THE OLYMPIAD CORNER

No. 296

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The problems from this issue come from the Italian Team Selection Test, the British Mathematical Olympiad, the Macedonian Mathematical Olympiad, and the Chinese Mathematical Olympiad. Our thanks go to Adrian Tang for sharing the material with the editors.

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 octobre 2012**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

OC31. Trouver toutes les paires (p, q) de nombres premiers tels que $pq \mid (5^p + 5^q)$.

OC32. Soit ABC un triangle acutangle avec $\angle B = \angle C$. Soit O le centre de son cercle circonscrit et H son orthocentre. Montrer que le centre du cercle BOH se situe sur la droite AB.

OC33. Soit n et k deux entiers tels que $n \geq k \geq 1$. On considère un cercle de n ampoules électriques, toutes éteintes. A chaque tour, on peut changer le statut d'un ensemble quelconque de k ampoules consécutives. Parmi les 2^n combinaisons possibles, combien peut-on en engendrer

(a) si k est un premier impair ?

(b) si k est un entier impair ?

(c) si k est un entier pair ?

OC34. Soit m et n deux entiers avec $4 < m < n$, et $A_1A_2 \cdots A_{2n+1}$ un $2n+1$-gone régulier. Soit $P = \{A_1, A_2, \cdots, A_{2n+1}\}$. Trouver le nombre de m-gones convexes avec exactement deux angles droits internes et dont les sommets sont tous dans P.

OC35. Trouver toutes les paires d'entiers (x, y) telles que

$$y^3 = 8x^6 + 2x^3y - y^2\,.$$

OC36. Soit $\boldsymbol{a}$, $\boldsymbol{b}$ et $\boldsymbol{c}$ les longueurs des côtés opposés respectivement aux angles $\angle \boldsymbol{A}$, $\angle \boldsymbol{B}$ et $\angle \boldsymbol{C}$ d'un triangle obtusangle $\boldsymbol{ABC}$. Montrer que

$$\boldsymbol{a^3 \cos A + b^3 \cos B + c^3 \cos C < abc}\,.$$

OC37. Trouver tous les entiers $\boldsymbol{n}$ rendant possible le coloriage de toutes les arêtes et diagonales d'un $\boldsymbol{n}$-gone convexe avec $\boldsymbol{n}$ couleurs satisfaisant les conditions suivantes :

(i) Chacune des arêtes et diagonales est coloriée par une seule couleur ;

(ii) Pour tout ensemble de trois couleurs distinctes, il existe un triangle dont les sommets sont des sommets du $\boldsymbol{n}$-gone et les trois arêtes sont respectivement coloriées par les trois couleurs de cet ensemble.

OC38. Soit $\boldsymbol{a}$, $\boldsymbol{b}$ et $\boldsymbol{c}$ trois nombres réels positifs tels que $\boldsymbol{ab + bc + ca = \frac{1}{3}}$. Montrer que l'inégalité suivante est satisfaite :

$$\boldsymbol{\frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \geq \frac{1}{a + b + c}}\,.$$

OC39. Étant donné un entier positif $\boldsymbol{n}$, soit $\boldsymbol{b(n)}$ le nombre d'entiers positifs dont les représentations binaires apparaissent comme blocs d'entiers consécutifs dans la représentation binaire de $\boldsymbol{n}$. Par exemple $\boldsymbol{b(13) = 6}$, puisque $\boldsymbol{13 = 1101_2}$, qui contient comme blocs consécutifs les représentations binaires de $\boldsymbol{13 = 1101_2}$, $\boldsymbol{6 = 110_2}$, $\boldsymbol{5 = 101_2}$, $\boldsymbol{3 = 11_2}$, $\boldsymbol{2 = 10_2}$ et $\boldsymbol{1 = 1_2}$.

Montrer que si $\boldsymbol{n \leq 2500}$, alors $\boldsymbol{b(n) \leq 39}$, et déterminer les valeurs de $\boldsymbol{n}$ pour lesquelles on a égalité.

OC40. Soit $\boldsymbol{M}$ et $\boldsymbol{N}$ les intersections de deux cercles $\boldsymbol{\Gamma_1}$ et $\boldsymbol{\Gamma_2}$. Soit $\boldsymbol{AB}$ la tangente commune aux deux cercles, la plus rapprochée de $\boldsymbol{M}$, disons $\boldsymbol{A \in \Gamma_1}$ et $\boldsymbol{B \in \Gamma_2}$. Soit respectivement $\boldsymbol{C}$ et $\boldsymbol{D}$ les points symétriques de $\boldsymbol{A}$ et $\boldsymbol{B}$ par rapport à $\boldsymbol{M}$. Soit respectivement $\boldsymbol{E}$ et $\boldsymbol{F}$ les intersections du cercle circonscrit de $\boldsymbol{DCM}$ et des cercles $\boldsymbol{\Gamma_1}$ et $\boldsymbol{\Gamma_2}$.

Montrer que les rayons des cercles circonscrits des triangles $\boldsymbol{MEF}$ et $\boldsymbol{NEF}$ sont d'égale longueur.

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OC31. Find all pairs $\boldsymbol{(p, q)}$ of prime numbers such that $\boldsymbol{pq \mid (5^p + 5^q)}$.

OC32. Let $\boldsymbol{ABC}$ be an acute-angled triangle with $\boldsymbol{\angle B = \angle C}$. Let the circumcentre be $\boldsymbol{O}$ and the orthocentre be $\boldsymbol{H}$. Prove that the centre of the circle $\boldsymbol{BOH}$ lies on the line $\boldsymbol{AB}$.

OC33. Let n and k be integers such that $n \geq k \geq 1$. There are n light bulbs placed in a circle. They are all turned off. Each turn, you can change the state of any set of k consecutive light bulbs.

How many of the 2^n possible combinations can be reached

(a) if k is an odd prime?

(b) if k is an odd integer?

(c) if k is an even integer?

OC34. Let m, n be integers with $4 < m < n$, and $A_1A_2\cdots A_{2n+1}$ be a regular $2n+1$-gon. Let $P = \{A_1, A_2, \cdots, A_{2n+1}\}$. Find the number of convex m-gons with exactly two acute internal angles whose vertices are all in P.

OC35. Find all pairs of integers (x, y) such that

$$y^3 = 8x^6 + 2x^3y - y^2 .$$

OC36. The obtuse-angled triangle ABC has sides of length a, b, and c opposite the angles $\angle A$, $\angle B$ and $\angle C$ respectively. Prove that

$$a^3 \cos A + b^3 \cos B + c^3 \cos C < abc .$$

OC37. Find all integers n such that we can colour all the edges and diagonals of a convex n-gon by n given colours satisfying the following conditions:

(i) Every one of the edges or diagonals is coloured by only one colour;

(ii) For any three distinct colours, there exists a triangle whose vertices are vertices of the n-gon and the three edges are coloured by the three colours, respectively.

OC38. Let a, b, c be positive real numbers such that $ab + bc + ca = \frac{1}{3}$. Prove the inequality:

$$\frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \geq \frac{1}{a + b + c} .$$

OC39. Given a positive integer n, let $b(n)$ denote the number of positive integers whose binary representations occur as blocks of consecutive integers in the binary expansion of n. For example $b(13) = 6$ because $13 = 1101_2$, which contains as consecutive blocks the binary representations of $13 = 1101_2$, $6 = 110_2$, $5 = 101_2$, $3 = 11_2$, $2 = 10_2$ and $1 = 1_2$.

Show that if $n \leq 2500$, then $b(n) \leq 39$, and determine the values of n for which equality holds.

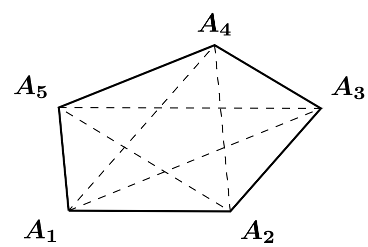
OC40. Let M and N be the intersection of two circles, Γ_1 and Γ_2. Let AB be the line tangent to both circles closer to M, say $A \in \Gamma_1$ and $B \in \Gamma_2$. Let C be the point symmetrical to A with respect to M, and D the point symmetrical to B with respect to M. Let E and F be the intersections of the circle circumscribed around DCM and the circles Γ_1 and Γ_2, respectively.

Show that the circles circumscribed around the triangles MEF and NEF have radii of the same length.

First we look at the remainder of the solutions for the 48th IMO Bulgarian Team, First Selection Test, given at [2010: 275] that we started last issue.

2. Let $A_1A_2A_3A_4A_5$ be a convex pentagon such that the triangles $A_1A_2A_3$, $A_2A_3A_4$, $A_3A_4A_5$, $A_4A_5A_1$, $A_5A_1A_2$ have the same area. Prove that there exists a point M such that the triangles A_1MA_2, A_2MA_3, A_3MA_4, A_5MA_1 have the same area.

Solved by Titu Zvonaru, Cománești, Romania, shortened by the editor.



Triangles $A_5A_1A_2$ and $A_1A_2A_3$ have the same base and area, whence $A_1A_2 || A_5A_3$. Similarly we deduce that each side of the given pentagon is parallel to a diagonal. If in a convex pentagon each side is parallel to a diagonal, then the ratio of a diagonal to the corresponding parallel side is the golden section $\varphi = \frac{1+\sqrt{5}}{2}$. For a simple proof of this result see the solution to problem M133 [2005: 278-279]. Recall that affine transformations preserve the ratios of segment lengths along parallel lines, so that the affine transformation that takes the first three vertices of the regular pentagon $A_1'A_2'A_3'A_4'A_5'$ to $A_1A_2A_3$ will take the vertex A_4' (which is the point on the line through A_1' parallel to $A_2'A_3'$ for which the directed segment $A_1'A_4'$ is φ times the length of the directed segment $A_2'A_3'$) to the point A_4. Similarly, A_5' is taken to A_5. Of course, the centre of gravity M' of the regular pentagon has the property that the areas of the five triangles $A_i'M'A_{i+1}'$ are equal. (In fact the triangles are congruent.) Our affine transformation takes M' to the centre of gravity M of the given pentagon. Because affine transformations preserve ratios of areas, the areas of A_iMA_{i+1} are equal, as required.

3. Prove that there are no distinct positive integers x and y such that

$$x^{2007} + y! = y^{2007} + x! \,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Geupel.

We prove the more general result that for each integer $n \geq 2$ such that $n \neq 2^m - 1$ ($m \in \mathbb{N}$) the function $f : \mathbb{N} \to \mathbb{Z} : f(x) = x! - x^n$ is injective.

