

SKOLIAD No. 135

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Please send your solutions to problems in this Skoliad by **August 15, 2012**. A copy of *CRUX with Mayhem* will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

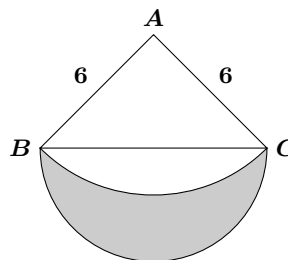
Our contest this month is the Calgary Mathematical Association 35th Junior High School Mathematics Contest, Part B, 2011. Our thanks go to Robert Woodrow and Bill Sands, both of University of Calgary, Alberta, for providing us with this contest and for permission to publish it.

L'Association mathématique de Calgary 35^e Compétition Junior de Mathématique Ronde finale, partie B, 2011.

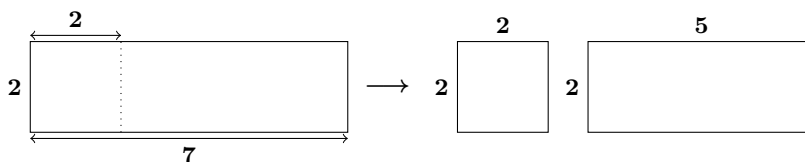
1. Ariel a acheté une certaine quantité d'abricots. **90%** du poids d'un abricot est constitué d'eau. Elle sèche les abricots jusqu'à ce que **60%** du poids d'un abricot soit constitué d'eau. **15 kg** se sont ainsi évaporés. Quel était le poids initial des abricots (en kg) ?

2. Un groupe de dix amis vont ensemble au cinéma. Un autre groupe de neuf amis vont aussi au même cinéma. Quatorze des ces **19** personnes ont acheté chacune en plus une confection régulière de popcorn. Au total il s'est avéré que le coût combiné du ticket de cinéma plus les popcorn était le même pour chacun des deux groupes. Le prix du ticket de cinéma est **6\$**. Trouver tous les prix possibles d'une confection régulière de popcorn.

3. Dans la figure, $|AB| = 6$, $|AC| = 6$, et $\angle BAC$ est un angle droit. On tire deux arcs de cercle passant par B et C : un arc est centré en A et l'autre est un demi-cercle de diamètre BC . Quelle est l'aire du triangle $\triangle ABC$? Quelle est la longueur de BC ? Calculer l'aire comprise entre les deux arcs de cercle, c'est-à-dire l'aire de la partie hachurée de la figure.



4. Étant donné un rectangle différent d'un carré, un *découpage carré* consiste à découper le rectangle en deux parties, dont une est un carré (l'autre étant un rectangle possiblement égal à un carré). Par exemple, un découpage carré d'un rectangle 2×7 donne un carré 2×2 et un rectangle 2×5 , comme montré.



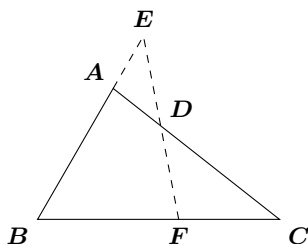
On part d'un rectangle 40×2011 . À chaque étape, on effectue un découpage carré de la partie rectangulaire qui n'est pas carrée. On continue jusqu'à ce que toutes les parties soient des carrés. Combien de carrés y a-t-il au total?

5. Cinq équipes, A , B , C , D , et E , participent à un tournoi de hockey où chaque équipe joue contre chacune autre exactement une fois. Tout match résulte en une victoire pour une équipe et une défaite pour l'autre ou bien un match nul. Le tableau

équipe	victoires	défaites	nuls
A	3		
B	1		1
C	1		
D	0		
E			4

comportait à l'origine tous les résultats du tournoi, mais quelques cases ont été effacées. En dépit de l'information manquante, l'issue de chaque match peut être déterminée de façon unique. Pour chacun des dix jeux, de déterminer qui a gagné ou si elle était un match nul.

6. Soit ABC un triangle dont les longueurs des côtés sont données par $|AB| = 5$, $|AC| = 7$, et $|BC| = 8$. Le point D appartenant au côté AC est tel que $|AB| = |CD|$. On prolonge le côté BA au-delà de A jusqu'à un point E tel que $|AC| = |BE|$. Appelons F le point d'intersection de la droite ED et du côté BC .



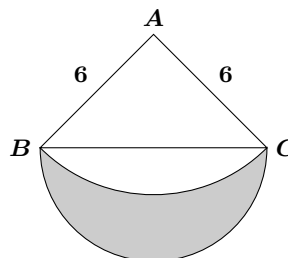
Trouver les longueurs de AD , AE , BF , et FC .

Calgary Mathematical Association
35th Junior High School Mathematics Contest
Final Round, Part B, 2011.

