SOLUTIONS

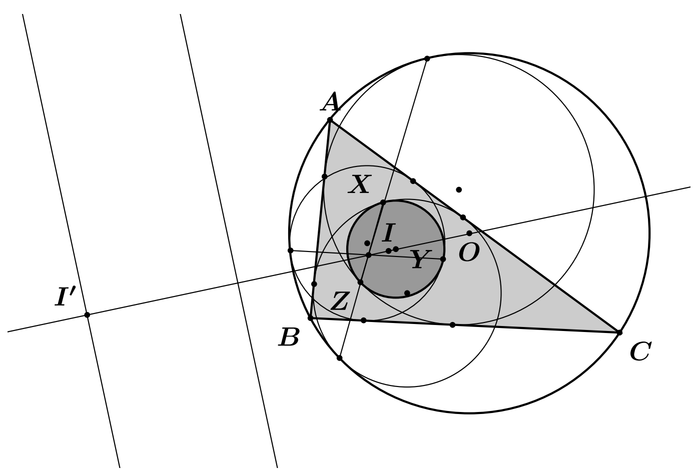
No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3542★. [2010 : 240, 242; 2011 : 244] *Proposed by Cosmin Pohoaţă, Tudor Vianu National College, Bucharest, Romania.*

The mixtilinear incircles of a triangle ABC are the three circles each tangent to two sides and to the circumcircle internally. Let Γ be the circle tangent to each of these three circles internally. Prove that Γ is orthogonal to the circle passing through the incentre and the isodynamic points of the triangle ABC.

[*Ed.: Let* Γ_A *be the circle passing through* A *and the intersection points of the internal and external angle bisectors at* A *with the line* BC*. The isodynamic points are the two points that* Γ_A*,* Γ_B*, and* Γ_C *have in common.*]

Solution by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.



1. The circle Γ has barycentric equation

$$(a+b+c)^2(a^2+b^2+c^2-2ab-2bc-2ca)(a^2yz+b^2zx+c^2xy)$$
$$+\,8abc(x+y+z)\left(\sum_{\text{cyclic}} bc(b+c-a)x\right)=0.$$

Proof. The mixtilinear incircles are defined by the equations

$$
\begin{aligned}
(a+b+c)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)f_a(x,y,z) &= 0,\\
(a+b+c)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)f_b(x,y,z) &= 0,\\
(a+b+c)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)f_c(x,y,z) &= 0,
\end{aligned}
$$

where

$$
\begin{aligned}
f_a(x,y,z) &= 4b^2c^2x+c^2(c+a-b)^2y+b^2(a+b-c)^2z,\\
f_b(x,y,z) &= c^2(b+c-a)^2x+4c^2a^2y+a^2(a+b-c)^2z,\\
f_c(x,y,z) &= b^2(b+c-a)^2x+a^2(c+a-b)^2y+4a^2b^2z.
\end{aligned}
$$

Their radical center is the point defined by

$$f_a(x,y,z)=f_b(x,y,z)=f_c(x,y,z).$$

Solving these equations we obtain the radical center in homogeneous barycentric coordinates

$$(a^2(b^2+c^2-a^2-4bc):b^2(c^2+a^2-b^2-4ca):c^2(a^2+b^2-c^2-4ab)).$$

This point divides OI in the ratio $2R:-r$. The lines joining this radical center to the points of tangency with the circumcircle intersect the respective mixtilinear incircles again at the points

$$
\begin{aligned}
X &= (a(a^2+2a(b+c)-3(b-c)^2):\ 2b^2(c+a-b):\ 2c^2(a+b-c)),\\
Y &= (2a^2(b+c-a):\ b(b^2+2b(c+a)-3(c-a)^2):\ 2c^2(a+b-c)),\\
Z &= (2a^2(b+c-a):\ 2b^2(c+a-b):\ c(c^2+2c(a+b)-3(a-b)^2)).
\end{aligned}
$$

Γ is the circle containing these three points.

Note: The center of Γ is the point

$$(a^2(b^2+c^2-a^2+8bc):b^2(c^2+a^2-b^2+8ca):c^2(a^2+b^2-c^2+8ab)),$$

which divides OI in the ratio $4R:r$.

2. The line $\sum_{\text{cyclic}} bc(b+c-a)x=0$ is the perpendicular bisector of II', where I' is the inversive image of the incenter I in the circumcircle.

   Proof. The polar of I in the circumcircle is the line $\sum_{\text{cyclic}} bc(b+c)x=0$. Replacing (x,y,z) by $2(a+b+c)(x,y,z)-(x+y+z)(a,b,c)$, we obtain the image of this polar under the homothety $\mathsf{h}\left(I,\frac{1}{2}\right)$. This gives the line in question. Since the pedal of I on its polar is the inversive image I', the line is the perpendicular bisector of II'.