Firstly, we show that f is increasing for $x \geq 2n$. Indeed, for $x \geq 2n$ we have $x! \geq (1 \cdot x)(2 \cdot (x-1)) \cdots (n \cdot (x+1-n)) \geq x \cdot x \cdots x = x^n$. Taking into account that

$$\left(1+\frac{1}{x}\right)^n \leq \left(1+\frac{1}{2n}\right)^{2n} \leq e,$$

we obtain

$$\begin{aligned} f(x+1) \;&=\; \left(1+\frac{1}{x}\right)^n (x!-x^n) + \left[x+1-\left(1+\frac{1}{x}\right)^n\right] x! \\ \geq\; f(x) &+ \left[x+1-\left(1+\frac{1}{2n}\right)^{2n}\right] x! \;\geq\; f(x)+(x+1-e)x! \;>\; f(x), \end{aligned}$$

which completes the proof that f is increasing for $x \geq 2n$.

We prove our initial claim by contradiction.

Assume that we have $x \leq y$ and $f(x) = f(y)$. For $k \in \mathbb{Z}$ and p prime, let $d_p(k)$ denote the greatest $\alpha \in \mathbb{Z}$ such that $p^\alpha \mid k$. Let p be any prime divisor of x. Then, $p \mid x!$, $p \mid x^n$, and $p \mid y!$. Hence, $p \mid x^n - x! + y!$, that is $p \mid y^n$ and therefore $p \mid y$. From $y! - x! = x![(x+1)(x+2)\cdots y - 1]$, we see that $d_p(x!) = d_p(y! - x!) = d_p(y^n - x^n) \geq n$.

By

$$d_p(x!) = \left\lfloor \frac{x}{p} \right\rfloor + \left\lfloor \frac{x}{p^2} \right\rfloor + \cdots \leq x\left(\frac{1}{p}+\frac{1}{p^2}+\cdots\right) = \frac{x}{p-1},$$

we deduce $x \geq (p-1)d_p(x!) \geq (p-1)n$. Because $f(z)$ is increasing for $z \geq 2n$, we must have $p = 2$ and $x = 2^m$. Thus, $n \leq 2^m < 2n$. From

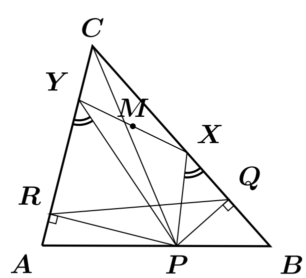
$$d_2(y!-x!) = d_2(x!) = 2^m\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^m}\right) = 2^m - 1$$

and $n \mid d_2(y^n - x^n)$, we conclude $n \mid 2^m - 1$. Consequently $n = 2^m - 1$, a contradiction.

4. Given a point P on the side AB of a triangle ABC, consider all pairs of points (X, Y), $X \in BC$, $Y \in AC$ such that $\angle PXB = \angle PYA$. Prove that the mid-points of the segments XY lie on a straight line.

Solved by Titu Zvonaru, Comăneşti, Romania.

Let Q, R be the projections of P onto BC and AC, respectively, and let M be the midpoint of XY.



We denote $m = CR$, $n = CQ$, $P = RQ$, $\alpha = \angle PXB = \angle PYA$. By Stewart's Theorem we obtain

$$\begin{aligned} & QC^2 \cdot RY - QY^2 \cdot CR + QR^2 \cdot CY = CR \cdot CY \cdot YR \\ \Leftrightarrow \quad & m \cdot QY^2 = n^2 \cdot RY + p^2(m - RY) - m(m - RY) \cdot RY \\ \Leftrightarrow \quad & m \cdot QY^2 = RY(n^2 - p^2 - m^2) + mp^2 + m \cdot RY^2 \\ \Leftrightarrow \quad & m \cdot QY^2 = RY(-2mp\cos\angle CRQ) + mp^2 + m \cdot RY^2 \end{aligned}$$

Since the quadrilateral $CRPQ$ is cyclic, we deduce

$$\cos\angle CRQ = \cos\angle CPQ = \frac{PQ}{CP} = \frac{RX\tan\alpha}{CR}\,.$$

It follows that

$$QY^2 = -2p \cdot \frac{RX \cdot RY \cdot \tan\alpha}{CP} + p^2 + RY^2 \tag{1}$$

and similarly

$$RX^2 = -2p \cdot \frac{RX \cdot RY \cdot \tan\alpha}{CP} + p^2 + QX^2\,. \tag{2}$$

By (1) and (2) we have

$$\begin{aligned} QY^2 - RY^2 &= RX^2 - QX^2 \\ \Leftrightarrow 2(RX^2 + RY^2) - XY^2 &= 2(QX^2 + QY^2) - XY^2 \\ \Leftrightarrow RM^2 &= QM^2\,, \end{aligned}$$

hence the point M lies on the perpendicular bisector of QR.

5. The real numbers a_i, b_i, $1 \le i \le n$, are such that

$$\sum_{i=1}^{n} a_i^2 = 1\,, \quad \sum_{i=1}^{n} b_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^{n} a_i b_i = 0\,.$$

Prove that

$$\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 \le n\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; and Henry Ricardo, Tappan, NY, USA. We use the write-up of Apostolopoulos.

Let $x = \sum_{i=1}^{n} a_i$ and $y = \sum_{i=1}^{n} b_i$. By Cauchy-Schwarz Inequality we have

$$\begin{aligned}(x^2+y^2)^2 &= \left[\sum_{i=1}^{n}(a_ix+b_iy)\right]^2 \le n\sum_{i=1}^{n}(a_ix+b_iy)^2\\ &= n\sum_{i=1}^{n}(a_i^2x^2+b_i^2y^2+2a_ib_ixy)\\ &= n\left(x^2\sum_{i=1}^{n}a_i^2+y^2\sum_{i=1}^{n}b_i^2+2xy\sum_{i=1}^{n}a_ib_i\right)\\ &= n(x^2+y^2)\end{aligned}$$

so $(x^2+y^2)^2 \le n(x^2+y^2)$, namely

$$x^2+y^2\le n \Leftrightarrow \left(\sum_{i=1}^{n}a_i\right)^2+\left(\sum_{i=1}^{n}b_i\right)^2\le n.$$

6. For a finite set S denote by $\mathcal{P}(S)$ the set of all subsets of S. The function $f:\mathcal{P}(S)\to\mathbb{R}$ is such that

$$f(X\cap Y) \;=\; \min(f(X),f(Y))$$

for any two subsets $X,Y\in\mathcal{P}(S)$. Find the largest number of distinct values that f can take.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Given

$$S=\{a_1,a_2,\ldots,a_n\},$$

define, for $i=1,2,\ldots,n$,

$$X_i=S\backslash\{a_i\}.$$

Set $x_i=f(X_i)$, and define $M=f(S)$. This defines f on all other subsets of S since all others can be formed from intersections of these. Moreover, these other function values are in $\{x_1,x_2,\ldots,x_n,M\}$. Hence, f has at most $n+1$ distinct values. In fact, each subset of S can be obtained in a unique way (apart from order) from intersections of the X_i and S; hence, any set of choices of the x_i and of M gives an allowable f. Accordingly, f can have $n+1$ distinct values.

Next we turn to the solutions of problems of the 48$^{\text{th}}$ IMO Bulgarian Team, Second Selection Test, given at [2010; 276].

2. Find all positive integers m such that

$$\frac{2^m\alpha^m-(\alpha+\beta)^m-(\alpha-\beta)^m}{3\alpha^2+\beta^2}$$

is an integer for all integer values of α, β with $\alpha\beta \neq 0$.

Solved by George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Apostolopoulos.

Denote $x = \alpha + \beta$, $y = \alpha - \beta$. Then

$$\frac{(2\alpha)^m - (\alpha+\beta)^m - (\alpha-\beta)^m}{3\alpha^2+\beta^2} = \frac{(x+y)^m - x^m - y^m}{x^2+xy+y^2},$$

where $\alpha\beta \neq 0 \Leftrightarrow x^2 \neq y^2$. Now we consider $(x+y)^m - x^m - y^m$ and x^2+xy+y^2 as polynomials with one variable x. Do the division and we get that $(x+y)^m - x^m - y^m = (x^2+xy+y^2)f(x,y) + g(y)x + h(y)$, where $f(x,y)$, $g(y)$, $h(y)$ are polynomials with integer coefficients. Suppose m satisfies that $x^2+xy+y^2 \mid (x+y)^m - x^m - y^m$. Fix y, let x vary in the set of positive integers. We have $x^2+xy+y^2 \mid g(y)x+h(y)$. But for very large x, $|x^2+xy+y^2| > |g(y)x+h(y)|$, then $g(y)x+h(y) = 0 \Rightarrow g(y) = h(y) = 0$. We deduce that for every y, $g(y) = h(y) = 0$. Thus g and h are both zero polynomials. On the other hand if g and h are zero polynomials, it is clear that $x^2+xy+y^2 \mid (x+y)^m - x^m - y^m$ for $x^2 \neq y^2$. Thus m is valid $\Leftrightarrow g \equiv h \equiv 0$.

Next let $w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, then $w^2+w+1 = 0$, $w^3 = 1$. We claim that $g \equiv h \equiv 0 \Leftrightarrow (w+1)^m - w^m - 1 = 0$. If $g \equiv h \equiv 0$, then $(x+y)^m - x^m - y^m = (x^2+xy+y^2)f(x,y)$. This holds for all x and y. Take $x = wy$, then $x^2+xy+y^2 = 0 \Rightarrow (x+y)^m - x^m - y^m = 0 \Rightarrow y^m[(w+1)^m - w^m - 1] = 0$, choose $y \neq 0$ then $(w+1)^m - w^m - 1 = 0$. If $(w+1)^m - w^m - 1 = 0$ we take $x = wy$, then $(x+y)^m - x^m - y^m = 0$ and $x^2+xy+y^2 = 0$. We deduce that $g(y)wy + h(y) = 0$. Since $g(y)$ and $h(y)$ are real numbers and w is not, $g(y)y$ must be zero. Then $g(y) \equiv 0$ for all y. Finally $h(y) \equiv 0$.

Now we only need to find m such that

$$(w+1)^m - w^m - 1 = 0\,, \quad w+1 = -w^2\,, \quad (-1)^m w^{2m} - w^m - 1 = 0\,.$$

Case 1: If $m \equiv 0 \pmod 3$, then $(-1)^m w^{2m} = (-1)^m$, $w^m = 1$, and $(-1)^m - 2 = 0$, no solution.

Case 2: If $3 \nmid m$ we have

$$(-1)^m \omega^{2m} - \omega^m - 1 = 0\,.$$

Also, since ω is a third root of unity and $3 \nmid m$, we have

$$\omega^{2m} + \omega^m + 1 = 0\,.$$

By adding these two relations we get

$$\omega^{2m}[1 + (-1)^m] = 0\,.$$

Thus

$$(-1)^m = -1 \Rightarrow 2 \nmid m\,.$$

Thus

$$3 \nmid m;\ 2 \nmid m \Rightarrow m = \pm 1 \pmod 6 .$$

Finally all valid m are $m \equiv \pm 1 \pmod 6$.

