# MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.



## Mayhem Problems

*Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le* **15 février 2012**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

*Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.*

*La rédaction souhaite remercier Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.*

**M488**. *Proposé par l'Équipe de Mayhem.*

Un triangle a les sommets $(x_1, y_1)$, $(x_2, y_2)$ et $(x_3, y_3)$.

(a) Si $x_1 < x_2 < x_3$ et $y_3 < y_1 < y_2$, déterminer la surface du triangle.

(b) Démontrer que si on laisse tomber les conditions sur $x_1$, $x_2$, $x_3$, $y_1$, $y_2$ et $y_3$, alors l'expression que vous avez fournie en (a) donne soit la surface, soit $-1$ fois la surface.

**M489**. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Démontrer que si $m$ et $n$ sont des entiers positifs relativement premiers tels que

$$m\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2010}\right)=n,$$

alors $2011$ divise $n$.

**M490**. *Proposé par Johan Gunardi, étudiant, SMPK 4 BPK PENABUR, Jakarta, Indonésie.*

Pour $n$ entier positif, soit $S(n)$ la somme des chiffres dans l'expression décimale (base $10$) de $n$. Soit $m$ un entier positif donné ; démontrer qu'il existe $n$ entier positif tel que $m = \dfrac{S(n^2)}{S(n)}$.

**M491**. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Soient $a$, $b$ et $c$ des constantes, pas nécessairement distinctes. Résoudre l'équation ci-bas :

$$\frac{(x-a)^2}{(x-a)^2-(b-c)^2}+\frac{(x-b)^2}{(x-b)^2-(c-a)^2}+\frac{(x-c)^2}{(x-c)^2-(a-b)^2}=1\,.$$

**M492**. *Proposé par Pedro Henrique O. Pantoja, étudiant, UFRN, Brésil.*

Démontrer que

$$\sum_{k=0}^{2009}(k+1)![6^k(6k+11)-k-1]=2011!\left(6^{2010}-1\right)\,.$$

**M493**. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Déterminer tous les entiers positifs $x$ qui satisfont à l'équation

$$\frac{x+\left[\sqrt{x}+\sqrt{x+1}\right]}{\left[\sqrt{4x+1}+4022\right]}+\frac{x}{\left[\sqrt{4x+2}\right]+4022}=1\,,$$

où $[x]$ est la partie entière de $x$.

**M494**. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit $z$ un nombre complexe tel que $|z|=2$. Déterminer la valeur minimum de $\left|z-\dfrac{1}{z}\right|$.

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**M488**. *Proposed by the Mayhem Staff.*

A triangle has vertices $(x_1,y_1)$, $(x_2,y_2)$, and $(x_3,y_3)$.

(a) If $x_1<x_2<x_3$ and $y_3<y_1<y_2$, determine the area of the triangle.

(b) Show that, if the conditions on $x_1$, $x_2$, $x_3$, $y_1$, $y_2$, and $y_3$ are dropped, the expression from (a) gives either the area or $-1$ times the area.

**M489**. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Prove that if $m$ and $n$ are relatively prime positive integers such that

$$m\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2010}\right)=n,$$

then $2011$ divides $n$.

**M490**. *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

For any positive integer $n$, let $S(n)$ denote the sum of the digits of $n$ (in base $10$). Given a positive integer $m$, prove that there exists a positive integer $n$ such that $m=\dfrac{S(n^2)}{S(n)}$.

**M491**. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let $a$, $b$, and $c$ be given constants, not necessarily distinct. Solve the equation below:

$$\frac{(x-a)^2}{(x-a)^2-(b-c)^2}+\frac{(x-b)^2}{(x-b)^2-(c-a)^2}+\frac{(x-c)^2}{(x-c)^2-(a-b)^2}=1\,.$$

**M492**. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Prove that

$$\sum_{k=0}^{2009}(k+1)![6^k(6k+11)-k-1]=2011!\left(6^{2010}-1\right)\,.$$

**M493**. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Find all positive integers $x$ that satisfy the equation

$$\frac{x+\left[\sqrt{x}+\sqrt{x+1}\right]}{\left[\sqrt{4x+1}+4022\right]}+\frac{x}{\left[\sqrt{4x+2}\right]+4022}=1\,,$$

where $[x]$ is the integer part of $x$.

**M494**. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

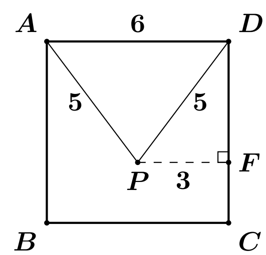
Let $z$ be a complex number such that $|z|=2$. Find the minimum value of $\left|z-\dfrac{1}{z}\right|$.