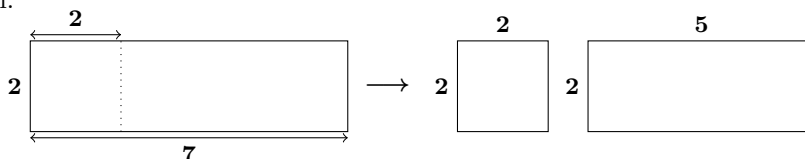
1. Ariel purchased a certain amount of apricots. 90% of the apricot weight was water. She dried the apricots until just 60% of the apricot weight was water. 15 kg of water was lost in the process. What was the original weight of the apricots (in kg)?

2. A group of ten friends all went to a movie together. Another group of nine friends also went to the same movie together. Fourteen of these nineteen people each bought a regular bag of popcorn as well. It turned out that the total cost of the movie plus popcorn for one of the two groups was the same as for the other group. A movie ticket costs \$6. Find all possible costs of a regular bag of popcorn.

3. In the diagram, $|AB| = 6$, $|AC| = 6$, and $\angle BAC$ is a right angle. Two arcs are drawn: a circular arc with centre A and passing through B and C , and a semi-circle with diameter BC . What is the area of $\triangle ABC$? What is the length of BC ? Find the area between the two arcs; that is, find the area of the shaded region in the diagram.



4. Given a non-square rectangle, a *square-cut* is a cutting-up of the rectangle into two pieces, a square and a rectangle (which may or may not be a square). For example, performing a square-cut on a 2×7 rectangle yields a 2×2 square and a 2×5 rectangle, as shown.

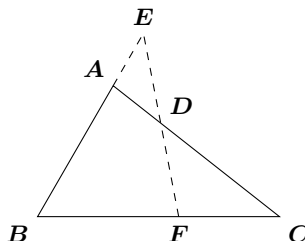


You are initially given a 40×2011 rectangle. At each stage, you make a square-cut on the non-square piece. You repeat this until all pieces are squares. How many square pieces are there at the end?

5. Five teams, A , B , C , D , and E , participate in a hockey tournament where each team plays against each other team exactly once. Each game either ends in a win for one team and a loss for the other, or ends in a tie for both teams. The table originally showed all of the results of the tournament, but some of the entries in the table have been erased. The result of each game played can be uniquely determined. For each of the ten games, determine who won or if it was a tie.

Team	Wins	Losses	Ties
A	3		
B	1		1
C	1		
D	0		
E			4

6. A triangle ABC has sides $|AB| = 5$, $|AC| = 7$, and $|BC| = 8$. Point D is on side AC such that $|AB| = |CD|$. We extend the side BA past A to a point E such that $|AC| = |BE|$. Let the line ED intersect side BC at a point F .



Find the lengths of AD , AE , BF , and FC .

Next follow solutions to the City Competition of the Croatian Mathematical Society, 2010, secondary level, grade 1, given in Skoliad 129 at [2010:481–483].

1. Let n be a positive integer and a a non-zero real number. Reduce the fraction

$$\frac{a^{3n+1} - a^4}{a^{2n+3} + a^{n+4} + a^5}.$$

Solution by Harris Lin, student, Killarney Secondary School, Vancouver, BC.

If $n = 1$, the expression equals $\frac{a^4 - a^4}{a^5 + a^5 + a^5} = 0$.

If $n = 2$, the expression equals

$$\frac{a^7 - a^4}{a^7 + a^6 + a^5} = \frac{a^4(a^3 - 1)}{a^5(a^2 + a + 1)},$$

but $a^3 - 1^3 = (a - 1)(a^2 + a + 1)$, so the expression equals

$$\frac{a^4(a - 1)(a^2 + a + 1)}{a^5(a^2 + a + 1)} = \frac{a - 1}{a}.$$

If $n = 3$, the expression equals

$$\frac{a^{10} - a^4}{a^9 + a^7 + a^5} = \frac{a^4(a^6 - 1)}{a^5(a^4 + a^2 + 1)},$$

but, again,

$$a^6 - 1 = (a^2)^3 - 1^3 = (a^2 - 1)((a^2)^2 + (a^2) + 1) = (a^2 - 1)(a^4 + a^2 + 1),$$

so the expression equals

$$\frac{a^4(a^2 - 1)(a^4 + a^2 + 1)}{a^5(a^4 + a^2 + 1)} = \frac{a^2 - 1}{a}.$$

Now hazard the guess that

$$\frac{a^{3n+1} - a^4}{a^{2n+3} + a^{n+4} + a^5} = \frac{a^{n-1} - 1}{a}.$$

The equation holds if and only if

$$(a^{3n+1} - a^4)a = (a^{n-1} - 1)(a^{2n+3} + a^{n+4} + a^5),$$

so

$$\begin{aligned} a^{3n+2} - a^5 &= a^{n-1}a^{2n+3} + a^{n-1}a^{n+4} + a^{n-1}a^5 - a^{2n+3} - a^{n+4} - a^5 \\ &= a^{3n+2} + a^{2n+3} + a^{n+4} - a^{2n+3} - a^{n+4} - a^5 \\ &= a^{3n+2} - a^5, \end{aligned}$$

but this is obviously true, so the guess is correct.