3. Find all integers $n \geq 3$ such that: for any two positive integers $m < n-1$, $r < n-1$ there exist m distinct elements of the set $\{1, 2, \ldots, n-1\}$ whose sum is congruent to r modulo n.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Curtis.

We claim that the n's with the desired property are exactly the odd $n \geq 3$.

1. Suppose that k is a positive integer, $n = 2k+2$, $m = 2k$, $r = k+1$, and $S_n = \{1, 2, \ldots, n-1\}$. For $j \in S_n$, let x_j denote the sum of all elements of S_n except the j^{th}. Then $x_j = (k+1)(2k+1) - j$. Thus,

$$\begin{aligned} x_j \equiv r \bmod n \quad &\Leftrightarrow \quad (k+1)(2k+1) - j \equiv (k+1) \bmod 2k+2 \\ &\Leftrightarrow \quad (k+1)(2k) \equiv j \bmod 2(k+1) \\ &\Leftrightarrow \quad 2(k+1) \text{ divides } 2k(k+1) - j \end{aligned}$$

   Since $2(k+1)$ divides $2k(k+1)$, n must divide j. But $1 \leq j \leq n-1$, a contradiction. Hence, no even n has the desired property.

2. Now suppose that k is a positive integer and $n = 2k+1$. Let $r \in S_n$.

   (a) Suppose $m = 2l+1$, where l is a positive integer less than k. Let

$$A = \begin{cases} \{1, 2, 3, \ldots, r-1, r+1, \ldots, l+1\} & \text{if} \quad r \leq l \\ \{1, 2, 3, \ldots, l\} & \text{if} \quad l+1 \leq r \leq 2k-l \\ \{1, 2, \ldots, 2k-r, 2k+2-r, \ldots, l+1\} & \text{if} \quad r \geq (2k+1) - l. \end{cases}$$

   Then

$$\sum_{i \in A} [i + (n-i)] + r$$

   is a sum of m distinct elements of S_n. This sum is congruent to $r \bmod n$ since $i + (n-i) \equiv 0 \bmod n$ for each i.

   (b) Suppose $m = 2l$ and $r \geq 3$. Let

$$A = \begin{cases} \{2, 3, \ldots, r-2, r, r+1, \ldots, l+1\} & \text{if} \quad 3 \leq r \leq l \\ \{2, 3, \ldots, l\} & \text{if} \quad l+1 \leq r \leq 2k-l \\ \{2, 3, \ldots, 2k+1-r, 2k+3-r, \ldots, l+1\} & \text{if} \quad r \geq (2k+1) - l. \end{cases}$$

   Then

$$1 + (r-1) + \sum_{i \in A} [i + (n-i)]$$

   is a sum of m distinct elements of S_n. The sum is congruent to $r \bmod n$.

(c) Suppose $m = 2l$ and $r = 1$. Then

$$2 + 2k + \sum_{i=3}^{l+1}[i + (n - i)] \equiv 1 \bmod n.$$

(d) Suppose $m = 2l \leq 2k - 4$ and $r = 2$. Then

$$3 + 2k + \sum_{i=4}^{l+2}[i + (n - i)] \equiv 2 \bmod n.$$

(e) Suppose $m = 2l = 2k - 2 = n - 3$ and $r = 2$. Since

$$\sum_{i \in S_n} i = \frac{(n-1)n}{2} = kn \equiv 0 \bmod n,$$

$$\sum_{\substack{i \in S_n \\ i \neq 1, 2k-2}} i \equiv 0 - (2k - 1) \equiv 2 \bmod n.$$

Thus, for $n = 2k + 1 \geq 3$, and $1 \leq m, r < n - 1$, there exist m distinct elements of S_n with sum congruent to $r \bmod n$.

4. Solve the system

$$\begin{cases} x^2 + yu &= (x+u)^n \\ x^2 + yz &= u^4 \end{cases}$$

where x, y and z are prime numbers and u is a positive integer.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution by Curtis.

From the second equation, $yz = (u^2 + x)(u^2 - x)$. Since y and z are prime, the only factors of the left-hand side are 1, y, z, and yz. Thus there are at most four possibilities.

1. The case $u^2 + x = 1$, $u^2 - x = yz$ is impossible since u and x are both positive.

2. If $u^2 + x = yz$ and $u^2 - x = 1$, then $x = (u + 1)(u - 1)$. Since x is prime, we must have $u - 1 = 1$ and $u + 1 = x$. Thus, $u = 2$ and $x = 3$. From $u^2 + x = yz$, we obtain $7 = yz$, which is impossible.

3. If $u^2 + x = y$ and $u^2 - x = z$, the first equation gives

$$x^2 + u(u^2 + x) = (x + u)^n,$$

or

$$x^2 + u^3 + ux = (x + u)^n.$$

If $n \geq 3$, then $(x + u)^n \geq (x + u)^3 > x^2 + u^3 + ux$. Hence any solution must have $n < 3$.

(a) If $n = 1$, then $x^2 + u^3 + ux = x + u$, which can be rewritten as $x(x-1)+u(u^2-1)+ux = 0$ which is impossible since the left-hand side is positive.

(b) If $n = 2$, then $x^2+u^3+ux = x^2+2ux+u^2$, which can be rewritten as $u[u(u-1)-x] = 0$. Since $u > 0$, we have $x = u(u-1)$. Since x is prime, $u = 2$ and $x = 2$, implying that $y = 6$ and $z = 2$.

4. If $u^2 + x = z$ and $u^2 - x = y$, the first equation gives

$$x^2 + u(u^2 - x) = (x + u)^n.$$

As before, if $n \geq 3$, the right-hand side is greater than the left-hand side.

(a) If $n = 1$, then $x^2 + u(u^2 - x) = x + u$, which can be rewritten as $x(x - 1) + u(u^2 - x - 1) = 0$. But $u^4 - x^2 = yz \geq 4$ so that $u^4 \geq x^2 + 4$ and $u^2 > x$. Thus, $u^2 - x - 1 \geq 0$, and $x(x - 1) + u(u^2 - x - 1) > 0$, a contradiction.

(b) If $n = 2$, then $x^2+u(u^2-x) = (x+u)^2$, which can be rewritten as $u[u(u-1)-3x] = 0$. Thus $u(u-1) = 3x$ and $u-1 \in \{1, 3, x, 3x\}$.

- If $u - 1 = 1$, then $u = 2$, and $3x = 2$, which is impossible.
- If $u - 1 = 3$, then $u = 4$, and $x = 4$, which is not a prime.
- If $u - 1 = x$, then $x(x + 1) = 3x$, implying that $x = 2$. In this case, $u = 3$, $y = 7$, and $z = 11$.
- If $u - 1 = 3x$, then $u = 1$, so that $x = 0$, a contradiction.

In summary, the only solutions are

$$(x, y, z, u, n) \in \{(2, 6, 2, 2, 2), (2, 7, 11, 3, 2)\}.$$

And now we look at solutions to problems of the 2007 Mediterranean Mathematical Competition, given at [2010: 277].

1. Let $x \leq y \leq z$ be real numbers, such that $xy + yz + zx = 1$. Prove that $xz < \frac{1}{2}$. Is it possible to improve the value of the constant $\frac{1}{2}$?

Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Edward T.H. Wang and Dexter Wei, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Wang and Wei.

If $x = 0$, then $xz = 0 < \frac{1}{2}$. If $x > 0$, then $2xz = xz + xz \leq yz + zx < xy + yz + zx < 1$ so $xz < \frac{1}{2}$. Suppose $x < 0$. If $z \geq 0$, then $xz \leq 0 < \frac{1}{2}$ so it suffices to consider the case when $x \leq y \leq z < 0$.

Set $r = -z$, $s = -y$, and $t = -z$. Then r, s, and t are all positive such that $t \le s \le r$ and $ts + sr + rt = 1$. Hence, $tr < \frac{1}{2}$ by what we showed above. Then $xz < \frac{1}{2}$ follows.

Now we prove that $\frac{1}{2}$ is the best upper bound for xz by showing that for any $\varepsilon > 0$, there exists x, y, z satisfying $x \le y \le z$, $xy + yz + zx = 1$ and $xz > \frac{1}{2} - \varepsilon$.

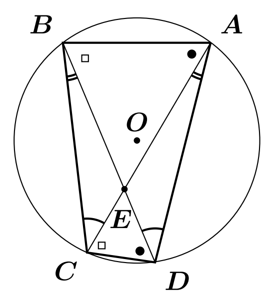
If $\varepsilon \ge \frac{1}{2}$, simply take $x = y = z = \frac{\sqrt{3}}{3}$. Hence we may assume that $0 < \varepsilon < \frac{1}{2}$.

We set $x = y = \frac{\sqrt{2\varepsilon}}{2}$ and $z = \frac{z-\varepsilon}{2\sqrt{2\varepsilon}}$. Then $y \le z$ is equivalent to $2\varepsilon < 2 - \varepsilon$ or $\varepsilon < \frac{2}{3}$ which is true.

Next, $xy + yz + zx = x^2 + 2xz = \frac{\varepsilon}{2} + \frac{2-\varepsilon}{2} = 1$. Finally, $xz > \frac{1}{2} - \varepsilon$ is equivalent, in succession, to $\frac{2-\varepsilon}{4} > \frac{1}{2} - \varepsilon$; $\frac{1}{2} - \frac{\varepsilon}{4} > \frac{1}{2} - \varepsilon$; $\frac{\varepsilon}{4} < \varepsilon$ which is clearly true and our proof is complete.

2. The quadrilateral $ABCD$ is convex and cyclic, and the diagonals AC and BD intersect at the point E. Given that $AB = 39$, $AE = 45$, $AD = 60$ and $BC = 56$, determine the length of CD.

Solved by George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We use the solution of Apostolopoulos.



The triangles BEC and AED are similar, so

$$\frac{BE}{AE} = \frac{BC}{AD} \Rightarrow BE = \frac{AE \cdot BC}{AD} = \frac{45 \cdot 56}{60} = 42\,.$$

Also, the triangles AEB and CED are similar, so $\frac{CD}{AB} = \frac{ED}{AE} = \frac{CE}{BE}$, namely $\frac{CD}{29} = \frac{ED}{45} = \frac{CE}{42} = l > 0$. Using Ptolemy's Theorem yields:

$$\begin{aligned} & AB \cdot CD + AD \cdot BC = AC \cdot BD \\ \Rightarrow \quad & 39(39l) + 60 \cdot 56 = (45 + CE)(42 + ED) \\ \Rightarrow \quad & 39^2 l + 60 \cdot 56 = (45 + 42l)(42 + 45l) \\ \Leftrightarrow \quad & 315l^2 + 378l - 245 = 0\,, \end{aligned}$$

so

$$l = \frac{7}{15} \quad \text{or} \quad l < 0$$

namely $l = \frac{7}{15}$, thus $CD = 39l = 39 \cdot \frac{7}{15} = 18.2$.