3. One obtains the equation of the pencil of circles generated by a circle and a line by setting equal to zero a linear combination of the circle's formula and the product of the line's formula times that of the line at infinity: $x+y+z$. (Circles are conics that pass through the conjugate imaginary points on the line at infinity; all circles will contain those two points while the pencil generated by a circle and line will consist of all circles through the common pair of points of that circle and line—points which are possibly imaginary or coincident.)

From the equation of Γ in (1), we conclude that it is a member in the pencil of circles generated by the circumcircle $a^2yz + b^2zx + c^2xy = 0$ and the perpendicular bisector of II' with equation computed in (2).

Now, any circle through I and I' is orthogonal to the circumcircle.

Since the isodynamic points are inverse in the circumcircle, the circle through I, I' and one of them must also contain the other. In other words, the circle $\mathcal{C}$ through I and the isodynamic points contains I', and is orthogonal to every circle in the pencil generated by the circumcircle and the perpendicular bisector of II'.

Since Γ is a member of this pencil, it is orthogonal to circle $\mathcal{C}$.

No other solutions were received.

3556. [2010 : 315, 317] *Proposed by Arkady Alt, San Jose, CA, USA.*

For any acute triangle with side lengths a, b, and c, prove that

$$(a+b+c)\min\{a,b,c\} \;\le\; 2ab+2bc+2ca-a^2-b^2-c^2\,.$$

I. Solution by Edmund Swylan, Riga, Latvia.

Note that $a+b+c = (-a+b+c)+(a-b+c)+(a+b-c),$ where each of the three terms on the right are positive. Now,

$$\begin{aligned}
-a^2+ab+ca &= (-a+b+c)a,\\
ab-b^2+bc &= (a-b+c)b,\\
ca+bc-c^2 &= (a+b-c)c.
\end{aligned}$$

The result follows, for any triangle, by adding these equations, replacing the final factors on the right by $\min\{a,b,c\}$, and using the first identity above.

II. Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Taking $a \le b \le c$, the required inequality may be successively rewritten:

$$\begin{aligned}
ab+2bc+ca-2a^2-b^2-c^2 &\ge 0,\\
2a(b-a)+(c-b)(a+b-c) &\ge 0.
\end{aligned}$$

By the triangle inequality, both terms are nonnegative, proving the claim.

III. Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy

We introduce the well-known change of variables $a = x + z$, $b = x + y$, $c = y + z$, or $x = \frac{a+b-c}{2} \geq 0$, $y = \frac{b+c-a}{2} \geq 0$ and $z = \frac{a+c-b}{2} \geq 0$. Taking $a \leq b \leq c$ gives $y \geq x, z$ and the required inequality becomes

$$2(x + y + z)(x + z) \leq 4(xy + yz + xz),$$

which is equivalent to $(xy+zy) \geq x^2+z^2$, which holds because of the inequalities between x, y, z. The result follows.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Cománeşti, Romania; and the proposer.

Most solvers observed that the result holds regardless of whether the triangle is acute. Arslanagić, Geupel and Malikić also noted the case of equality, $a = b = c$, but all appear to have missed two separate (degenerate) cases of equality, namely (i) $c = 2a = 2b$ and (ii) $a = 0$, $b = c$.

3563. [2010 : 316, 318] *Proposed by Mikhail Kochetov and Sergey Sadov, Memorial University of Newfoundland, St. John's, NL.*

A square $n \times n$ array of lamps is controlled by an $n \times n$ switchboard. Flipping a switch in position (i, j) changes the state of all lamps in row i and in column j.

(a) Prove that for even n it is possible to turn off all the lamps no matter what the initial state of the array is. Demonstrate how to do it with the minimum number of switches.

(b) Prove that for odd n it is possible to turn off all the lamps if and only if the initial state of the array has the following property: either the number of ON lamps in every row and every column is odd, or the number of ON lamps in every row and every column is even. If this property holds, provide an algorithm to turn off all the lamps.

Solution by Steffen Weber, student, Martin-Luther-Universität, Halle, Germany.

The order in which switches are flipped does not affect the final state, and flipping a switch twice has no effect. A series of flips, avoiding this redundancy, may be coded as an $n \times n$ $(0, 1)$-matrix in which "1" represents a flip in the corresponding position. It follows that there are at most 2^{n^2} distinct transformations.

(a) If n is even, flipping all switches which are in row i or column j changes only the state of lamp (i, j). Doing this once for every ON lamp turns OFF all lamps. It follows that there are at least 2^{n^2} distinct transformations, and so *exactly* 2^{n^2} distinct transformations, in one-to-one correspondence with the coded matrices described above, which give the unique minimum series of flips attaining each transformation.

Eliminating redundant flips from the above series of flips to turn all lamps OFF we have that the unique minimum series of flips is attained by flipping each switch (i, j) if and only if there are initially an odd number of ON lamps among the $2n - 1$ positions in row i and column j.