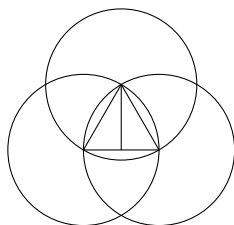
2. Find a positive integer which when multiplied by **9** gives an integer between **1100** and **1200**, and when multiplied by **13** gives an integer between **1500** and **1600**.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Since $122 \cdot 9 = 1098 < 1100$ and $124 \cdot 13 = 1612 > 1600$, the only integer that could satisfy the conditions is **123**. Now, $123 \cdot 9 = 1107$ and $123 \cdot 13 = 1599$, so **123** does indeed satisfy the conditions.

3. Three circles, each with radius **2**, are given in the plane such that the centre of each lies on the intersection of the other two. Determine the area of the intersection of the three disks bounded by those circles.

Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.



The three centres form an equilateral triangle with side length **2**. The intersection of the three disks consists of this triangle and three congruent segments. Using the Pythagorean Theorem, the triangle has height $\sqrt{2^2 - 1^2} = \sqrt{3}$. Thus the area of the triangle is $\frac{2\sqrt{3}}{2} = \sqrt{3}$.

Each of the segments are a part of a sector in a circle of radius **2**, \triangle . Since the triangle is equilateral, the sector angle is 60° , so the area of the sector is $\frac{60}{360}\pi 2^2 = \frac{2}{3}\pi$. The triangle has area $\sqrt{3}$, so each segment has area $\frac{2}{3}\pi - \sqrt{3}$.

Therefore the area of the intersection is $\sqrt{3} + 3(\frac{2}{3}\pi - \sqrt{3}) = 2\pi - 2\sqrt{3}$.

4. Consider the integer **n**. Let **m** be the integer obtained from **n** by removing its ones digit. If $n - m = 2010$, find **n**.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Introduce letters for the digits of **n** and **m** so the problem has the form

$$\begin{array}{r} abcd \\ - abc \\ \hline 2010 \end{array} .$$

Since **abc** is at most **999**, the digit **a** must be **2** or **3**. However, if $a = 3$, then $n - m > 3000 - 399 = 2601 > 2010$, so $a = 2$. Now the problem has the form

$$\begin{array}{r} 2bcd \\ - 2bc \\ \hline 2010 . \end{array}$$

Therefore, b must be **2** or **3**. If $b = 3$, then $n - m > 2300 - 239 = 2061 > 2010$, so $b = 2$, and the problem has the form

$$\begin{array}{r} 22cd \\ - 22c \\ \hline 2010 . \end{array}$$

Again, c must be **3** or **4**, but if $c = 4$, then $n - m > 2240 - 224 = 2016 > 2010$, so $c = 3$, and the problem has the form

$$\begin{array}{r} 223d \\ - 223 \\ \hline 2010 . \end{array}$$

Now d must be **3**. Thus $n = 2233$.

Alternatively, let d denote the ones digit of n . Then $n = 10m + d$, so $2010 = n - m = 9m + d$. Since $2010 \div 9 \approx 223.3$, $m = 223$. Finally, $d = 2010 - 9m = 3$, and $n = 2233$ as above.

5. A bag contains a sufficient number of red, white, and blue balls. Each child in a given group takes three balls at random from the bag. What is the smallest number of children in the group that ensures that two of them have taken the same combination of balls, that is, the same number of balls of each colour?

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

The possible colour combinations are *RRR*, *WWW*, *BBB*, *RRW*, *RWW*, *RRB*, *BBR*, *WWB*, *WBB*, *RWB*. Since there are **10** colour combinations, **10** children could have all different combinations, but with **11** children, at least two would have the same combination. Thus the answer is **11** children.

6. If $a^2 + 2b^2 = 3c^2$, prove that

$$\left(\frac{a+b}{b+c} + \frac{b-c}{b-a} \right) \cdot \frac{a+2b+3c}{a+c}$$

is a positive integer.

Solution by Harris Lin, student, Killarney Secondary School, Vancouver, BC.