3. In the triangle ABC, the angle $\alpha = \angle A$ and the side $a = |BC|$ are given. It is known that $a = \sqrt{rR}$, where r is the inradius and R is the circumradius of ABC. Determine all such triangles, that is, compute the sides b and c of all such triangles.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Amengual Covas.

Let $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$.

From $a = \sqrt{rR}$, we obtain

$$r = \frac{a^2}{R}. \tag{1}$$

Note that the area of $\triangle ABC$ may be expressed as $\frac{abc}{4R}$, and also as rs. Equating those we get $abc = 4sRr = 4sa^2$. Thus

$$bc = 4as. \tag{2}$$

Then $\cos^2\frac{A}{2} = \frac{s(s-a)}{bc} = \frac{s(s-a)}{4as} = \frac{s-a}{4a}$. Since $\cos\alpha = 2\cos^2\frac{A}{2} - 1$, this is equivalent to $\cos\alpha = \frac{s-3a}{2a}$ or $s = a(3+2\cos\alpha)$, so that (2) becomes

$$bc = 4a^2(3+2\cos\alpha). \tag{3}$$

We also have

$$\begin{aligned} b+c &= 2s-a \\ &= 2a(3+2\cos\alpha)-a \\ &= a(5+4\cos\alpha). \end{aligned} \tag{4}$$

We solve the system (3) and (4) by considering b and c as roots of a quadratic equation with coefficients determined by the product (3) and the sum (4) of the roots:

$$x^2 - a(5+4\cos\alpha)x + 4a^2(3+2\cos\alpha) = 0. \tag{5}$$

The requirement that b and c be sides of $\triangle ABC$ forces the discriminant of (5)

$$a^2[(5+4\cos\alpha)^2 - 16(3+2\cos\alpha)] = a^2[(4\cos\alpha+1)^2 - 24]$$

to be non-negative. This is true only if $\cos\alpha \geq \frac{\sqrt{24}-1}{4}$, that is, only if $\alpha \leq 12°54'15''$. If this holds, the roots of (5)

$$x = \frac{1}{2}a[5+4\cos\alpha \pm \sqrt{(4\cos\alpha+1)^2-24}]$$

yield the length of the sides CA and AB of $\triangle ABC$.

4. Let $x > 1$ be a noninteger number. Prove that

$$\left(\frac{x+\{x\}}{[x]} - \frac{[x]}{x+\{x\}}\right) + \left(\frac{x+[x]}{\{x\}} - \frac{\{x\}}{x+[x]}\right) > \frac{9}{2},$$

where $[x]$ and $\{x\}$ represents the integer and the fractional part of x.

Solved by Mohammed Aassila, Strasbourg, France; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Henry Ricardo, Tappan, NY, USA; Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the improved results of Bataille.

Let $L = \left(\frac{x+\{x\}}{\lfloor x\rfloor} - \frac{\lfloor x\rfloor}{x+\{x\}}\right) + \left(\frac{x+\lfloor x\rfloor}{\{x\}} - \frac{\{x\}}{x+\lfloor x\rfloor}\right)$. We show that $L > \frac{16}{3}$, slightly improving the proposed inequality. Using $\lfloor x\rfloor + \{x\} = x$, we calculate

$$\frac{x+\{x\}}{\lfloor x\rfloor} + \frac{x+\lfloor x\rfloor}{\{x\}} = \frac{x^2+\{x\}^2+\lfloor x\rfloor^2}{\{x\}\lfloor x\rfloor}$$

and

$$\frac{\lfloor x\rfloor}{x+\{x\}} + \frac{\{x\}}{x+\lfloor x\rfloor} = \frac{x^2+\{x\}^2+\lfloor x\rfloor^2}{(x+\{x\})(x+\lfloor x\rfloor)} = \frac{x^2+\{x\}^2+\lfloor x\rfloor^2}{2x^2+\{x\}\lfloor x\rfloor}.$$

It follows that

$$L = \frac{2x^2(x^2+\{x\}^2+\lfloor x\rfloor^2)}{\{x\}\lfloor x\rfloor(2x^2+\{x\}\lfloor x\rfloor)}.$$

Now, recalling that $2(a^2+b^2) > (a+b)^2$ for distinct positive real numbers a, b, we see that

$$2x^2(x^2+\{x\}^2+\lfloor x\rfloor^2) > x^2(2x^2+(\{x\}+\lfloor x\rfloor)^2) = 3x^4$$

(note that $\{x\} \in [0,1)$ and $\lfloor x\rfloor \geq 1$ so that $\{x\} \neq \lfloor x\rfloor$). In addition, from AM-GM,

$$\{x\}\lfloor x\rfloor(2x^2+\{x\}\lfloor x\rfloor) < \frac{(\{x\}+\lfloor x\rfloor)^2}{4}\cdot\left(2x^2+\frac{(\{x\}+\lfloor x\rfloor)^2}{4}\right) = \frac{9x^4}{16}$$

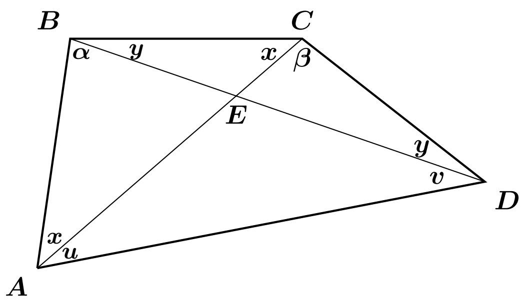
and we finally deduce

$$L > (3x^4)\cdot\frac{16}{9x^4} = \frac{16}{3}.$$

We return to the files of solutions from our readers and the 24$^{\text{th}}$ Balkan Mathematical Olympiad 2007 given at [2010: 277–278].

1. Let $ABCD$ be a convex quadrilateral with $AB = BC = CD$, $AC \neq BD$ and let E be the intersection point of its diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120^\circ$.

Solved by Mohammed Aassila, Strasbourg, France; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Amengual Covas.



Let $\angle DAE = u$, $\angle EDA = v$, $\angle ABD = \alpha$ and $\angle ACD = \beta$. Letting the base angles in isosceles triangles ABC and BCD be x and y, respectively, we have $u + v = x + y$ in $\triangle AED$, $\triangle BEC$ and $x + \alpha = y + \beta$ in $\triangle ABE$, $\triangle CDE$, respectively, because of the vertically opposite angles at E. Therefore,

$$\begin{aligned} \angle BAD + \angle ADC &= (x+u) + (v+y) \\ &= (u+v) + (x+y) \qquad (1) \\ &= 2(x+y) \end{aligned}$$

We must have $\alpha \neq \beta$. Suppose $\alpha = \beta$: Then the condition $x+\alpha = y+\beta$ implies $x = y$ and we would have $\angle ABC = \angle BCD$, making $\triangle ABC$ and $\triangle BCD$ congruent which contradicts the assumption $AC \neq BD$.

Now, by the law of sines,

$$\begin{aligned} \frac{AE}{\sin\alpha} &= \frac{AB}{\sin\angle BEA} \\ &= \frac{CD}{\sin\angle DEC} \quad (\angle BEA \text{ and } \angle DEC \text{ are vertically opposite angles}) \\ &= \frac{DE}{\sin\beta}. \end{aligned}$$

so

$$\begin{aligned} AE = DE \quad &\Leftrightarrow \quad \sin\alpha = \sin\beta \\ &\Leftrightarrow \quad \alpha + \beta = 180^\circ \quad (\text{since } a \neq \beta) \\ &\Leftrightarrow \quad \angle BAD + \angle ADC + x + y = 180^\circ \\ &\qquad (\text{since the angles of } ABCD \text{ add up to } 360^\circ) \\ &\Leftrightarrow \quad \angle BAD + \angle ADC + \frac{1}{2}(\angle BAD + \angle ADC) = 180^\circ \quad \text{by (1)} \\ &\Leftrightarrow \quad \angle BAD + \angle ADC = 120^\circ \end{aligned}$$

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$f(f(x) + y) = f(f(x) - y) + 4f(x)y.$$

Solved by Mohammed Aassila, Strasbourg, France; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA. We give the solution by Curtis.

We claim that the only solutions are the 0 function and functions $f(x) = x^2 + b$, where b is a real number. It is readily verified that these satisfy the functional equation, so we assume that f not identically 0 satisfies the functional equation. Choose $\bar{x} \in \mathbb{R}$ such that $f(\bar{x}) \neq 0$. Let $z \in \mathbb{R}$, and set $\bar{y} = \dfrac{z}{8f(\bar{x})}$. Then

$$f\left(f(\bar{x}) + \frac{z}{8f(\bar{x})}\right) - f\left(f(\bar{x}) - \frac{z}{8f(\bar{x})}\right) = \frac{z}{2}.$$

Let $x_1 = f(\bar{x}) + \bar{y}$ and $x_2 = f(\bar{x}) - \bar{y}$. Then

$$z = 2[f(x_1) - f(x_2)].$$

With

$$v = f(x_1) - 2f(x_2),$$

we have

$$z = f(x_1) + v,$$

so that

$$\begin{aligned} f(z) = f(f(x_1) + v) &= f(f(x_1) - v) + 4f(x_1)v \\ &= f(2f(x_2)) + 4f(x_1)v. \end{aligned}$$

Let $b = f(0)$, and in the functional equation, let $x = x_2$ and $y = f(x_2)$. Then

$$f(2f(x_2)) = b + 4[f(x_2)]^2.$$

Thus,

$$\begin{aligned} f(z) = b + 4[f(x_2)]^2 + 4f(x_1)v &= b + 4[f(x_2)]^2 + 4f(x_1)[f(x_1) - 2f(x_2)] \\ &= b + 4[f(x_1) - f(x_2)]^2 \\ &= b + z^2. \end{aligned}$$

3. Find all positive integers n such that there is a permutation σ of the set $\{1, 2, \ldots, n\}$ such that the number below is a rational number

$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\cdots + \sqrt{\sigma(n)}}}}$$

Ed.: A permutation of the set $\{1, 2, \ldots, n\}$ is a one-to-one function of this set to itself.

Solved by Mohammed Aassila, Strasbourg, France; and George Apostolopoulos, Messolonghi, Greece. We give the version of Apostolopoulos.

