proposed by Antonio Ledesma López, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain.*

The integer 20114022 is divisible by 2011. Determine if there exists a positive integer that is divisible by 2011 and whose digits add to 2011.

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

The answer is yes. The positive integer $\mathbf{201120112011 \cdots 201110055}$, which contains $\mathbf{500}$ groups with digits $\mathbf{2011}$ and $\mathbf{10055 = 5 \times 2011}$ as its final digits, is divisible by $\mathbf{2011}$ because a direct division would show that it is equal to $\mathbf{2011 \times 100010001 \cdots 1000100005}$, where the last number of this product contains $\mathbf{499}$ groups with digits $\mathbf{1000}$ and the digits $\mathbf{100005}$ as its final digits. Another calculation shows that the digits add to

$$(2+0+1+1)+(2+0+1+1)+\cdots+(2+0+1+1)+(1+0+0+5+5)$$
$$= \overbrace{4+4+4+\cdots+4}^{500 \text{ times}} + 11 = 2011\,.$$

Other solutions can be obtained in a similar way, for example $\mathbf{20112011 \cdots 201112066}$, made up of $\mathbf{499}$ groups with digits $\mathbf{2011}$ and $\mathbf{12066 = 3 \times 2011}$ as its final digits.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; GLORIA FANG, student, University of Toronto Schools, Toronto, ON; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

M466. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Determine all pairs (m, n) of positive integers such that $2^m - 2 = n!$.

I. Solution by Sally Li, student, Marc Garneau Collegiate Institute, Toronto, ON.

The condition is equivalent to $2^{m-1} - 1 = \frac{n!}{2}$. Since $\frac{n!}{2}$ is non-zero, it must be odd, thus n can only be 1, 2 or 3. Trying each case we find the only solutions are $m = n = 2$ or $m = n = 3$.

II. Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

We have $2^m - 2 = n!$ or $2^m = 2 + n!$. If $n \geq 4$, then $2 + n! \geq 26$, thus $m > 4$. Hence, $2^m \equiv 0 (\text{mod } 4)$ and $n! + 2 \equiv 2 (\text{mod } 4)$, a contradiction.

Therefore, $n \leq 3$. By direct calculation we get $(m, n) = (3, 3)$ or $(m, n) = (2, 2)$.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; GLORIA FANG, student, University of Toronto Schools, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer. One incorrect solution was received.

M467. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Determine all real numbers x for which

$$(x - 2010)^3 + (2x - 2010)^3 + (4020 - 3x)^3 = 0\,.$$

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

First, we observe that the given equation can be written as

$$(x - 2010)^3 + (2x - 2010)^3 = (3x - 4020)^3\,.$$

Putting $a = x - 2010$, $b = 2x - 2010$, and $c = 3x - 4020$, we have $a + b = c$ and $a^3 + b^3 = c^3$ from which immediately follows $(a + b)^3 = a^3 + b^3$ or $3ab(a + b) = 0$. To get the solutions we consider the following two cases:

(i) $a + b = 3x - 4020 = 0$ from which we obtain the solution $x = 1340$;

(ii) $ab = (x - 2010)(2x - 2010) = 0$ from which we obtain the solutions $x = 2010$ and $x = 1005$.

Since the polynomial has degree three and we have found three solutions, on account of the Fundamental Theorem of Algebra, the given equation does not have more roots and we are done.

Also solved by MIHÁLY BENCZE, Brasov, Romania; FLORENCIO CANO VARGAS, Inca, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; GLORIA FANG, student, University of Toronto Schools, Toronto, ON; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

Bencze generalized the problem by writing it as $(ax - b)^3 + (cx - d)^3 + (b + d - (a + c)x)^3 = 0$*, with* $ac(a + c) \neq 0$*, which has solutions* $x_1 = \frac{b}{a}$*,* $x_2 = \frac{d}{c}$ *and* $x_3 = \frac{b+d}{a+c}$*. Thus setting* $a = 1$*,* $b = 2010$*,* $c = 2$ *and* $d = 2010$ *yields the given problem and its solutions.*

M468. *Proposed by Gheorghe Ghiţă, M. Eminescu National College, Buzău, Romania.*

Determine all pairs (p, q) of prime numbers for which each of $p + q$, $p + q^2$, $p + q^3$, and $p + q^4$ is a prime number.