(b) If n is odd, flipping any switch changes the parity of the number of ON lamps in every row and column. If all lamps are OFF this parity is 0 for every row and column, so as required, a solution is possible only if, in the initial state, the number of ON lamps in every row and column has the same parity.

If this condition is met, follow the procedure described in (a) for n even to turn OFF all lamps in the initial $(n - 1) \times (n - 1)$ block. Because the parity property is preserved, either every lamp in the n^{th} row and column is OFF (and we're done), or all lamps in those positions are ON; in the latter case, flipping switch (n, n) turns off these remaining lamps.

Also solved by SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; OLIVER GEUPEL, Brühl, NRW, Germany; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

Some solvers observed that the solution described for odd n is also optimum. Some solvers explicitly derived relationships between the state matrix and the (coded) transformation, using arithmetic modulo 2 or representing both states and transformations (acting additively) as vectors in the space of $n \times n$ matrices over $\mathbb{Z}_2$.

The proposer notes that this problem belongs to a popular class of "switchboard problems" also known as "all ones" and "lights out" (the name of a commercial game). Problems of such type are found in many math competitions and popular math journals. The problem was inspired by [6]. However, optimality was not shown for even n and the solution for odd n was inelegant. Some other references are given below.

References

[1] Kvant, Problem M-665, v. 12, No 9 (1981), p. 26; Solution: v. 13, No 10 (1982), pp. 23–24.

[2] Math. Intelligencer, Problem 88-8, v. 10, No. 3 (1988); Solution: v. 11, No. 2 (1989), pp. 31–32.

[3] Amer. Math. Monthly, Problem 10197, v. 99 (1992), p. 162; Solution: v. 100 (1993), p. 806 .

[4] M. Anderson, T. Feil, *Turning lights out with linear algebra*, Math. Magazine, v. 71, No. 4 (1998), pp. 300–303.

[5] A. Shen, *Lights out*, Math Intelligencer, v. 22, No. 13 (2000), pp. 20–21.

[6] P. Araújo, *How to turn all the lights out*, Elem. Math. v. 55 (2000), pp. 135–141.

3564. [2010 : 396, 398] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, d be positive real numbers. Prove that

$$a^3 + b^3 + c^3 + d^3 + \frac{32abcd}{a+b+c+d} \ \geq\ 3(abc + bcd + cda + dab)\,.$$

Solution by Joe Howard, Portales, NM, USA.

This is Problem 8 of [1], left as an exercise. We will follow the solution to Problem 3, from the same article.

Without loss of generality we can assume that $d = \min\{a, b, c, d\} = 1$. Then, the inequality is:

$$a^3 + b^3 + c^3 + 1 + \frac{32abc}{a+b+c+1} \geq 3(abc + ab + ac + bc)\,,$$

or equivalently

$$(a+b+c+1)^4 + 32abc \geq 3(a+b+c+1)^2(ab+bc+ca+a+b+c)\,.$$

Let $p = a+b+c$, $q = ab+bc+ca$ and $r = abc$. Then $p \geq 3$ and $q \geq 3$. We need to prove

$$(p+1)^4 + 32r \geq 3(p+1)^2(p+q)\,.$$

The inequality $3q \leq p^2$ is equivalent to $ab + ac + bc \leq a^2 + b^2 + c^2$, which is easy to prove. Thus, we can find some $t \geq 0$ so that

$$q = \frac{p^2 - t^2}{3}\,.$$

Now, by Theorem 1 in [1] we get

$$r \geq \frac{p^3 - 3pt^2 - 2t^3}{27}\,.$$

To complete the proof it suffices to show that

$$(p+1)^4 - 3p(p+1)^2 - 3(p+1)^2\frac{p^2 - t^2}{3} + \frac{32}{27}(p^3 - 3pt^2 - 2t^3) \geq 0\,. \quad (1)$$

This simplifies to

$$(5p+3)(p-3)^2 + t^2(27 + 27p^2 - 42p - 64t) \geq 0\,.$$

Using $p \geq 3$ we get $14p^2 \geq 42p$, while from $\frac{p^2-t^2}{3} = q \geq 3$ we get

$$p^2 \geq t^2 + 9 \geq 6t\,.$$

Thus

$$27p^2 = 14p^2 + 13p^2 \geq 42p + 78t \geq 42p + 64t\,,$$

which proves (1), and hence completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer . There was one incorrect solution.

References

[1] P. V. Thuận, Lê Vĩ, *A useful inequality revisited*, Crux, Vol 35(3), 2009, pp. 164-171

3565. [2010 : 396, 398] *Proposed by Max Diaz, student, San Juan Bosco High School, Huancayo, Junin, Peru.*

Find all positive integers n such that $\sigma(\tau(n)) = n$, where $\tau(m)$ and $\sigma(m)$ are, respectively, the number of positive divisors of the integer m and the sum of all the positive divisors of the integer m.