Suppose that for some $n \in \mathbb{N}^*$ we have

$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\cdots + \sqrt{\sigma(n)}}}} = v_1 \in \mathbb{Q}$$

then $\sigma(1) + \sqrt{\sigma(2) + \sqrt{\cdots + \sqrt{\sigma(n)}}}$ is a rational number. Similar, for each $k \in \{1, 2, \ldots, n\}$ the number $v_k = \sqrt{\sigma(k) + \sqrt{\sigma(k+1) + \sqrt{\cdots + \sqrt{\sigma(n)}}}}$ is a rational number. Define $\alpha_k = \sqrt{n + \sqrt{n + \sqrt{\cdots + \sqrt{n}}}}$, for each $k \in \mathbb{N}^*$. By an easy induction, we can prove that $\alpha_k < \sqrt{n} + 1$, for each $k \in \mathbb{N}^*$, as a result we will have $v_1 < \alpha_n < \sqrt{n} + 1$. If $l > 0$ is such that $l^2 \leq n < (l+1)^2$ then for some $i \in \{1, 2, \ldots, n\}$ we have $\sigma(i) = l^2$.

Case 1. $i \neq n$.

Then we have

$$l < \sqrt{l^2 + \sqrt{\sigma(i+1) + \sqrt{\cdots + \sqrt{\sigma(n)}}}} < \sqrt{n} + 1 < l + 2$$
$$\Rightarrow \sqrt{l^2 + \sqrt{\sigma(i+1) + \sqrt{\cdots + \sqrt{\sigma(n)}}}} = l + 1$$
$$\Rightarrow 2l + 1 = \sqrt{\sigma(i+1) + \sqrt{\cdots + \sqrt{\sigma(n)}}} < \sqrt{n} + 1 < l + 2 \Rightarrow l < 1\,,$$

a contradiction.

Case 2. $i = n$.

If $l > 1$, then $(l^2 - 1) \in \{\sigma(1), \sigma(2), \ldots, \sigma(n-1)\}$. Suppose that $j < n$ is such that $\sigma(j) = l^2 - 1$. Similarly to Case 1, we get

$$l < \sqrt{l^2 - 1 + \sqrt{\sigma(j+1) + \sqrt{\cdots + \sqrt{l^2}}}} < \sqrt{n} + 1 < l + 2$$
$$\Rightarrow \sqrt{l^2 - 1 + \sqrt{\sigma(j+1) + \sqrt{\cdots + \sqrt{l^2}}}} = l + 1$$
$$\Rightarrow 2l + 2 = \sqrt{\sigma(j+1) + \sqrt{\cdots + \sqrt{l^2}}} < \sqrt{n} + 1 < l + 2\,,$$

a contradiction.

If $l = 1$, then $n \in \{1, 2, 3\}$. We conclude that for $n = 1$ and for $n = 3$ there are permutations that satisfy the terminus relation. For $n = 1$, we have $\sqrt{1} = 1$ and for $n = 3$, we have $\sqrt{2 + \sqrt{3 + \sqrt{1}}} = 2$. But for $n = 2$ no such permutation exists.

So $n = 1$, $n = 3$.

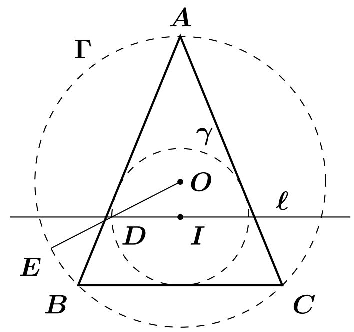
Next we turn to the Indian Team Selection Test 2007 given at [2010: 278–279].

1. Let ABC be a triangle with $AB = AC$, and let Γ be its circumcircle. The incircle γ of ABC moves (slides) on BC in the direction of B. Prove that when γ touches Γ internally, it also touches the altitude through A.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Smeenk.

We denote $\Gamma = \Gamma(O, R)$, $\gamma = \gamma(I, r)$.

Line ℓ passes through I and is parallel to BC. D lies on ℓ, and $ID = r$. OD intersects Γ at E.



It suffices to show that $DE = r$, and so $OD = R - r$. $OI = R\cos\alpha - r$, $ID = r$, and we are to show

$$(R-r)^2 = (R\cos\alpha - r)^2 + r^2 \qquad (1)$$

As $\beta = \gamma$ we have

$$\begin{aligned} r &= R(\cos\alpha + \cos\beta + \cos\gamma - 1) \\ &= R(-\cos 2\beta + 2\cos\beta - 1) \\ &= 2R\cos\beta(1 - \cos\beta). \end{aligned} \qquad (2)$$

Substituting (2) in (1) we see that it holds and we are done.

2. Consider the quadratic polynomial $p(x) = x^2 + ax + b$, where a, b are in the interval $[-2, 2]$. Determine the range of the real roots of $p(x) = 0$ as a and b vary over $[-2, 2]$.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

The polynomial $p(x)$ has real roots if and only if its discriminant $D = a^2 - 4b$ is nonnegative. Set

$$x_+ = \frac{-a + \sqrt{a^2 - 4b}}{2} \quad \text{and} \quad x_- = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

Since $-2 \le b \le \frac{1}{4}a^2$, we have $0 \le a^2 - 4b \le a^2 + 8$. Hence,

$$-\frac{a}{2} \le x_+ \le \frac{-a + \sqrt{a^2 + 8}}{2} \quad \text{and} \quad \frac{-a - \sqrt{a^2 + 8}}{2} \le x_- \le \frac{-a}{2}.$$

The function $f(t) = \frac{1}{2}(-t + \sqrt{t^2 + 8})$ is continuous and decreasing on $[-2, 2]$; hence

$$-1 \le x_+ \le 1 + \sqrt{3}.$$

The function $g(t) = \frac{1}{2}(-t - \sqrt{t^2 + 8})$ is also continuous and decreasing on $[-2, 2]$; hence,

$$-1 - \sqrt{3} \le x_- \le 1.$$

Thus the range of the real roots of $p(x)$ is $[-1 - \sqrt{3}, 1 + \sqrt{3}]$.

3. Let triangle ABC have side lengths a, b, c; circumradius R, and internal angle bisector lengths w_a, w_b, w_c. Prove that

$$\frac{b^2+c^2}{w_a}+\frac{c^2+a^2}{w_b}+\frac{a^2+b^2}{w_c}\ >\ 4R\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Apostolopoulos.

It is well-known that $w_a=\dfrac{2bc\cos\dfrac{A}{2}}{b+c}$, and thus we have

$$\begin{aligned}\sum_{\text{cyclic}}\frac{b^2+c^2}{w_a}>4R \quad&\Leftrightarrow\quad \sum_{\text{cyclic}}\frac{(b^2+c^2)(b+c)}{2bc\cos\frac{A}{2}}>4R\\ &\Leftrightarrow\quad \sum_{\text{cyclic}}\frac{(b^2+c^2)(b+c)}{4Rbc\cos\frac{A}{2}}>2\\ &\Leftrightarrow\quad \sum_{\text{cyclic}}\frac{(b^2+c^2)(b+c)\sin\frac{A}{2}}{2abc}>1.\end{aligned}$$

Note that $b^2+c^2\geq 2bc$, $b+c>a$. It suffices to prove that $\displaystyle\sum_{\text{cyclic}}\sin\frac{A}{2}>1$. Let r be the inradius of the triangle. Then $\displaystyle\sum_{\text{cyclic}}\cos A+1+\frac{r}{R}>1$. This inequality holds for all triangles. Note that $\dfrac{\pi-A}{2}$, $\dfrac{\pi-B}{2}$, $\dfrac{\pi-C}{2}$ are also the three angles of a triangle thus $\displaystyle\sum_{\text{cyclic}}\cos\frac{\pi-A}{2}>1$. That is $\displaystyle\sum_{\text{cyclic}}\sin\frac{A}{2}>1$.

5. Show that in a non-equilateral triangle, the following are equivalent:

(a) The angles of the triangle are in arithmetic progression;

(b) The common tangent to the nine-point circle and the in-circle is parallel to the Euler line.

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Bataille.

We show that both (a) and (b) are equivalent to: (c) one of the angles of the triangle is 60°.

Let ABC be the given non-equilateral triangle with $BC=a, CA=b, AB=c$.

If (c) holds, then the angles of ΔABC are of the form 60°, $60^\circ+\alpha$, $60^\circ-\alpha$, hence (a) holds. Conversely, if, say, $B=\frac{A+C}{2}$, then $B=\frac{180^\circ-B}{2}$, hence $B=60^\circ$. Thus, (a) $\Longleftrightarrow$ (c).

Now, let $O, H, I,$ and N denote the circumcentre, the orthocentre, the incentre, and the centre of the nine-point circle, respectively. As usual, let R and s be the circumradius and the semi-perimeter of ΔABC. It is well-known that the incircle is internally tangent to the nine-point circle. It follows that (b) is equivalent to $OH \perp IN$. Since $\overrightarrow{NI} = \overrightarrow{OI} - \overrightarrow{ON} = \overrightarrow{OI} - \frac{1}{2}\overrightarrow{OH}$, the condition $OH \perp IN$ is equivalent to $\overrightarrow{OH} \cdot \overrightarrow{OI} = \frac{OH^2}{2}$. Using $2s\overrightarrow{OI} = a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC}$ and $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, a simple computation shows that $\overrightarrow{OH} \cdot \overrightarrow{OI} = \frac{OH^2}{2}$ is itself equivalent to

$$s = -a\cos 2A - b\cos 2B - c\cos 2C \qquad (1)$$

(note that for example $\overrightarrow{OA} \cdot \overrightarrow{OB} = R^2 \cos 2C$, C being acute or not).

We successively rewrite (1) as

$$\begin{aligned} a(1 + 2\cos 2A) + b(1 + 2\cos 2B) + c(1 + 2\cos 2C) &= 0 \\ \sin A + 2\sin A\cos 2A + \sin B \qquad\qquad & \\ +2\sin B\cos 2B + \sin C + 2\sin C\cos 2C &= 0 \\ \sin 3A + \sin 3B + \sin 3C &= 0 \end{aligned}$$

(the latter because $2\sin A\cos 2A = \sin 3A - \sin A$, etc.).

As a result, (b) is equivalent to $\sin 3A + \sin 3B + \sin 3C = 0$, or, with the help of the familiar trig formulas, to $4\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2} = 0$. Finally, (b) is equivalent to $90^\circ = \frac{3A}{2}$ or $\frac{3B}{2}$ or $\frac{3C}{2}$ and (b) $\iff$ (c).

7. Let a, b, c be nonnegative real numbers such that $a+b \le c+1$, $b+c \le a+1$ and $c+a \le b+1$. Prove that

$$a^2 + b^2 + c^2 \;\le\; 2abc + 1\,.$$

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We use the solution by Alt.

First note that $a, b, c \le 1$. Indeed,

$$c + a \le b + 1 \Rightarrow c + 2a \le a + b + 1 \le c + 1 + 1 \Rightarrow 2a \le 2 \Rightarrow a \le 1\,,$$

and, similarly, $b, c \le 1$.

