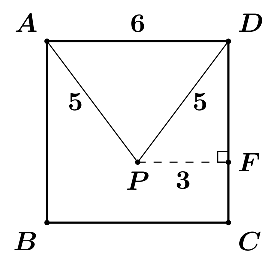
# Mayhem Solutions

## M451. *Proposed by the Mayhem Staff.*

Square $ABCD$ has side length $6$. Point $P$ is inside the square so that $AP = DP = 5$. Determine the length of $PC$.

*Solution by Florencio Cano Vargas, Inca, Spain.*




We draw the line parallel to $BC$ that passes through $P$. Let $F$ be the intersection point of this line with $DC$. Since $PA = PD$, the point $P$ lies on the perpendicular bisector of $AD$ and then $PF = \frac{AD}{2} = 3$. Triangle $DPF$ is a right-angled triangle, so:

$$DF = \sqrt{DP^2 + PF^2} = 4\,.$$

Moreover, triangle $CFP$ is also right angled, hence:

$$PC = \sqrt{PF^2 + CF^2} = \sqrt{3^2 + 2^2} = \sqrt{13}\,.$$

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JORGE ARMERO JIMÉNEZ, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; SCOTT BROWN, Auburn University, Montgomery, AL, USA; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; GERHARDT HINKLE, Student, Central High School, Springfield, MO, USA; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; HEEYOON KIM, Conyers, GA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; JAGDISH MADNANI, Bangalore, India; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.*

## M452. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

(a) Suppose that $x = \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}$ for some real number $a > 0$. Prove that $x^2 - a = x$.

(b) Determine the integer equal to

$$\frac{\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}}{\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}} - \sqrt{42 + \sqrt{42 + \sqrt{42 + \cdots}}}\,.$$

*Solution by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

(a) By squaring both sides of the equation we have,

$$x^2 = a + \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}} \Rightarrow x^2 - a = x\,.$$

(b) From part (a) we have that $\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}$ is the positive root of the equation $x^2 - x - 30 = 0$. We can factor the equation to get $(x-6)(x+5) = 0$, so $x = -5$ or $x = 6$. Since $x$ is positive, then

$$\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}} = 6\,.$$

Similarly, we get

$$\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}} = 3 \text{ and } \sqrt{42 + \sqrt{42 + \sqrt{42 + \cdots}}} = 7\,.$$

Hence

$$\frac{\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}}{\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}} - \sqrt{42 + \sqrt{42 + \sqrt{42 + \cdots}}} = \frac{6}{3} - 7 = -5\,.$$

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SCOTT BROWN, Auburn University, Montgomery, AL, USA; FLORENCIO CANO VARGAS, Inca, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; MARC FOSTER, student, Angelo State University, San Angelo, TX, USA; G.C. GREUBEL, Newport News, VA, USA; GERHARDT HINKLE, Student, Central High School, Springfield, MO, USA; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; HEEYOON KIM, Conyers, GA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ANDRÉS PLANELLS CÁRCEL, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer. One incorrect solution was received.*

**M453**. *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

Let $ABCD$ be a parallelogram. Sides $AB$ and $AD$ are extended to points $E$ and $F$ so that $E$, $C$, and $F$ lie on a straight line. In problem M447, we saw that $BE \cdot DF = AB \cdot AD$. Prove that

$$\sqrt{AE + AF} \;\geq\; \sqrt{AB} + \sqrt{AD}\,.$$

*Solution by George Apostolopoulos, Messolonghi, Greece.*

We have

$$\begin{aligned}(BE-DF)^2\geq 0 &\iff BE^2+DF^2-2BE\cdot DF\geq 0\\ &\iff BE^2+DF^2+2BE\cdot DF-4BE\cdot DF\geq 0\\ &\iff (BE+DF)^2\geq 4BE\cdot DF\\ &\iff (BE+DF)^2\geq 4AB\cdot AD\\ &\iff BE+DF\geq 2\sqrt{AB\cdot AD}\\ &\iff (AE-AB)+(AF-AD)\geq 2\sqrt{AB\cdot AD}\\ &\iff AE+AF\geq AB+AD+2\sqrt{AB\cdot AD}\\ &\iff AE+AF\geq \sqrt{AB}^2+\sqrt{AD}^2+2\sqrt{AB\cdot AD}\\ &\iff (\sqrt{AE+AF})^2\geq(\sqrt{AB}+\sqrt{AD})^2\end{aligned}$$

So $\sqrt{AE+AF}\geq\sqrt{AB}+\sqrt{AD}$.

*Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FLORENCIO CANO VARGAS, Inca, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; HEEYOON KIM, Conyers, GA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; JORGE SEVILLA LACRUZ, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.*

**M454**. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Determine all real numbers $x$ with $16^x+1=2^x+8^x$.

*Solution by Marc Foster and Travis B. Little, students, Angelo State University, San Angelo, TX, USA.*

The equation may be re-written in the following ways:

$$\begin{aligned}16^x+1&=8^x+2^x\,,\\ 2^x8^x+1&=8^x+2^x\,,\\ (2^x-1)(8^x-1)&=0\,.\end{aligned}$$

This implies that either $2^x=1$ or $8^x=1$. In either case, the solution is $x=0$, which is the solution to the original equation.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; HEEYOON KIM, Conyers, GA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta,* *Indonesia; LUIZ ERNESTO LEITÃO, Pará, Brazil; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; ARTURO PARDO PÉREZ, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.*

*Most of the other solvers converted everything to base* $\mathbf{2}$ *and solved the related equation* $y^4 + 1 = y + y^3$.

**M455**. *Proposed by Gheorghe Ghiţă, M. Eminescu National College, Buzău, Romania.*

Suppose that $n$ is a positive integer.

(a) If the positive integer $d$ is a divisor of each of the integers $n^2 + n + 1$ and $2n^3 + 3n^2 + 3n - 1$, prove that $d$ is also a divisor of $n^2 + n - 1$.

(b) Prove that the fraction $\dfrac{n^2 + n + 1}{2n^3 + 3n^2 + 3n - 1}$ is irreducible.

*Solution by Florencio Cano Vargas, Inca, Spain.*

(a) We can write:

$$n^2 + n - 1 = (2n^3 + 3n^2 + 3n - 1) - 2n(n^2 + n + 1)$$

Then if $d$ divides $2n^3 + 3n^2 + 3n - 1$ and $n^2 + n + 1$, then it also divides $n^2 + n - 1$.

(b) Let $d$ be a positive common divisor of the numerator and denominator. From (a) we know that $d$ also divides $n^2 + n - 1$. But $n^2 + n + 1$ and $n^2 + n - 1$ only differ by two units, so then $d \leq 2$. To discard the possibility $d = 2$, we note that $n^2 + n + 1 = n(n + 1) + 1$ is always odd since either $n$ or $n + 1$ is even. Then the only possibility that remains is $d = 1$ and the fraction is irreductible.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; JACLYN CHANG, student, University of Calgary, Calgary, AB (part (a) only); SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; LEI WANG, Missouri State University, Springfield, MO, USA; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer. One incorrect solution was received.*

**M456**. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $f$ and $g$ be real-valued functions with $g$ an odd function, $f(x) \leq g(x)$ for all real numbers $x$, and $f(x + y) \leq f(x) + f(y)$ for all real numbers $x$ and $y$. Prove that $f$ is an odd function.

*Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

From $f(x) \le g(x)$ for any $x$, and from the oddness of $g(x)$ we get

$$f(-x) \le g(-x) = -g(x) \le -f(x)\,.$$

Moreover, we have

$$f(0) = f(x + (-x)) \le f(x) + f(-x) \Longrightarrow f(-x) \ge f(0) - f(x)\,, \quad (1)$$

and

$$f(x+0) \le f(0) + f(x) \Longrightarrow f(0) \ge 0\,.$$

Since $f(0) \ge 0$, then from (1) we can conclude that $f(-x) \ge -f(x)$. Then we have

$$-f(x) \le f(-x) \le -f(x)\,.$$

Hence $f(x)$ is an odd function.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; JAVIER GARCÍA CAVERO, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.*

# Problem of the Month

Ian VanderBurgh

Here is a neat problem that has an easy-to-understand and appreciate real-life context and leads us to a good discussion of two different methods of solution.

**Problem** (2010 AMC 8) Everyday at school, Jo climbs a flight of $\mathbf{6}$ stairs. Jo can take the stairs $\mathbf{1}$, $\mathbf{2}$, or $\mathbf{3}$ at a time. For example, Jo could climb $\mathbf{3}$, then $\mathbf{2}$, then $\mathbf{1}$. In how many ways can Jo climb the stairs?
(A) 13 (B) 18 (C) 20 (D) 22 (E) 24

I can just picture hundreds of Grade 8 students trying this after writing this contest in 2010... There are actually quite a lot of ways of climbing the stairs! The first step (ahem, no pun intended) is to read and understand the problem. As part of this, we should fiddle to see what ways we can find. We are looking for ways of adding combinations of $\mathbf{1}$s, $\mathbf{2}$s and $\mathbf{3}$s to get $\mathbf{6}$. Try playing with this for a couple of minutes!