Solution by Oliver Geupel, Brühl, NRW, Germany.

The solutions are 1, 3, 4, and 12.

Note that if m is a positive integer and $m = ab$ where $1 \leq a \leq b \leq m$, then $a \leq \sqrt{m}$. Thus, for each $d = 1, 2, \ldots, \lfloor\sqrt{m}\rfloor$, there is at most one positive integer d' such that $dd' = m$. Hence, we have

$$\tau(m) \leq 2\lfloor\sqrt{m}\rfloor \leq 2\sqrt{m}. \tag{1}$$

Furthermore, if d is a positive divisor of m with $d < m$, then clearly $d \leq \dfrac{m}{2}$, so $d \leq \left\lfloor \dfrac{m}{2} \right\rfloor$.

Hence,

$$\sigma(m) \leq m + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} k \leq m + \frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{m}{2}+1\right) = \frac{1}{8}(m^2 + 10m). \tag{2}$$

Now, if n is a solution to the given problem, then by (1), (2) and the fact that $\frac{1}{8}(m^2 + 10m)$ is an increasing function, we have

$$\begin{aligned} n = \sigma(\tau(n)) &\leq \max_k \left\{\sigma(k) | 1 \leq k \leq 2\sqrt{n}\right\} \\ &\leq \frac{1}{8}\left((2\sqrt{n})^2 + 10(2\sqrt{2})\right) = \frac{1}{2}(n + 5\sqrt{n}), \end{aligned}$$

from which we easily deduce that $n \leq 25$.

Direct checking for $n = 1, 2, \ldots, 25$ then reveal that only 1, 3, 4 and 12 satisfy the given condition and our proof is complete.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; HENRY RICARDO, Tappan, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3566. [2010 : 396, 398] *Proposed by an unknown proposer.*

Given points A and C on a circle with centre O, choose B on the shorter arc AC. Let ℓ be the line tangent to the circle at B, and let P and Q be the points where ℓ intersects the bisectors of $\angle AOB$ and $\angle BOC$, respectively. Prove that if $E = AC \cap OQ$, then PE is perpendicular to OQ.

Similar solutions by Václav Konečný, Big Rapids, MI, USA; and by Kee-Wai Lau, Hong Kong, China.

We assume that the circle has unit radius, and set $\alpha = \angle AOP = \angle POB$ and $\gamma = \angle BOQ = \angle QOC$. In triangle OEC we have $\angle EOC = \angle QOC = \gamma$ and

$$\angle OCE = \angle OCA = \angle OAC = \frac{180^\circ - 2(\alpha+\gamma)}{2} = 90^\circ - (\alpha+\gamma),$$

whence $\angle OEC = 90^\circ + \alpha$. By the sine law,

$$OE = \frac{\cos(\alpha+\gamma)}{\cos\alpha}.$$

Moreover, from the right triangle OPB,

$$OP = \frac{1}{\cos\alpha};$$

thus $\frac{OE}{OP} = \cos(\alpha+\gamma)$. But in triangle POE, $\angle POE = \alpha+\gamma$; therefore ΔPOE is a right triangle with hypotenuse OP and right angle at E. That is, PE is perpendicular to OQ, as desired.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece(2 solutions); CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; DAG JONSSON, Uppsala, Sweden; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALBERT STADLER, Herrliberg, Switzerland; and PETER Y. WOO, Biola University, La Mirada, CA, USA(2 solutions).

All the submitted solutions were short and neat. Other nice methods came down to showing that PE is an altitude of triangle OPQ, or that the quadrilateral $OAPE$ is cyclic with a right angle at A.

3567. [2010 : 396, 398] *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Prove that

$$\int_0^\infty \frac{e^{-x}\left(1-e^{-2x}\right)\left(1-e^{-4x}\right)\left(1-e^{-6x}\right)}{x(1-e^{-14x})}\,dx \;=\; \ln 2\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let I denote the given integral.

With the substitution $u = e^{-x}$, we easily obtain

$$I = -\int_0^1 \frac{(1-u^2)(1-u^4)(1-u^6)}{(1-u^{14})\ln u}du.$$

We apply the following known result [1]:

$$\int_0^1 \frac{(1-u^p)(1-u^q)(1-u^r)u^{s-1}}{(1-u^t)\ln u}du = \ln\left\{\frac{\Gamma\left(\frac{p+s}{t}\right)\Gamma\left(\frac{q+s}{t}\right)\Gamma\left(\frac{r+s}{t}\right)\Gamma\left(\frac{p+q+r+s}{t}\right)}{\Gamma\left(\frac{p+q+s}{t}\right)\Gamma\left(\frac{q+r+s}{t}\right)\Gamma\left(\frac{p+r+s}{t}\right)\Gamma\left(\frac{s}{t}\right)}\right\},$$

where Γ denotes the gamma function and p, q, r, s, t are all positive. With $p = 2$, $q = 4$, $r = 6$, $s = 1$, and $t = 14$ we then obtain

$$I = \ln\left\{\frac{\Gamma\left(\frac{1}{14}\right)\Gamma\left(\frac{9}{14}\right)\Gamma\left(\frac{11}{14}\right)}{\Gamma\left(\frac{3}{14}\right)\Gamma\left(\frac{5}{14}\right)\Gamma\left(\frac{13}{14}\right)}\right\}.$$