Let $x := 1 - a$, $y := 1 - b$, $z := 1 - c$. Then $x, y, z \in [0, 1]$, $a = 1 - x$, $b = 1 - y$, $c = 1 - z$

$$\begin{cases} a + b \le c + 1 \\ b + c \le a + 1 \\ c + a \le b + 1 \end{cases} \iff \begin{cases} z \le x + y \\ x \le y + z \\ y \le z + x \end{cases} \iff |x - y| \le z \le x + y,$$

and the original inequality becomes

$$\begin{aligned}
&(1-x)^2+(1-y)^2+(1-z)^2 \le 2(1-x)(1-y)(1-z)+1\\
&\iff x^2+y^2+z^2 \le 2(xy+yz+zx)-2xyz\\
&\iff x^2+y^2-2xy \le 2z(x+y-xy)-z^2\\
&\iff (x-y)^2 \le 2z(x+y-xy)-z^2\,. \qquad (1)
\end{aligned}$$

Let $f(z) = 2z(x+y-xy) - z^2$. Since $z \in [|x-y|, x+y]$ then $\min_z f(z) = \min\{f(|x-y|), f(x+y)\}$, and, therefore,

$$\begin{aligned}
(1) &\iff (x-y)^2 \le \min_z \left(2z(x+y-xy)-z^2\right)\\
&\iff (x-y)^2 \le \min\{f(|x-y|), f(x+y)\} \qquad (2)\\
&\iff \begin{cases} (x-y)^2 \le f(|x-y|) \\ (x-y)^2 \le f(x+y) \end{cases}\\
&\iff \begin{cases} (x-y)^2 \le 2|x-y|(x+y-xy)-(x-y)^2 \\ (x-y)^2 \le 2(x+y)(x+y-xy)-(x-y)^2 \end{cases}\\
&\iff \begin{cases} (x-y)^2 \le |x-y|(x+y-xy) \\ x^2+y^2 \le (x+y)(x+y-xy) \end{cases}
\end{aligned}$$

We have

$$\begin{aligned}
x^2+y^2 \le (x+y)(x+y-xy) &\iff (x+y)xy \le 2xy\\
&\iff 0 \le xy(2-x-y)
\end{aligned}$$

and

$$(x-y)^2 \le |x-y|(x+y-xy) \iff 0 \le |x-y|(x+y-xy-|x-y|)\,.$$

Since $x, y \in [0,1]$ then the inequality $0 \le xy(2-x-y)$ obviously holds and

$$\begin{aligned}
|x-y| \le x+y-xy &\iff xy-x-y \le x-y \le x+y-xy\\
&\iff \begin{cases} xy-x \le x \\ -y \le y-xy \end{cases} \iff \begin{cases} 0 \le x(2-y) \\ 0 \le y(2-x) \end{cases}\,.
\end{aligned}$$

8. Given a finite string S of symbols a and b, we write $\triangle(S)$ for the number of a's in S minus the number of b's. (For example, $\triangle(abbabba) = -1$.) We call a string S balanced if every substring (of consecutive symbols) T of S has the property $-1 \leq \triangle(T) \leq 2$. (Thus $abbabba$ is not balanced, as it contains the substring $bbabb$ and $\triangle(bbabb) = -3$). Find, with proof, the number of balanced strings of length n.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Oliver Geupel, Brühl, NRW, Germany. We give Geupel's solution.

We prove that the number of balanced strings of length n is $n+1$. If a string S is balanced, then it does not contain the string bb as a substring, because $\triangle(bb) = -2$. Moreover, S contains at most one substring of the form aa, and the remaining part of S must alternate between the symbols a and b. Conversely, any string with not more than one occurrence of the substring aa and alternating symbols in the remaining parts of the string is balanced. We have two such strings with no substring aa and $n-1$ strings with exactly one substring aa. This completes the proof.

9. Define the functions f, g, h on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ as follows:

$$\begin{aligned} f(x,y,z) &= (3x+2y+2z, 2x+2y+z, 2x+y+2z)\,, \\ g(x,y,z) &= (3x+2y-2z, 2x+2y-z, 2x+y-2z)\,, \\ h(x,y,z) &= (3x-2y+2z, 2x-y+2z, 2x-2y+z)\,. \end{aligned}$$

Given a primitive Pythagorean triplet (x,y,z), with $x > y > z$, prove that (x,y,z) can be uniquely obtained by repeated application of f, g, h to the triple $(5,4,3)$. For example: $(697,528,455) = f \circ h \circ g \circ h(5,4,3)$.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Representing f, g, and h by the matrices F, G, and H given by

$$F = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad G = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & -1 \\ 2 & 1 & -2 \end{bmatrix}, \quad H = \begin{bmatrix} 3 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix},$$

their inverses are given by

$$F^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ -2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}, \quad G^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ -2 & 2 & 1 \\ 2 & -1 & -2 \end{bmatrix},$$

$$H^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ 2 & -1 & -2 \\ -2 & 2 & 1 \end{bmatrix}.$$

Let (x,y,z) be a primitive Pythagorean triple with $x > y > z$. Then there exist relatively prime opposite-parity integers s, t with $s > t$ such that $x = s^2 + t^2$ and either $y = 2st$ and $z = s^2 - t^2$ or $y = s^2 - t^2$ and $z = 2st$.

Case 1. Suppose first that $y = 2st$ and $z = s^2 - t^2$. Then $2st > s^2 - t^2$, so that $t^2 > s^2 - 2st = s(s - 2t) > t(s - 2t)$. This implies that $3t > s$. We have

- $F^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = F^{-1}\left(\begin{bmatrix} s^2 + t^2 \\ 2st \\ s^2 - t^2 \end{bmatrix}\right) = \begin{bmatrix} t^2 + (s - 2t)^2 \\ t^2 - (s - 2t)^2 \\ 2t(s - 2t) \end{bmatrix}$
  $(3t > s > 2t)$;
- $G^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = G^{-1}\left(\begin{bmatrix} s^2 + t^2 \\ 2st \\ s^2 - t^2 \end{bmatrix}\right) = \begin{bmatrix} t^2 + (2t - s)^2 \\ t^2 - (2t - s)^2 \\ 2t(2t - s) \end{bmatrix}$
  $(2t > s > t)$;
- $H^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = H^{-1}\left(\begin{bmatrix} s^2 + t^2 \\ 2st \\ s^2 - t^2 \end{bmatrix}\right) = \begin{bmatrix} t^2 + (2t - s)^2 \\ 2t(2t - s) \\ t^2 - (2t - s)^2 \end{bmatrix}$
  $(2t > s > t)$.

Since s and t are relatively prime, $s \neq 2t$ unless $s = 2$ and $t = 1$; if $s > 2t$, then the first right-hand side is a primitive Pythagorean triple closer to the origin in $Z \times Z \times Z$ than the original; if $s < 2t$, the second and third right-hand sides are primitive Pythagorean triples closer to the origin than the original. We note that with $[x'\ y'\ z']^T = F^{-1}([x\ y\ z]^T)$, $y' > z'$ if and only if $t^2 - (s - 2t)^2 > 2t(s - 2t)$, which is equivalent to $y > z$. Likewise, with $[x'\ y'\ z']^T = G^{-1}([x\ y\ z]^T)$, $y' > z'$ if and only if $t^2 - (2t - s)^2 > 2t(2t - s)$, which is equivalent to $s^2 - 6st + 7t^2 < 0$. This inequality holds if and only if $(3 - \sqrt{2})t < s < (3 + \sqrt{2})t$. This also implies that with $[x'\ y'\ z']^T = H^{-1}([x\ y\ z]^T)$, $y' > z'$ if and only if $s < (3 - \sqrt{2})t$. Hence, when y is even and z is odd, we find the previous Pythagorean triple by applying F^{-1} if $3t > s > 2t$, G^{-1} if $2t > s > (3 - \sqrt{2})t$, and H^{-1} if $(3 - 2\sqrt{2})t > s > t$.

Case 2. Similarly, if $y = s^2 - t^2$ and $z = 2st$, then $s^2 - t^2 > 2st$, implying that $s(s - 2t) > t^2 > 0$, so that $s > 2t$. We have

- $F^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = F^{-1}\left(\begin{bmatrix} s^2 + t^2 \\ s^2 - t^2 \\ 2st \end{bmatrix}\right) = \begin{bmatrix} t^2 + (s - 2t)^2 \\ 2t(s - 2t) \\ t^2 - (s - 2t)^2 \end{bmatrix}$
  $(3t > s > 2t)$;
- $G^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = G^{-1}\left(\begin{bmatrix} s^2 + t^2 \\ s^2 - t^2 \\ 2st \end{bmatrix}\right) = \begin{bmatrix} (s - 2t)^2 + t^2 \\ 2(s - 2t)t \\ (s - 2t)^2 - t^2 \end{bmatrix}$
  $(s > 3t)$;
- $H^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = H^{-1}\left(\begin{bmatrix} s^2 + t^2 \\ s^2 - t^2 \\ 2st \end{bmatrix}\right) = \begin{bmatrix} (s - 2t)^2 + t^2 \\ (s - 2t)^2 - t^2 \\ 2(s - 2t)t \end{bmatrix}$
  $(s > 3t)$.

Suppose that $(s,t)\neq(2,1)$. If $s<3t$, then the first right-hand side is a primitive Pythagorean triple closer to the origin than the original triple. If $s>3t$, the second and third right-hand sides are primitive Pythagorean triples closer to the origin than the original triple. With $[x'\ y'\ z']^T=F^{-1}([x\ y\ z]^T)$, $y'>z'$ if and only if $2t(s-2t)>t^2-(s-2t)^2$, which is equivalent to $y>z$. Similarly, $y'>z'$ if and only if $s<(3+\sqrt{2})t$ when $[x'\ y'\ z']^T=G^{-1}([x\ y\ z]^T)$, and $y'>z'$ if and only if $s>(3+\sqrt{2})t$, when $[z'\ y'\ z']^T=H^{-1}([x\ y\ z]^T)$. Hence, if y is odd and z is even, we find the previous triple by applying F^{-1} if $3t>s>2t$; G^{-1} if $(3+\sqrt{2})t>s>3t$, and H^{-1} if $s>(3+\sqrt{2})t$.

Thus, repeatedly applying F^{-1}, G^{-1}, and H^{-1} to a primitive Pythagorean triple yields a sequence of primitive Pythagorean triples, or equivalently a sequence of lattice points in the (s,t)-plane successively closer to the origin, terminating when $s=2$, $t=1$, which corresponds to the triple $(5,4,3)$. At each step, there is a unique choice of F^{-1}, G^{-1}, or H^{-1} for which $x'>y'>z'$. Reversing the process yields a unique path to (x,y,z) from $(5,4,3)$.

11. Find all pairs of integers (x,y) such that $y^2=x^3-p^2x$, where p is a prime such that $p\equiv 3\pmod 4$.

Solution based on an approach of George Apostolopoulos, Messolonghi, Greece, modified by the editor.

The equation can be rewritten as $y^2=(x-p)(x+p)x$. There are two cases.