# Mayhem Solutions

## M451. *Proposed by the Mayhem Staff.*

Square $ABCD$ has side length $6$. Point $P$ is inside the square so that $AP = DP = 5$. Determine the length of $PC$.

*Solution by Florencio Cano Vargas, Inca, Spain.*




We draw the line parallel to $BC$ that passes through $P$. Let $F$ be the intersection point of this line with $DC$. Since $PA = PD$, the point $P$ lies on the perpendicular bisector of $AD$ and then $PF = \frac{AD}{2} = 3$. Triangle $DPF$ is a right-angled triangle, so:

$$DF = \sqrt{DP^2 + PF^2} = 4\,.$$

Moreover, triangle $CFP$ is also right angled, hence:

$$PC = \sqrt{PF^2 + CF^2} = \sqrt{3^2 + 2^2} = \sqrt{13}\,.$$

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JORGE ARMERO JIMÉNEZ, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; SCOTT BROWN, Auburn University, Montgomery, AL, USA; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; GERHARDT HINKLE, Student, Central High School, Springfield, MO, USA; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; HEEYOON KIM, Conyers, GA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; JAGDISH MADNANI, Bangalore, India; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.*

## M452. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

(a) Suppose that $x = \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}$ for some real number $a > 0$. Prove that $x^2 - a = x$.

(b) Determine the integer equal to

$$\frac{\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}}{\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}} - \sqrt{42 + \sqrt{42 + \sqrt{42 + \cdots}}}\,.$$

*Solution by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

(a) By squaring both sides of the equation we have,

$$x^2 = a + \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}} \Rightarrow x^2 - a = x\,.$$

(b) From part (a) we have that $\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}$ is the positive root of the equation $x^2 - x - 30 = 0$. We can factor the equation to get $(x-6)(x+5) = 0$, so $x = -5$ or $x = 6$. Since $x$ is positive, then

$$\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}} = 6\,.$$

Similarly, we get

$$\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}} = 3 \text{ and } \sqrt{42 + \sqrt{42 + \sqrt{42 + \cdots}}} = 7\,.$$

Hence

$$\frac{\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}}{\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}} - \sqrt{42 + \sqrt{42 + \sqrt{42 + \cdots}}} = \frac{6}{3} - 7 = -5\,.$$

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SCOTT BROWN, Auburn University, Montgomery, AL, USA; FLORENCIO CANO VARGAS, Inca, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; MARC FOSTER, student, Angelo State University, San Angelo, TX, USA; G.C. GREUBEL, Newport News, VA, USA; GERHARDT HINKLE, Student, Central High School, Springfield, MO, USA; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; HEEYOON KIM, Conyers, GA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ANDRÉS PLANELLS CÁRCEL, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer. One incorrect solution was received.*

**M453**. *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

Let $ABCD$ be a parallelogram. Sides $AB$ and $AD$ are extended to points $E$ and $F$ so that $E$, $C$, and $F$ lie on a straight line. In problem M447, we saw that $BE \cdot DF = AB \cdot AD$. Prove that

$$\sqrt{AE + AF} \;\geq\; \sqrt{AB} + \sqrt{AD}\,.$$

*Solution by George Apostolopoulos, Messolonghi, Greece.*

We have

$$\begin{aligned}(BE-DF)^2 \geq 0 &\iff BE^2+DF^2-2BE\cdot DF \geq 0\\ &\iff BE^2+DF^2+2BE\cdot DF-4BE\cdot DF \geq 0\\ &\iff (BE+DF)^2 \geq 4BE\cdot DF\\ &\iff (BE+DF)^2 \geq 4AB\cdot AD\\ &\iff BE+DF \geq 2\sqrt{AB\cdot AD}\\ &\iff (AE-AB)+(AF-AD) \geq 2\sqrt{AB\cdot AD}\\ &\iff AE+AF \geq AB+AD+2\sqrt{AB\cdot AD}\\ &\iff AE+AF \geq \sqrt{AB}^2+\sqrt{AD}^2+2\sqrt{AB\cdot AD}\\ &\iff (\sqrt{AE+AF})^2 \geq (\sqrt{AB}+\sqrt{AD})^2\end{aligned}$$

So $\sqrt{AE+AF} \geq \sqrt{AB}+\sqrt{AD}$.

*Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FLORENCIO CANO VARGAS, Inca, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; HEEYOON KIM, Conyers, GA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; JORGE SEVILLA LACRUZ, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.*

**M454**. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Determine all real numbers $x$ with $16^x+1=2^x+8^x$.

*Solution by Marc Foster and Travis B. Little, students, Angelo State University, San Angelo, TX, USA.*

The equation may be re-written in the following ways:

$$\begin{aligned}16^x+1&=8^x+2^x\,,\\ 2^x8^x+1&=8^x+2^x\,,\\ (2^x-1)(8^x-1)&=0\,.\end{aligned}$$

This implies that either $2^x=1$ or $8^x=1$. In either case, the solution is $x=0$, which is the solution to the original equation.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; HEEYOON KIM, Conyers, GA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta,* *Indonesia; LUIZ ERNESTO LEITÃO, Pará, Brazil; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; ARTURO PARDO PÉREZ, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.*

*Most of the other solvers converted everything to base* $\mathbf{2}$ *and solved the related equation* $y^4 + 1 = y + y^3$.

**M455**. *Proposed by Gheorghe Ghiţă, M. Eminescu National College, Buzău, Romania.*

Suppose that $n$ is a positive integer.

(a) If the positive integer $d$ is a divisor of each of the integers $n^2 + n + 1$ and $2n^3 + 3n^2 + 3n - 1$, prove that $d$ is also a divisor of $n^2 + n - 1$.

(b) Prove that the fraction $\dfrac{n^2 + n + 1}{2n^3 + 3n^2 + 3n - 1}$ is irreducible.

*Solution by Florencio Cano Vargas, Inca, Spain.*

(a) We can write:

$$n^2 + n - 1 = (2n^3 + 3n^2 + 3n - 1) - 2n(n^2 + n + 1)$$

Then if $d$ divides $2n^3 + 3n^2 + 3n - 1$ and $n^2 + n + 1$, then it also divides $n^2 + n - 1$.

(b) Let $d$ be a positive common divisor of the numerator and denominator. From (a) we know that $d$ also divides $n^2 + n - 1$. But $n^2 + n + 1$ and $n^2 + n - 1$ only differ by two units, so then $d \leq 2$. To discard the possibility $d = 2$, we note that $n^2 + n + 1 = n(n + 1) + 1$ is always odd since either $n$ or $n + 1$ is even. Then the only possibility that remains is $d = 1$ and the fraction is irreductible.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; JACLYN CHANG, student, University of Calgary, Calgary, AB (part (a) only); SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; LEI WANG, Missouri State University, Springfield, MO, USA; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer. One incorrect solution was received.*

**M456**. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $f$ and $g$ be real-valued functions with $g$ an odd function, $f(x) \leq g(x)$ for all real numbers $x$, and $f(x + y) \leq f(x) + f(y)$ for all real numbers $x$ and $y$. Prove that $f$ is an odd function.

*Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

From $f(x) \leq g(x)$ for any $x$, and from the oddness of $g(x)$ we get

$$f(-x) \leq g(-x) = -g(x) \leq -f(x)\,.$$

Moreover, we have

$$f(0) = f(x + (-x)) \leq f(x) + f(-x) \Longrightarrow f(-x) \geq f(0) - f(x)\,, \qquad (1)$$

and

$$f(x+0) \leq f(0) + f(x) \Longrightarrow f(0) \geq 0\,.$$

Since $f(0) \geq 0$, then from (1) we can conclude that $f(-x) \geq -f(x)$. Then we have

$$-f(x) \leq f(-x) \leq -f(x)\,.$$

Hence $f(x)$ is an odd function.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; JAVIER GARCÍA CAVERO, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.*

# Problem of the Month

Ian VanderBurgh

Here is a neat problem that has an easy-to-understand and appreciate real-life context and leads us to a good discussion of two different methods of solution.

**Problem** (2010 AMC 8) Everyday at school, Jo climbs a flight of **6** stairs. Jo can take the stairs **1**, **2**, or **3** at a time. For example, Jo could climb **3**, then **2**, then **1**. In how many ways can Jo climb the stairs?
(A) 13 (B) 18 (C) 20 (D) 22 (E) 24

I can just picture hundreds of Grade 8 students trying this after writing this contest in 2010... There are actually quite a lot of ways of climbing the stairs! The first step (ahem, no pun intended) is to read and understand the problem. As part of this, we should fiddle to see what ways we can find. We are looking for ways of adding combinations of **1**s, **2**s and **3**s to get **6**. Try playing with this for a couple of minutes!