Since it is known [2] that

$$\frac{\Gamma\left(\frac{1}{14}\right)\Gamma\left(\frac{9}{14}\right)\Gamma\left(\frac{11}{14}\right)}{\Gamma\left(\frac{3}{14}\right)\Gamma\left(\frac{5}{14}\right)\Gamma\left(\frac{13}{14}\right)} = 2,$$

the result follows.

Also solved by MOHAMMED AASSILA, Strasbourg, France; MICHEL BATAILLE, Rouen, France; and the proposer.

References

[1] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, 5^{th} edition, 1996; formula #32.

[2] The American Mathematical Monthly, Problem 11426: *Gamma Products*, v. 116(2009), p.365; solution in v. 117(2010), p. 842.

3569★. [2010 : 397, 399] *Proposed by Jian Liu, East China Jiaotang University, Nanchang City, China.*

Let the point P lie inside the triangle ABC and let the point Q lie outside the triangle. Let w_1, w_2, w_3 denote the lengths of the angle bisectors of $\angle BPC$, $\angle CPA$, $\angle APB$, respectively. Does the inequality

$$PA\cdot QA + PB\cdot QB + PC\cdot QC \;\geq\; 4(w_1w_2 + w_2w_3 + w_3w_1)$$

hold? [*At* `http://www.emis.de/journals/JIPAM/article1162.html?sid=1162` *the proposer's inequality is proved when Q lies* inside *the triangle.*]

Comment by Oliver Geupel, Brühl, NRW, Germany.

The cited article does not contain the promised inequality. Its main result is the weaker inequality

$$PA \cdot QA + PB \cdot QB + PC \cdot QC \geq 4(r_1r_2 + r_2r_3 + r_3r_1), \qquad (1)$$

where r_i denotes the distances from P to the sides of the triangle, and both P and Q are interior points. We prove the following

Lemma. For each point Q outside the triangle there exists a point Q' on the boundary of the triangle such that $QA > QA',\ QB > QB'$, and $QC > QC'$.

Proof. Drawing rays from the vertices of the triangle orthogonal to the sides partitions the plane outside the triangle into six regions: Three regions S_A, S_B, S_C outwardly on the sides and three regions T_A, T_B, T_C outwardly on the vertices. If Q lies in an "S" region, define Q' to be the orthogonal projection of Q onto the adjacent side of the triangle. If Q lies in an "T" region define Q' to be the adjacent vertex.

Editor's comment. Geupel's lemma implies that inequality (1) holds for all points Q in the plane. The status of the required result for angle bisectors, however, remains in doubt until somebody produces either a correct reference or a valid proof. Of course, Geupel's lemma shows that if the desired inequality holds when Q is an interior point, then it holds for arbitrary Q.

No solutions were received.

3570. [2010 : 397, 399] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let r, r_a, r_b, r_c, and R be, respectively, the inradius, the exradii, and the circumradius of triangle ABC with side lengths a, b, c. Prove that

$$\frac{r_a^2}{a^2+r_a^2} + \frac{r_b^2}{b^2+r_b^2} + \frac{r_c^2}{c^2+r_c^2} \;\geq\; \frac{4R+r}{4R-r}\,.$$

Solution by Kee-Wai Lau, Hong Kong, China.

Applying the Cauchy-Schwarz inequality to vectors

$$\left(\frac{r_a}{\sqrt{a^2+r_a^2}}, \frac{r_b}{\sqrt{b^2+r_b^2}}, \frac{r_c}{\sqrt{c^2+r_c^2}}\right) \text{ and } \left(\sqrt{a^2+r_a^2}, \sqrt{b^2+r_b^2}, \sqrt{c^2+r_c^2}\right)\,,$$

we have

$$\frac{r_a^2}{a^2+r_a^2} + \frac{r_b^2}{b^2+r_b^2} + \frac{r_c^2}{c^2+r_c^2} \geq \frac{(r_a+r_b+r_c)^2}{\left(a^2+r_a^2\right)+\left(b^2+r_b^2\right)+\left(c^2+r_c^2\right)}\,. \qquad (1)$$

Let s be the semiperimeter. We need the following known results.