Case 1. $p\nmid y$, Then $(p,(x-p)x(x+p))=1$. When x is even, then $x-p$, $x+p$ and x are pairwise relatively prime and so must all be squares. Since x is a multiple of 4 and $p\equiv 3\pmod 4$, it follows that $x+p\equiv 3\pmod 4$, and so cannot be square. Thus there are no solutions when x is even.

When x is odd, then $(x-p,x+p)=2$ so that $\frac{x-p}{2}$, $\frac{x+p}{2}$ and x are pairwise relatively prime. Their product is the integer $\left(\frac{y}{2}\right)^2$, so they are all squares. Let $x=r^2$, $x-p=2s^2$, $x+p=2t^2$. Then $t^2-s^2=p$, $t=\frac{p+1}{2}$ and $s=\frac{p-1}{2}$. Thus

$$r^2=(x-p)+p=\frac{(p-1)^2}{2}+p=\frac{p^2+1}{2},$$

whereupon $p^2-2r^2=-1$. The positive solutions of this pellian equation are given by $p_n+q_n\sqrt{2}=(1+\sqrt{2})(3+2\sqrt{2})^n$ for n a nonnegative integer. Since $p_{n+1}=6p_n-p_{n-1}$, it can be verified that $p_n\equiv(-1)^n\pmod 8$. The first few solutions are $(1,1)$, $(7,5)$, $(41,29)$, $(239,169)$. The equation is solvable if and only if n is odd and $p=p_n$ is a prime. For example, we obtain $(p,x,y)=(7,25,2\times 3\times 4\times 5)=(7,25,120)$ and $(p,x,y)=(239,169^2,2\times 119\times 120\times 169)=(239,28561,4826640)$.

Case 2. $p\mid y$. If $y=0$, then $(x,y)=(p,0),(-p,0),(0,0)$ are solutions. We show that there are no other solutions.

Suppose that $y\neq 0$. Since $p\mid y$, then $(x-p,x+p,x)=p$ and $p^2\mid y$. Dividing both sides of the equation by p^3, we have that $pb^2=(a-1)a(a+1)$, where $x=pa$, $y=p^2b$. Since $b\neq 0$, $a-1>0$.

The prime p divides exactly one of $a-1$, $a+1$, a. Since any pair of these three integers is co-prime, two of them must be squares that differ by either 1 or 2. But this is possible only when $a=0$ and $a=1$, both of which are excluded. The result follows.

Comment by the editor. The condition that $p \equiv 3 \pmod 4$ seems artificial. When p is any odd prime and does not divide y, we deduce as above that $x-p$, x and $x+p$ are all square. The only pair of squares that differ by p are $\left(\frac{p-1}{2}\right)^2$ and $\left(\frac{p+1}{2}\right)^2$. Thus there is no integer x for which all of $x-p$, x and $x+p$ are square and so there are no solutions when x is even and y is not a multiple of p.

When x is odd, we can pursue the foregoing argument to find solutions when $p \equiv 1 \pmod 8$, such as $(p,x,y) = (41, 29^2, 2\times 20\times 21\times 29) = (41, 841, 24360)$.

The reader is invited to examine the case that $p=2$.

12. Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the equation

$$f(x+y)+f(x)f(y) \;=\; (1+y)f(x)+(1+x)f(y)+f(xy)\,,$$

for all $x,y\in\mathbb{R}$.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Substituting $y=1$ into

$$f(x+y)+f(x)f(y) = (1+y)f(x)+(1+x)f(y)+f(xy) \qquad (1)$$

gives

$$f(x+1) = 3f(x)+f(1)[1+x-f(x)]. \qquad (2)$$

Substituting $y=-1$ into (1) gives

$$f(x-1) = f(-x)+f(-1)[1+x-f(x)]. \qquad (3)$$

Substituting $y=0$ into (1) gives

$$f(0)[f(x)-x-2]=0. \qquad (4)$$

Hence, either $f(0)=0$ or $f(x)=x+2$ for all real x. Substituting this last possibility into (1) yields the contradiction $2xy=0$ for all $x,y\in\mathbb{R}$. Thus

$$f(0)=0.$$

Setting $x=1$ and $y=-1$ in (1) gives $f(1)f(-1)=2f(-1)+f(-1)$, so that $f(-1)[3-f(1)]=0$. Hence, either

$$f(1)=3 \quad \text{or} \quad f(-1)=0.$$

We note for future reference that substituting, respectively, $y=x$ and $y=-x$ in (1) gives

$$f(2x)+[f(x)]^2 = 2(1+x)f(x)+f(x^2) \qquad (5)$$

and

$$f(x)f(-x) = (1-x)f(x) + (1+x)f(-x) + f(-x^2). \qquad (6)$$

Case 1. If $f(1) = 3$, then setting $y = 1$ gives $f(x+1) + 3f(x) = 2f(x) + 3(1+x) + f(x)$, which is equivalent to $f(x+1) = 3(1+x)$. Thus, for all real x,

$$f(x) = 3x.$$

Case 2. Now suppose $f(-1) = 0$ and $f(1) \neq 3$. From (3), we have

$$f(x-1) = f(-x),$$

so that, by replacing x with $x+1$,

$$f(x) = f(-x-1). \qquad (7)$$

Set $y = -x-1$ in (1). Then

$$[f(x)]^2 = -xf(x) + (1+x)f(x) + f(-x-x^2).$$

Hence,

$$f(-x-x^2) = [f(x)]^2 - f(x).$$

In particular, if $x = 1$, we have $f(-2) = [f(1)]^2 - f(1)$. But from (7), we have $f(-2) = f(1)$. Thus, $f(1) = [f(1)]^2 - f(1)$, so that $f(1)[f(1) - 2] = 0$. Therefore, either

$$f(1) = 0 \quad \text{or} \quad f(1) = 2.$$

Subcase (a). Suppose $f(1) = 0$. Equations (2) and (3) give

$$f(x+1) = 3f(x) \qquad (8)$$

and

$$f(x-1) = f(-x), \qquad (9)$$

respectively. Applying (8) with $x = 1$ yields

$$f(2) = 0. \qquad (10)$$

Thus, substituting $y = 2$ in (1) gives

$$f(x+2) = 3f(x) + f(2x). \qquad (11)$$

On the other hand, applying (8) twice yields

$$f(x+2) = 9f(x). \qquad (12)$$

From (11) and (12), we have

$$f(2x) = 6f(x). \qquad (13)$$

Replacing x with $x-1$ in (8) gives $f(x)=3f(-x)$; combining with (9) gives

$$f(-x)=\frac{1}{3}f(x). \tag{14}$$

Applying (14) and then (5) to (6) gives

$$\begin{aligned}
\frac{1}{3}[f(x)]^2 &= (1-x)f(x)+\frac{1}{3}(1+x)f(x)+\frac{1}{3}f(x^2)\\
\frac{1}{3}[f(x)]^2 &= (1-x)f(x)+\frac{1}{3}(1+x)f(x)\\
&\qquad +\frac{1}{3}\{f(2x)+[f(x)]^2-2(1+x)f(x)\}.
\end{aligned}$$

Simplifying gives

$$f(2x)=2(2x-1)f(x). \tag{15}$$

From (13) and (15), we obtain

$$\begin{aligned}
6f(x)&=2(2x-1)f(x)\\
4(2-x)f(x)&=0.
\end{aligned}$$

Since $f(2)=0$, this last equation implies that $f(x)$ is identically 0.

Subcase (b). Now suppose that $f(1)=2$. From (2) and (3),

$$f(x+1)=f(x)+2(1+x) \tag{16}$$

and

$$f(x-1)=f(-x). \tag{17}$$

Replacing x with $x-1$ in (16) and combining with (17) gives

$$f(-x)=f(x)-2x. \tag{18}$$

Applying (18) and then (5) to (6) gives

$$\begin{aligned}
f(x)[f(x)-2x] &= (1-x)f(x)+(1+x)[f(x)-2x]+f(x^2)-2x^2\\
f(x)[f(x)-2x] &= (1-x)f(x)+(1+x)[f(x)-2x]+f(2x)\\
&\qquad +[f(x)]^2-2(1+x)f(x)-2x^2\\
[f(x)]^2-2xf(x) &= 2f(x)-2x(1+x)+f(2x)+[f(x)]^2\\
&\qquad -2(1+x)f(x)-2x^2\\
0 &= -2x(1+x)+f(2x)-2x^2\\
f(2x) &= 4x^2+2x\\
f(2x) &= (2x)^2+(2x)
\end{aligned}$$

for all real x, so that

$$f(x)=x^2+x.$$

In summary, there are three possibilities, each of which is readily verified to satisfy (1).

- If $f(1) = 3$, then $f(x) = 3x$.
- If $f(-1) = 0$ and $f(1) = 0$, then $f(x) \equiv 0$.
- If $f(-1) = 0$ and $f(1) = 2$, then $f(x) = x^2 + x$.

Next we turn to the files of readers' solutions to problems given in the October 2010 number of the *Corner* and the Olimpiada Nacional Escolar de Matematica 2009, Level 1, given at [2010: 372–373].

1. If P, E, R and U represent digits different from 0 and pairwise different such that $\overline{PER} + \overline{PRU} + \overline{PUE} + 2009 = \overline{PERU}$, find all the values that $P + E + R + U$ can take.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the version of Zvonaru.

The given condition is equivalent to

$$\begin{aligned} 100P + 10E + R + 100P + 10R + U + 100P + 10U + E + 2009 & \\ = \; & 1000P + 100E + 10R + U; \\ 700P + 89E \; = \; & 2009 + \overline{UR}. \end{aligned}$$

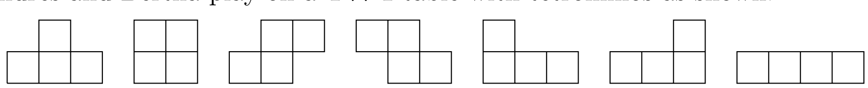
Since $2009 + \overline{UR} \le 2009 + 98$ and $89E \ge 89$, we deduce that $P \le 2$.

If $P = 1$, then we have $89E = 1309 + \overline{UR}$. We do not obtain a solution because $89E \le 89 \cdot 9 = 801$.

If $P = 2$, then we obtain $89E = 609 + \overline{UR}$. Since $609 + \overline{UR} \ge 609 + 12 = 621$ and $609 + \overline{UR} \le 609 + 98 = 707$, we deduce that $\frac{621}{89} \le E \le \frac{707}{89}$, hence $E = 7$.

It results that $\overline{PERU} = 2741$ and $P + E + R + U = 14$.

3. Andrés and Bertha play on a 4×4 table with tetrominos as shown.



Andrés begins the game placing 4 tetrominos of the same shape on the table without overlaps and leaving no empty space. Then Bertha must write on each square of the table one of the numbers 1, 2, 3 or 4 in such a way that each row and column has no two numbers repeated. Bertha wins if each tetromino on the chart covers 4 different numbers.