Solution by Florencio Cano Vargas, Inca, Spain.

Let p, q be prime numbers. If we require that $p + q > 2$ has to be prime, either $p = 2$ or $q = 2$ to ensure that $p + q$ is odd. Let us consider two cases separately:

Case 1: $q = 2$

In this case, p must be an odd prime number $p \geq 3$. Clearly, $p = 3$ is a solution since $p + q = 5$, $p + q^2 = 7$, $p + q^3 = 11$ and $p + q^4 = 19$ are prime numbers. For $p > 3$ we will study the values of the expressions modulo 3. Let $p \equiv t (\text{mod } 3)$ where $t = 1$ or $t = 2$ since $p > 3$ is a prime. Then

$$p + q \equiv t + 2 (\text{mod } 3)$$
$$p + q^2 \equiv t + 4 \equiv t + 1 (\text{mod } 3)$$

and therefore, either $p + q$ or $p + q^2$ is divisible by three, thus they both cannot be prime.

Case 2: $p = 2$

In this case, q must be an odd prime number $q \geq 3$. Clearly, $q = 3$ is a solution since $p + q = 5$, $p + q^2 = 11$, $p + q^3 = 29$ and $p + q^4 = 83$ are prime numbers. For $q > 3$ we will study the values of the expressions modulo 3. Let $q \equiv t (\text{mod } 3)$ where $t = 1$ or $t = 2$ since $q > 3$ is a prime, then

$$p + q \equiv 2 + t (\text{mod } 3)$$
$$p + q^2 \equiv 2 + t^2 (\text{mod } 3)\,.$$

If $t = 1$, $p + q$ is divisible by three and if $t = 2$, $p + q^2$ is divisible by three, therefore, either $p + q$ or $p + q^2$ is divisible by three, thus they both cannot be prime.

Summarizing, the only pairs (p, q) of prime numbers which satisfy the condition of the problem are $(2, 3)$ and $(3, 2)$.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; GLORIA FANG, student, University of Toronto Schools, Toronto, ON; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

M469. *Proposed by Antonio Ledesma López, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain.*

Prove that, for all real numbers x, we have $\left(2^{\sin x} + 2^{\cos x}\right)^2 \geq 2^{2-\sqrt{2}}$.

Solution by Gloria Fang, student, University of Toronto Schools, Toronto, ON.

By the AM-GM inequality we have that $\dfrac{2^{\sin x} + 2^{\cos x}}{2} \geq \sqrt{2^{\sin x} \cdot 2^{\cos x}}$, so $\left(2^{\sin x} + 2^{\cos x}\right)^2 \geq 4 \cdot 2^{\sin x} \cdot 2^{\cos x}$ thus $\left(2^{\sin x} + 2^{\cos x}\right)^2 \geq 2^{2+\sin x+\cos x}$. Using well known trigonometric identities we get

$$\begin{aligned}\sin x + \cos x &= \frac{\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x}{\frac{1}{\sqrt{2}}} \\ &= \frac{\cos \frac{\pi}{4} \sin x + \sin \frac{\pi}{4} \cos x}{\frac{1}{\sqrt{2}}} \\ &= \sqrt{2} \sin\left(\frac{\pi}{4} + x\right)\,.\end{aligned}$$

Since $-1 \leq \sin\left(\frac{\pi}{4} + x\right) \leq 1$, we must have $\sqrt{2} \sin\left(\frac{\pi}{4} + x\right) \geq -\sqrt{2}$. Thus

$$\left(2^{\sin x} + 2^{\cos x}\right)^2 \geq 2^{2+\sin x+\cos x} = 2^{2+\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)} \geq 2^{2-\sqrt{2}}\,.$$

Also solved by MIHÁLY BENCZE, Brasov, Romania; FLORENCIO CANO VARGAS, Inca, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

[Ed.: Note the solution could be shortened by noting that $\sin 2x = 2\sin(x)\cos(x) \leq 1 \Rightarrow (\sin(x) + \cos(x))^2 \leq 2 \Rightarrow 2 - \sqrt{2} \leq 2 + \sin(x) + \cos(x)$*, from which the result follows.]*