$$\sum_{\text{cyclic}} r_a = 4R + r, \tag{2}$$

$$\sum_{\text{cyclic}} r_a^2 = (4R + r)^2 - 2s^2, \tag{3}$$

$$\sum_{\text{cyclic}} a^2 = 2\left(s^2 - r^2 - 4Rr\right). \tag{4}$$

Formulas (2) and (3) appear on p.61 (items 99, 103) and formula (4) on p.52 (item 5) [1]. With these formulas, we can get

$$\begin{aligned}\frac{(r_a + r_b + r_c)^2}{(a^2 + r_a^2) + (b^2 + r_b^2) + (c^2 + r_c^2)} &= \frac{(4R + r)^2}{(4R + r)^2 - 2s^2 + 2\left(s^2 - r^2 - 4Rr\right)} \\ &= \frac{(4R + r)^2}{(4R + r)(4R - r)} = \frac{4R + r}{4R - r},\end{aligned}$$

and this with (1) completes the solution.

Also solved by JOE HOWARD, Portales, NM, USA; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

Geupel and the proposer note that the identity $a^2 + b^2 + c^2 + r_a^2 + r_b^2 + r_c^2 = 16R^2 - r^2$, *used in the last step of the featured solution, is interesting on its own.*

References

[1] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

3571. [2010 : 397, 399] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $n \geq 1$ be an integer. Among all increasing arithmetic progressions x_1, x_2, $\dots$, x_n such that $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, find the progression with the greatest common difference d.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Since the case $n = 1$ is degenerate, let us assume $n > 1$. An arithmetic progression $x_1, x_2, \dots, x_n$ with common difference $d > 0$ has the property that $1 = x_1^2 + x_2^2 + \cdots + x_n^2$ if and only if

$$1 = \sum_{k=0}^{n-1} (x_1 + kd)^2 = nx_1^2 + 2d\left(\sum_{k=0}^{n-1} k\right) x_1 + d^2 \left(\sum_{k=0}^{n-1} k^2\right).$$

Equivalently, x_1 is a real root of the following quadratic in x

$$1 = nx^2 + (n-1)nd \cdot x + \frac{1}{6}(n-1)n(2n-1)d^2,$$

which will happen if and only if $n > 1$ and $d > 0$ are such that its discriminant

$$(n-1)^2 n^2 d^2 - \frac{2}{3}(n-1)n^2(2n-1)d^2 + 4n$$

is non-negative and this is equivalent to saying $(n-1)n(n+1)d^2 \leq 12$. Consequently the greatest common difference is

$$d = \sqrt{\frac{12}{(n-1)n(n+1)}}\,,$$

and the first term of the progression is the solution of the quadratic above with this value of d, namely

$$x_1 = -\sqrt{\frac{3(n-1)}{n(n+1)}}.$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One incomplete solution was received.

3572. [2010 : 397, 399] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\left(\sum_{\text{cyclic}} \frac{ab}{c+ab}\right) + \frac{1}{4}\prod_{\text{cyclic}}\left(\frac{a+\sqrt{ab}}{a+b}\right) \geq 1\,.$$

Composite of similar solutions by Arkady Alt, San Jose, CA, USA; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and Albert Stadler, Herrliberg, Switzerland.

Note first that

$$\sum_{\text{cyclic}} \frac{ab}{c+ab} = \sum_{\text{cyclic}} \frac{ab}{c(a+b+c)+ab} = \sum_{\text{cyclic}} \frac{ab}{(c+a)(c+b)}$$
$$= \frac{1}{(a+b)(b+c)(c+a)} \sum_{\text{cyclic}} ab(a+b)\,.$$

Hence the given inequality is equivalent to

$$4\sum_{\text{cyclic}} ab(a+b) + \prod_{\text{cyclic}} (a+\sqrt{ab}) \geq 4(a+b)(b+c)(c+a),$$

or

$$\prod_{\text{cyclic}} (a+\sqrt{ab}) \geq 8abc\,.$$

By the AM-GM Inequality, we have

$$\prod_{\text{cyclic}}(a+\sqrt{ab}) = \sqrt{abc}\prod_{\text{cyclic}}(\sqrt{a}+\sqrt{b})$$
$$\geq 8\sqrt{abc}\sqrt{\sqrt{a}\sqrt{b}\sqrt{b}\sqrt{c}\sqrt{c}\sqrt{a}} = 8abc,$$

so our proof is complete. Clearly, equality holds if and only if $a = b = c = \frac{1}{3}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.