(a) Show that Bertha can always win the game.

(b) Andrés fills the table with 4 tetrominos where at least 2 are different. Is it true that in this situation, playing with the same rules, Bertha can always win?

Solved by Oliver Geupel, Brühl, NRW, Germany; and Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany (joint work).

Let $S_1, \ldots, S_7$ denote the given shapes from left to right. The table can not be covered with 4 pieces of shape S_3 or 4 pieces of shape S_4. The possible coverings with 4 pieces of the same shape S_1, S_2, S_5, S_6, S_7, respectively, are each similar to one of the assemblies shown in Figure 1 below. It is demonstrated in Figure 1 how Bertha can win the game (a) in each case.

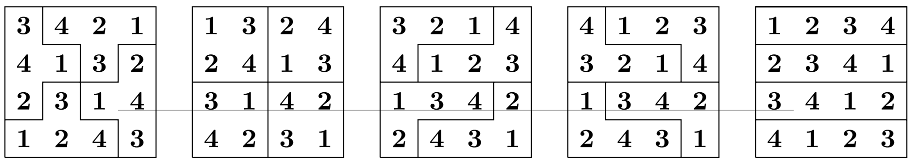


Figure 1

Andrés can win the game (b) if he puts the covering in Figure 2. For the proof, we use coordinates as in Figure 2. For example, (A, d) denotes the lower-left cell. With no loss of generality assume that Bertha writes the numbers in the upper-left piece according to Figure 2. In the lower-left piece, the number **1** can not occur in column A. Hence $(B, c) = 1$. Then (A, b) in the same piece cannot be **1**. Also (A, b) cannot be **3** or **4**, because both numbers occur in the same row. Thus $(A, b) = 2$. Also, $(D, a) = 4$ and $(C, a) = 3$. Now, the occurrence of **2** in the upper-right tetromino must be (D, c), with the consequence that there is no possible entry for (C, c). Therefore, Bertha loses the game (b).

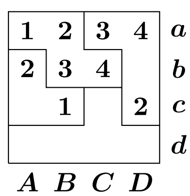


Figure 2

To complete this number of the *Corner* we look at solutions to the Olimpiada Nacional Escolar de Matematica 2009, Level 3 given at [2010 : 373–374].

1. For each positive integer N let $c(n)$ be the number of decimal digits of N. Let A be a set of positive integers such that if a and b are two distinct elements of A, then $c(a+b)+2 > c(a)+c(b)$. Find the largest number of elements that A can have.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the presentation by Curtis.

Note that $c(n)$ is a nondecreasing function, and that for each positive integer n, $10^{c(n)-1} \le n \le 10^{c(n)} - 1$. Hence, if $a \le b$, then

$$c(a+b) \le c(2b) \le c[2 \cdot (10^{c(b)} - 1)] \le c(10^{c(b)+1} - 1) = c(b) + 1.$$

Thus,

$$c(a+b) \le 1 + \max\{c(a), c(b)\}.$$

If $c(b) \ge c(a)$, then $3 + c(b) \ge c(a+b) + 2 > c(a) + c(b)$, so that $c(a) \in \{1, 2\}$. Similarly, if $c(a) > c(b)$, then $c(b) \in \{1, 2\}$. If $c(b) = c(a)$, then $c(b) = c(a) \in \{1, 2\}$.

Write the elements of A as

$$1 \le x_1 < x_2 < x_3 < \cdots .$$

If $i < j$, then $x_i < x_j$, implying that $c(x_i) \le c(x_j)$, so that $c(x_i) \in 1, 2$. Hence, A has at most one element greater than or equal to 100, so A is finite, and $\#(A) \le 100$. Write $A = \{x_i\}_{i=1}^{N}$. We have $N \le 100$. Suppose that $i < j$ and $c(x_i) = c(x_j) = 2$. Then $c(x_i + x_j) > 2$. In particular, the smallest two 2-digit numbers in A have sum at least 100. The table shows the maximum number of 2-digit numbers for various possible values of the smallest 2-digit number in A.

x_i	$\min x_j$	Max. num. of 2-digit num.
10	90	11
20	80	21
30	70	31
40	60	41
49	51	50
50	51	50
51	52	49

Thus, A can have at most nine 1-digit numbers, at most fifty 2-digit numbers, and at most one number with more than two digits, so $N \le 60$. In fact, A can have 60 numbers. One possible such set is

$$A = \{1, 2, 3, \ldots, 9, 50, 51, 52, \ldots, 99, 100\}.$$

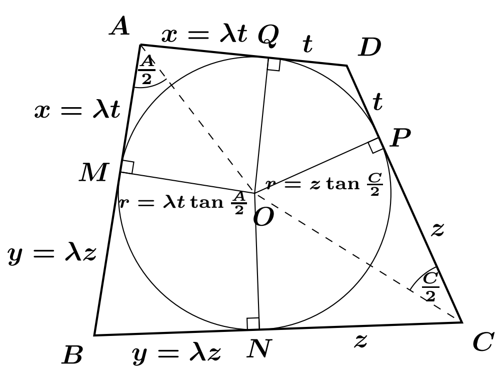
It is readily verified that any distinct pair $a, b \in A$ satisfies $c(a+b) + 2 > c(a) + c(b)$.

2. In a quadrilateral $ABCD$, a circle is drawn that is tangent to the sides AB, BC, CD and DA at the points M, N, P and Q respectively. Prove that if

$$(AM)(CP) = (BN)(DQ),$$

then $ABCD$ can be inscribed in a circle.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.



We put $AM = x$, $BN = y$, $CP = z$ and $DQ = t$. Then we have $xz = yt$, that is, $\frac{x}{t} = \frac{y}{z} = \lambda$, say, where λ is a real number. Thus, $x = \lambda t$ and $y = \lambda z$.

The tangents AM and AQ have the same length and similarly at the other vertices. Therefore

$$\begin{aligned} AB &= AM + MB = x + y = \lambda(t + z) \\ BC &= BN + NC = y + z = (\lambda + 1)z \\ CD &= CP + PD = z + t \\ DA &= DQ + QA = t + x = (\lambda + 1)t. \end{aligned}$$

By the law of cosines, applied to triangles ABD and BCD, we have

$$\begin{aligned} BD^2 &= [(\lambda + 1)t]^2 + [\lambda(t + z)]^2 - 2\lambda(\lambda + 1)t(t + z)\cos A \\ BD^2 &= [(\lambda + 1)z]^2 + (z + t)^2 - 2(\lambda + 1)z(z + t)\cos C \end{aligned}$$

or

$$(\lambda + 1)^2(t^2 - z^2) + (t + z)^2(\lambda^2 - 1) = 2(\lambda + 1)(z + t)(\lambda t \cos A - z \cos C)$$

that is

$$(\lambda+1)(z+t)[(\lambda+1)(t-z)+(\lambda-1)(t+z)] = 2(\lambda+1)(z+t)(\lambda t \cos A - z \cos C)$$

which simplifies to $\lambda t - z = \lambda t \cos A - z \cos C$ or $\lambda t(1 - \cos A) = z(1 - \cos C)$, which is equivalent to

$$2\lambda t \sin^2 \frac{A}{2} = 2z \sin^2 \frac{C}{2}\,. \tag{1}$$

We also have

$$\lambda t \tan \frac{A}{2} = z \tan \frac{C}{2} (= r) \tag{2}$$

where r is the radius of the inscribed circle in quadrilateral $ABCD$. From (1) and (2) we obtain, by division,

$$\begin{aligned} &\frac{2\lambda t \sin^2 \frac{A}{2}}{\lambda t \tan \frac{A}{2}} = \frac{2z \sin^2 \frac{C}{2}}{z \tan \frac{C}{2}} \Rightarrow \frac{2 \sin^2 \frac{A}{2}}{\tan \frac{A}{2}} = \frac{\sin^2 \frac{C}{2}}{\tan \frac{C}{2}} \\ &\Rightarrow 2 \sin \frac{A}{2} \cos \frac{A}{2} = 2 \sin \frac{C}{2} \cos \frac{C}{2} \end{aligned}$$

so

$$\sin A = \sin C\,. \tag{3}$$

Similarly, we find

$$\sin B = \sin D\,. \tag{4}$$

From (3), either $A = C$ or $A + C = 180°$. From (4), either $B = D$ or $B + D = 180°$.

Case (a). $A + D = 180°$ or $B + D = 180°$. In this case $ABCD$ is cyclic since opposite angles are supplementary.

Case (b). $A = C$ and $B = D$. Denote the center of the incircle of $ABCD$ by O. Right-angled triangles AMO, CPO and BNO, DQO are congruent with $x = z$ and $y = t$. Since $xz = yt$, we obtain $x = y = z = t$ making OM, ON, OP, OQ the perpendicular bisectors of segments AB, BC, CA, AD, respectively. Thus, $OA = OB = OC = OD$ making $ABCD$ cyclic.

3. (a) There are $\mathbf{8}$ points placed on a circle. We say that Juliana performs "operation T" if she chooses $\mathbf{3}$ such points and paints the sides of the triangle they determine in such a way that each painted triangle has at most one vertex in common with a previously painted triangle.

What is the greatest number of operations T that Juliana can make?

(b) If in part (a), if you have $\mathbf{7}$ points instead of $\mathbf{8}$ points, then what is the greatest number of operations T Juliana can make?

Solved by Oliver Geupel, Brühl, NRW, Germany.

We interpret the problem so that each painted triangle must have at most one vertex in common with *every* previously painted triangle. We claim that the greatest number of operations T is 8 in part (a) and 7 in part (b).

We start with the proof of part (a) and denote the points by the numbers $\mathbf{1, 2, \ldots, 8}$. A possible sequence of triangles of length 8 is

$$\mathbf{123,\ 456,\ 167,\ 148,\ 268,\ 347,\ 358,\ 257.}$$

On the other hand, consider a sequence of operations T. By assumption, each painted edge belongs to only one painted triangle. Therefore, each point from $\{\mathbf{1, 2, \ldots, 8}\}$ is adjacent with an even number of edges of painted triangles, hence, with not more than 6 edges. Then, the total number of painted edges is not greater than $\mathbf{8 \cdot 6/2 = 24}$. Consequently, we have not more than $\mathbf{24/3 = 8}$ triangles, which completes the proof of part (a).

It remains to prove part (b). Denote the points by the numbers $\mathbf{1, 2, \ldots, 7}$. A possible sequence of length 7 of triangles is

$$\mathbf{123,\ 345,\ 367,\ 146,\ 157,\ 247,\ 256.}$$

On the other hand, we have not more than $\binom{\mathbf{7}}{\mathbf{2}} = \mathbf{21}$ edges, hence not more than $\mathbf{21/3 = 7}$ steps, which completes the proof of part (b).

That completes the *Corner* for this issue.