What combinations did you get? Some that you might get include $\mathbf{3}+\mathbf{3}$, $\mathbf{3}+\mathbf{1}+\mathbf{1}+\mathbf{1}$, and $\mathbf{2}+\mathbf{3}+\mathbf{1}$. One question that we should immediately ask is whether re-arranging the order of a given sum makes a difference. Does it? Yes – for example, $\mathbf{2}+\mathbf{3}+\mathbf{1}$ (Jo takes **2** steps, then **3** steps, then **1** step) is different from $\mathbf{2}+\mathbf{1}+\mathbf{3}$ (**2** steps, then **1** step, then **3** steps) which are both different than $\mathbf{3}+\mathbf{1}+\mathbf{2}$, and so on. Can you find more ways to re-arrange this particular sum? The sum $\mathbf{3}+\mathbf{1}+\mathbf{1}+\mathbf{1}$ can also be re-arranged in a number of ways. How many can you find?

So it looks as if there are now two sub-problems – finding the different combinations of **1**s, **2**s and **3**s that give **6**, and then figuring out the number of ways in which we can re-arrange each of these combinations. Let's get a handle on the second sub-problem first.

To do this, we'll consider a slightly different context: How many different "words" can be made from the letters of AAAAB, AAAC, AABB, and ABC? (By a "word" in this case, we mean a rearrangement of the letters; it doesn't actually have to form a real word!) In each case, we could exhaustively list out the possibilities or look for a different approach:

- AAAAB: **5** words
  *List*: AAAAB, AAABA, AABAA, ABAAA, BAAAA
  *Alternate approach*: If we start with the four As (AAAA), there are then five possible positions for the B: either before the first A or after each of the four As. Thus, there are five words.

- AAAC: **4** words
  *List*: AAAC, AACA, ACAA, CAAA
  *Alternate approach*: Can you modify the previous argument to fit this case?

- AABB: **6** words
  *List*: AABB, ABAB, ABBA, BAAB, BABA, BBAA
  *Alternate approach*: While there are good ways to count the words in this case using more advanced mathematics like combinatorics, actually coming up with a simple explanation of why the answer is **6** without actually just doing it isn't that easy. Here's one try. Suppose that the word starts with A. Put in the A and the two Bs to get ABB; the remaining A can go in three places (right before the first B or after either B). So there are **3** words beginning with A. Can you see why there are also **3** words beginning with B?

- ABC: **6** words
  *List*: ABC, ACB, BAC, BCA, CAB, CBA
  *Alternate approach*: There are **3** possibilities for the first letter; for each of these, there are **2** possibilities for the second letter (all but the letter we already chose); the last letter is then completely determined. This tells us that there are $\mathbf{3}\times\mathbf{2}\times\mathbf{1}=\mathbf{6}$ possible words.

Now let's combine this information about re-arrangements with a systematic way of finding the different combinations.

**Solution 1.** Let's find the possible combinations in an organized way. We'll start by assuming that the order of steps doesn't actually matter, and then incorporate the order at the end. Coming up with a good way to find all of the combinations might require a bit of fiddling. One good method to use is to organize these by the number of **1**s.

Can there be six **1**s? Yes: $\mathbf{1+1+1+1+1+1=6}$.

Can there be five **1**s? No: if there are five **1**s, then Jo has only **1** step left for which she needs another **1**.

Can there be four **1**s? Yes: if there are four **1**s, then Jo has **2** steps left, which must be taken up by a **2**. (It can't be two **1**s since we're only allowed four **1**s.) This gives us $\mathbf{1+1+1+1+2}$.

Can there be three **1**s? Yes: if there are three **1**s, then Jo has **3** steps left. We can't divide the **3** into two pieces without using a **1**, so the only way is $\mathbf{1+1+1+3}$.

Can there be two **1**s? Yes: if there are two **1**s, then Jo has **4** steps left. To avoid using a **1**, this **4** must be $\mathbf{2+2}$. This gives us $\mathbf{1+1+2+2}$.

Can there be one **1**? Yes: with **5** steps left and not using a **1**, the remaining **5** must be $\mathbf{2+3}$. This gives us $\mathbf{1+2+3}$.

Can there be zero **1**s? Yes. If there are no **2**s, then there are only **3**s, so we have $\mathbf{3+3}$. If there is a **2**, then Jo has **4** steps left, which must be $\mathbf{2+2}$ since no **1**s are used. In this case, we have $\mathbf{3+3}$ or $\mathbf{2+2+2}$.