Note that

$$\frac{a+b}{b+c} \cdot \frac{a+2b+3c}{a+c} = \frac{a^2 + 2ab + 3ac + ab + 2b^2 + 3bc}{ab + bc + ac + c^2}.$$

Since $a^2 + 2b^2 = 3c^2$, this equals

$$\frac{3ab + 3ac + 3bc + 3c^2}{ab + bc + ac + c^2} = \frac{3(ab + ac + bc + c^2)}{ab + bc + ac + c^2} = 3.$$

Similarly,

$$\frac{b-c}{b-a} \cdot \frac{a+2b+3c}{a+c} = \frac{ab+2b^2+3bc-ac-2bc-3c^2}{ab+bc-a^2-ac}.$$

Since $a^2 = 3c^2 - 2b^2$, this equals

$$\frac{ab+bc-ac-a^2}{ab+bc-a^2-ac} = 1.$$

Thus

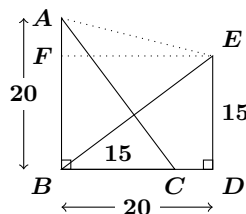
$$\left(\frac{a+b}{b+c} + \frac{b-c}{b-a}\right) \cdot \frac{a+2b+3c}{a+c} = 3 + 1 = 4.$$

7. A right triangle, $\triangle ABC$, with legs of lengths **15** and **20** and the right angle at vertex B is congruent to a triangle, $\triangle BDE$, with the right angle at vertex D . The point C lies strictly inside the segment \overline{BD} , and the points A and E are on the same side of the straight line BD .

- Find the distance between points A and E .
- Find the area of the intersection of $\triangle ABC$ and $\triangle BDE$.

Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.

- The triangles must sit as in the diagram on the left. Since $\angle ABC = \angle BDE = 90^\circ$, there exists a point F on AB such that $BDEF$ is a rectangle. Now $|AF| = 5$ and $|EF| = 20$, so the Pythagorean Theorem yields that $|AE|^2 = 20^2 + 5^2 = 425$. Thus $|AE| = \sqrt{425} = 5\sqrt{17}$.



- Now impose a coordinate system such that $A = (0, 20)$, $B = (0, 0)$, $C = (15, 0)$, $D = (20, 0)$, and $E = (20, 15)$. Then the line through B and E has the equation $y = \frac{15}{20}x = \frac{3}{4}x$. Moreover, the line through A and C has the equation $y = \frac{-20}{15}x + 20 = -\frac{4}{3}x + 20$. These two lines intersect when $\frac{3}{4}x = -\frac{4}{3}x + 20$, thus $\frac{9+16}{12}x = 20 \Rightarrow \frac{25}{12}x = 20$, so $x = \frac{48}{5}$, and $y = \frac{3}{4}x = \frac{36}{5}$.

Thus the intersection between $\triangle ABC$ and $\triangle BDE$ is itself a triangle with height $\frac{36}{5}$ and base **15**. Therefore the desired area is $\frac{1}{2} \cdot 15 \cdot \frac{36}{5} = 54$.

8. Let p and q be different odd prime numbers. Prove that the integer $(pq + 1)^4 - 1$ has at least four different prime divisors.

Solution by the editors.

$$\begin{aligned} \text{Since } a^2 - b^2 &= (a - b)(a + b), \\ (pq + 1)^4 - 1 &= ((pq + 1)^2 - 1)((pq + 1)^2 + 1) \\ &= (pq + 1 - 1)(pq + 1 + 1)((pq)^2 + 2pq + 1 + 1) \\ &= pq(pq + 2)(p^2q^2 + 2pq + 2). \end{aligned}$$

Since p and q are odd, $pq + 2$ is not divisible by either p or q . Let n denote $pq + 2$, and note that n is odd. Then $pqn + 2 = pq(pq + 2) + 2 = p^2q^2 + 2pq + 2$, so

$$(pq + 1)^4 - 1 = pqn(pqn + 2).$$

Note that $pqn + 2$ is not divisible by p or q since they are odd.

If n is not a power of a single prime, then n is divisible by at least two different primes, and $(pq + 1)^4 - 1$ is divisible by at least four different primes.

If n is a power of a prime, say $n = r^k$ where r is a prime and k is a positive integer, then r is odd because n is odd. Therefore $pqn + 2$ is not divisible by either of p , q , or r , so $pqn + 2$ contains a prime factor different from p , q , and r . Thus $(pq + 1)^4 - 1$ is divisible by at least four different primes.

*This problem was the Problem of the Month in the December 1999 issue of **CruX Mathematicorum**, [1999 : 495]. We encourage the reader to look up the solution at <http://cms.math.ca/cruX/>*

This issue's prize of one copy of **CruX Mathematicorum** for the best solutions goes to Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

We hope to receive more reader solutions to this issue's featured contest.