3574. [2010 : 398, 400, 548, 550] *Proposed by Michel Bataille, Rouen, France.*

Let x, y, and z be real numbers such that $x+y+z=0$. Prove that

$$\sum_{\text{cyclic}}\cosh x \;\leq\; \sum_{\text{cyclic}}\cosh^2\left(\frac{x-y}{2}\right) \;\leq\; 1+2\sum_{\text{cyclic}}\cosh x\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

Let $a := e^x, b = e^y, c = e^z$. Then $a,b,c > 0$ and $abc = e^{x+y+z} = 1$. Let $s := a+b+c, p := ab+ac+bc$. Then

$$\sum_{cyc}\cosh(x) = \frac{1}{2}\sum_{cyc}(a+bc) = \frac{s+p}{2}\,.$$

Let's observe that

$$\begin{aligned}\cosh\left(\frac{x-y}{2}\right) &= \frac{e^{\frac{x-y}{2}}+e^{\frac{y-x}{2}}}{2} = \frac{1}{2}\left(\frac{\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{b}}{\sqrt{a}}\right) \\ &= \frac{a+b}{2\sqrt{ab}} = \frac{(a+b)\sqrt{c}}{2}\,.\end{aligned} \tag{1}$$

Thus

$$\sum_{cyc}\cosh^2\left(\frac{x-y}{2}\right) = \frac{1}{4}\sum_{cyc}(a+b)^2c = \frac{1}{4}\sum_{cyc}a^2c+b^2c+2 = \frac{3+sp}{4}\,.$$

Also

$$\begin{aligned}\prod_{cyc}\cosh(x) &= \prod_{cyc}\frac{a+bc}{2} = \prod_{cyc}\frac{a^2+1}{2a} \\ &= \frac{1}{8}\prod_{cyc}(a^2+1) = \frac{2+p^2+s^2-2p-2s}{8}\,.\end{aligned}$$

Thus, the inequality to prove becomes

$$\frac{1}{2}(s+p) \leq \frac{3+sp}{4} \leq 1 + \frac{2+p^2+s^2-2p-2s}{4}, \qquad (2)$$

or equivalently

$$2(s+p) \leq 3+sp \leq 6+p^2+s^2-2(p+s). \qquad (3)$$

Observing that $p \geq 3\sqrt[3]{a^2b^2c^2} = 3$ and $s \geq 3\sqrt[3]{abc} = 3$ we obtain

$$sp+3-2(s+p) = (s-3)(p-3)+(s-3)+(p-3) \geq 0,$$

which proves the left hand side of (3).

To prove the RHS of (3) we note that

$$\begin{aligned} 6+&p^2+s^2-2(p+s)-(3+sp) \\ &= 3+p^2+s^2-2p-2s-sp = (p-s)^2+sp+3-2(s+p) \qquad (4) \\ &= (p-s)^2+(s-3)(p-3)+(s-3)+(p-3) \geq 0. \end{aligned}$$

This proves the RHS of (3), and thus completes the proof.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

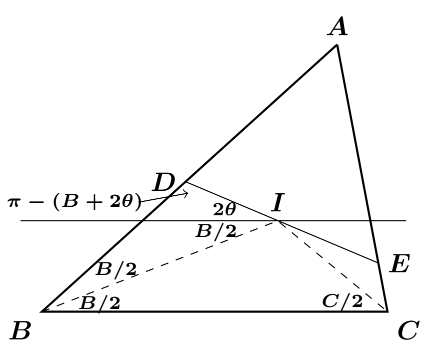
3575. [2010 : 398, 400] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle with incentre I. Characterize the lines through I intersecting the sides AB and AC at D and E, respectively, such that $DE = DB + EC$ and determine how many such lines there are in terms of $\angle B$ and $\angle C$.

Solution by Joel Schlosberg, Bayside, NY, USA.

For each line DE through I intersecting sides AB and AC, $\angle BID$ and $\angle CIE$ are external to ΔBIC; moreover, because $\angle BID + \angle CIE = 180° - \angle BIC = \frac{B+C}{2}$, each position of DE corresponds to a unique angle $\theta \in [-\frac{B}{4}, \frac{C}{4}] \subset [-45°, 45°]$ such that

$$\angle BID = \frac{B}{2} + 2\theta \quad \text{and} \quad \angle CIE = \frac{C}{2} - 2\theta.$$



Because $DE = DI + IE$, the requirement that $DE = DB + EC$ is equivalent to

$$\left(\frac{DI}{BI} - \frac{DB}{BI}\right)\frac{BI}{CI} = \frac{EC}{CI} - \frac{IE}{CI}. \tag{1}$$

By the Law of Sines applied to triangles DBI, BCI, and CEI, equation (1) is equivalent in turn to

$$\begin{aligned}
\frac{\sin\frac{B}{2} - \sin\left(\frac{B}{2}+2\theta\right)}{\sin(B+2\theta)} \cdot \frac{\sin\frac{C}{2}}{\sin\frac{B}{2}} &= \frac{\sin\left(\frac{C}{2}-2\theta\right) - \sin\frac{C}{2}}{\sin(C-2\theta)} \\
\frac{2\sin(-\theta)\cos\left(\frac{B}{2}+\theta\right)\sin\frac{C}{2}}{2\sin\left(\frac{B}{2}+\theta\right)\cos\left(\frac{B}{2}+\theta\right)\sin\frac{B}{2}} &= \frac{2\sin(-\theta)\cos\left(\frac{C}{2}-\theta\right)}{2\sin\left(\frac{C}{2}-\theta\right)\cos\left(\frac{C}{2}-\theta\right)}\,.
\end{aligned}$$

