RECURRING CRUX CONFIGURATIONS 2

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Triangles for which $2b = c + a$

This month we explore triangles ABC whose sides a, b, c are in arithmetic progression; we shall see that with the triangle labeled so that b is the intermediate side, having the sides in arithmetic progression is equivalent to requiring $\angle BIO = 90^\circ$, as well as IG parallel to AC, and many other noteworthy properties (where I, O, and G are the incentre, circumcentre, and centroid, respectively). As in last month's column, I will supply statements, references, and occasional hints, leaving the proofs as exercises.

Problem 268 [1977 : 190; 1978 : 78-79] (Proposed by Gali Salvatore = Léo Sauvé). Show that in ΔABC with $a \geq b \geq c$, the sides are in arithmetic progression if and only if

$$2\cot\frac{B}{2} = 3\left(\tan\frac{C}{2} + \tan\frac{A}{2}\right).$$

The featured solution made use of the half-angle identities

$$\tan\frac{A}{2}\tan\frac{C}{2} = \frac{s-b}{s} \quad \text{and} \quad \sum_{cyclic}\left(\tan\frac{A}{2}\tan\frac{B}{2}\right) = 1,$$

where s is the semiperimeter $(a+b+c)/2$. One solver, Charles W. Trigg, added the comment that in triangles where $c + a = 2b$, seven other relationships "follow easily":

1. $\cos C + \cos A = 4\sin^2\frac{B}{2}$.
2. $a\cos C - c\cos A = 2(a-c)$.
3. $ca = 6Rr$ (where R and r are the circumradius and inradius).
4. $\cos A = \frac{4c-3b}{2c}$.
5. $B < 60^\circ$ except when the triangle is equilateral.
6. $GI || BC$.
7. In the special case where $A = C + 90^\circ$, then a, b, c are in the ratio $(\sqrt{7}+1) : \sqrt{7} : (\sqrt{7}-1)$.

The editor Sauvé remarked that copies of Trigg's proofs were available from him on request; sadly both he and Trigg died long ago, so today it would probably be faster for the reader to discover the proofs for himself. Number 6, however, has appeared in this journal several times, twice as a "Klamkin Quickie" [1996 : 61] and [2001 : 79]. An even quicker proof appeared as D.L. MacKay's solution to Problem E411 in [1]; namely, *Prove that if the sides of a triangle form an* *arithmetic progression the line joining the centroid to the incentre is parallel to one side*: We have $b - a = c - b$ if and only if $s = 3b/2$, and

$$r = \frac{\Delta}{s} = \frac{2\Delta}{3b} = \frac{h}{3},$$

where Δ is the area of ΔABC and h is the altitude from B to CA. Thus the incentre and centroid are equidistant from the side BC. This proof (expanded somewhat) was reproduced as Problem 82 in [5]. A refined version of the same problem had appeared a couple years earlier in [2]. For that version, recall that the Nagel point (the common intersection point of the lines joining a vertex to the point where the opposite excircle touches a side) lies on the line GI. It turns out that in a triangle whose sides are in arithmetic progression, the common difference $|a - b| = |b - c|$ equals the distance from I to the Nagel point. The *MathWorld* web page has proposed calling GI the *Nagel line*, but the name seems not to have caught on yet.

Another solver of Problem 268, Leon Bankoff, submitted two striking results (which he saw—so he said— "out of the cornea of his eye"):

i. If $c + a = 2b$, then $\cot\frac{C}{2}k + \cot\frac{A}{2} = 2\cot\frac{B}{2}$;
ii. If $c^2 + a^2 = 2b^2$, then $\cot C + \cot A = 2\cot B$.

The first follows from Problem 268, while the second is Property 3 from our previous column on root-mean-square triangles.

Problem 2870 [2003 : 399; 2004 : 382-383] (Proposed by Toshio Seimiya). Given triangle ABC with incentre I, circumcentre O, and centroid G, suppose that $\angle AIO = 90°$. Prove that $IG||BC$.

The featured solution also proved the converse (for scalene triangles); this result combined with Trigg's Property 6 (above) implies that for scalene triangles,

> $\angle BIO \le 90°$ if and only if $2b \le c + a$, with equality holding only simultaneously.

This version is Problem 1506 in [7], where there are two published solutions; it also appeared as problem 2 on the Second Hong Kong Mathematical Olympiad 1999, with a solution in **CRUX with MAYHEM** [2005 : 520-521]. Amengual Covas added three further references, namely [3], [4], [6] which, he said, "provide other relationships."

Problem 3197 [2006 : 516, 518; 2007 : 501-502] (Proposed by Paul Deiermann). If AB is a fixed line segment, find the triangle ABC which has maximum area among those which satisfy $\angle AIO = \pi/2$. What is this maximum area?

Michel Bataille's featured solution plugged $2a = b + c$ into Heron's formula for the area and found that the maximum area is achieved for the triangle with sides

$$c = 1, a = \frac{3 + \sqrt{3}}{3}, \text{ and } b = 2a - 1.$$

Finally, triangles with sides in arithmetic progression are mentioned in a footnote to a problem dealing with lines through vertex B that are perpendicular to IO.

Problem 2246 (reworded) [1997 : 244; 1998 : 318-319] (Proposed by D.J. Smeenk). Given the nonequilateral triangle ABC, suppose that the line through B that is perpendicular to OI intersects the bisector of $\angle BAC$ at P, and that the line through P parallel to AC intersects BC at M. Show that I, G, and M are collinear.

The problem breaks down, of course, should ΔABC be equilateral (because then I and O would coincide). Note that $BI \perp IO$ implies that $P = I$, whence the conclusion that G is on the line IM gives yet another proof (although convoluted) that $BI \perp IO$ implies $GI || AC$. A comment attached to the problem pointed out that for M to be defined, IG and BC could not be parallel, which thus forbids $AI \perp IO$, whence the condition $2a \neq b + c$ should have been added to the statement of the problem.

References

[1] J.H. Butchart, Problem E411, *American Mathematical Monthly*, **47**:10 (Dec. 1940) 708-709. (Appeared in 1940, page 175).

[2] Walter B. Clarke, Problem 172, *National Mathematics Magazine*, **12**:5 (Feb. 1938) 249-250. (Appeared in 1938, page 194.)

[3] F. G.-M., *Exercices de géométrie*, 7th ed., pp. 464-465.

[4] F. G.-M., *Exercices de trigonométrie*, pp. 413-414, 481.

[5] Ross Honsberger, *Mathematical Morsels.* Dolciani Mathematical Expositions No. 3, 1978, pages 209-210.

[6] I. Shariguin, *Problemas de geometría Planimetría*, editorial Mir, 1989, pp. 96-97.

[7] Wu Wei Chao, Problem 1506, *Mathematics Magazine*, **70**:4 (Oct. 1997) 302-303. (Appeared in October 1996.)