So ignoring order, the possibilities are (i) $\mathbf{1+1+1+1+1+1}$, (ii) $\mathbf{1+1+1+1+2}$, (iii) $\mathbf{1+1+1+3}$, (iv) $\mathbf{1+1+2+2}$, (v) $\mathbf{1+2+3}$, (vi) $\mathbf{3+3}$, and (vii) $\mathbf{2+2}$. The combinations in (i), (vi) and (vii) can't be re-arranged in any other order. That gives us **1** way in each case.

How can we re-arrange the sums in (ii), (iii), (iv), and (v)? There are **5** ways of arranging the sum in (ii). This is because this sum can be related to the word AAAAB from before, with each A representing a **1** and B representing the **2**. Each re-arrangement of the sum in (ii) is the same as one of the words that we talked about earlier. Since there were **5** words, then there are **5** ways of arranging the sum.

Can you see how to relate the sums in (iii), (iv) and (v) to the words earlier? Try this out! You'll find that the sum in (iii) can be arranged in **4** ways, and the sum in each of (iv) and (v) can be arranged in **6** ways.

Therefore, there are $\mathbf{1+5+4+6+6+1+1=24}$ ways that Jo can climb the stairs. □

While there was a fair bit of work required to actually make that solution work, we didn't have to do anything really hard. But, we had to be very, very careful. Also, this method might not "scale up" very well to a larger number of steps, because of the number of cases that we had to consider.

Let's switch gears. Sometimes looking at smaller cases helps in one of two ways: either by showing us a pattern that might continue or more directly by allowing us to capitalize on these smaller cases.

What do smaller cases look like here? They are cases with fewer stairs. Let's try a few:

- With **1** stair, there is only **1** way for Jo to climb.
- With **2** stairs, there are **2** ways: $\mathbf{1+1}$ and **2**.
- With **3** stairs, there are **4** ways: $\mathbf{1+1+1}$, $\mathbf{1+2}$, $\mathbf{2+1}$, and **3**.
- With **4** stairs, there are **7** ways: $\mathbf{1+1+1+1}$, $\mathbf{1+1+2}$, $\mathbf{1+2+1}$, $\mathbf{2+1+1}$, $\mathbf{1+3}$, $\mathbf{3+1}$, and $\mathbf{2+2}$.

Do you notice anything about the number of ways in these four cases? Do you think that it is a coincidence that $\mathbf{1+2+4=7}$? In other words, is it a coincidence that the sum of the numbers of ways for **1**, **2** and **3** stairs gives us the number of ways for **4** stairs?

**Solution 2.** We have seen that with **1**, **2** and **3** stairs, there are **1**, **2** and **4** ways, respectively.

If Jo is to climb **4** stairs, then she starts by climbing **1** stair (leaving **3**) or by climbing **2** stairs (leaving **2**) or by climbing **3** stairs (leaving **1**).

If she starts by climbing **1** stair, then the number of ways that she can finish climbing is the number of ways to climb the remaining **3** stairs. In other words, the number of ways that she can climb the stairs staring with **1** stair is equal to the number of ways in which she can climb **3** stairs. (There are **4** ways to do this.)

If she starts by climbing **2** stairs, then the number of ways that she can finish climbing is the number of ways to climb the remaining **2** stairs. In other words, the number of ways that she can climb the stairs staring with **2** stairs is equal to the number of ways in which she can climb **2** stairs. (There are **2** ways.)

Similarly, the number of ways of climbing starting with **3** stairs is equal to the number of ways of climbing the remaining **1** stair. (There is **1** way.)

Therefore, the number of ways of climbing **4** stairs equals the sum of the number of ways of climbing **3**, **2** and **1** stairs, or $\mathbf{4+2+1=7}$.

What happens with **5** stairs? In this case, Jo starts with **1**, **2** or **3** stairs, leaving **4**, **3** or **2** stairs. Using a similar argument, the total number of ways of climbing **5** stairs equals the sum of the number of ways of climbing **4**, **3** and **2** stairs, or $\mathbf{7+4+2=13}$.

Continuing along these lines, for **6** stairs, the total number of ways will equal the sum of the number of ways of climbing **5**, **4** and **3** stairs, or $\mathbf{13+7+4=24}$. □

We've just used a method called *recursion* in this second solution. This can be a very powerful approach in cases where it will work. Recursion is particularly useful in fields like computer science.

I'll leave you with a challenge. Can you determine the number of ways that Jo could go up the stairs if there were **10** stairs? Which approach do think that you'd want to use?