Clearly $\theta = 0$ satisfies the condition; if $\theta \neq 0$ then $\sin(-\theta)$ is nonzero and can be canceled, yielding

$$\begin{aligned}
\sin\left(\frac{B}{2}+\theta\right)\sin\frac{B}{2} &= \sin\left(\frac{C}{2}-\theta\right)\sin\frac{C}{2} \\
\sin^2\frac{B}{2}\cos\theta + \sin\frac{B}{2}\cos\frac{B}{2}\sin\theta &= \sin^2\frac{C}{2}\cos\theta - \sin\frac{C}{2}\cos\frac{C}{2}\sin\theta,
\end{aligned}$$

whence,

$$\begin{aligned}
\tan\theta &= \frac{\sin^2\frac{C}{2} - \sin^2\frac{B}{2}}{\sin\frac{C}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\cos\frac{B}{2}} \\
&= \frac{-\left(1-2\sin^2\frac{C}{2}\right) + \left(1-2\sin^2\frac{B}{2}\right)}{2\sin\frac{C}{2}\cos\frac{C}{2} + 2\sin\frac{B}{2}\cos\frac{B}{2}} = \frac{-\cos C + \cos B}{\sin C + \sin B} \\
&= \frac{2\sin\frac{C+B}{2}\sin\frac{C-B}{2}}{2\sin\frac{C+B}{2}\cos\frac{C-B}{2}} = \tan\frac{C-B}{2}\,.
\end{aligned} \tag{2}$$

Since the tangent function is strictly increasing and bijective in the domain $(-90°,\ 90°)$, which contains both the interval $\left[-\frac{B}{4},\ \frac{C}{4}\right]$ and the angle $\frac{C-B}{2}$, equation (2) is equivalent to $\theta = \frac{C-B}{2}$ for $\theta \neq 0$ and $\theta \in \left[-\frac{B}{4},\ \frac{C}{4}\right]$. These conditions imply that

$$\angle B \neq \angle C, \quad \frac{\angle B}{\angle C} \in \left[\frac{1}{2}, 2\right], \quad \text{and} \quad \angle BID = \frac{B}{2} + 2\frac{C-B}{2} = C - \frac{B}{2}.$$

Our conclusion:

- If $\angle B = \angle C$ or $\frac{\angle B}{\angle C} \notin \left[\frac{1}{2}, 2\right]$, then there is a single line with the desired property; it satisfies $\angle BID = \frac{B}{2}$ and is therefore the parallel to BC through I.

- If $\angle B \neq \angle C$ and $\frac{\angle B}{\angle C} \in \left[\frac{1}{2}, 2\right]$, then there are two lines with the desired property; they satisfy $\angle BID = \frac{B}{2}$ and $\angle BID = C - \frac{B}{2}$.

Also solved by the proposer.

We also received one incorrect and one incomplete solution. The incomplete solution provided an appealing intuitive approach, as follows: One obvious line that satisfies the required equation is the line DE through I that is parallel to BC: in this case, because BI bisects $\angle B$, $\angle DIB = \angle CBI = \angle IBD$, whence $DB = DI$; similarly $EC = IE$, so that $DE = DB + EC$, as desired. Another solution is obtained by reflecting this DE (namely the parallel to BC through I) in the angle bisector AI to obtain $E'D'$, with E' the image of D on AC and D' the image of E on AB. Here

$$\begin{aligned} D'B + E'C &= AB + AC - (AD' + AE') = AB + AC - (AE + AD) \\ &= DB + EC = DE = D'E'. \end{aligned}$$

This second solution will not exist, however, if either D' or E' fall outside ΔABC. E' falls outside when C lies between E' and A; that is, if $\angle AE'I < \angle ACI$ or, in terms of $\angle B = \angle ADE = \angle AE'D' = \angle AE'I$, if $\angle B < \frac{\angle C}{2}$. Similarly D' falls outside if $\angle C < \frac{\angle B}{2}$. The two lines coincide when $\angle B = \angle C$. In all other cases there will be at least two solutions. It remains to show that there cannot be more than two solutions. For this our correspondent makes claims about the monotonicity of the lengths of the relevant segments, first as the point E moves along AC from its position where $EI||BC$ to the foot of the angle bisector BI, and then as it moves along AC to the vertex C; he claimed that these claims are "easy to prove." I suppose it should be comforting to know that the proof is easy; however, this editor has seen many very easy proofs that he was not clever enough to devise for himself, many of which were not only easy, but correct. Please, if the proof is easy but not routine, then supply the proof.