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From $f(x) \leq g(x)$ for any x , and from the oddness of $g(x)$ we get

$$f(-x) \leq g(-x) = -g(x) \leq -f(x).$$

Moreover, we have

$$f(0) = f(x + (-x)) \leq f(x) + f(-x) \implies f(-x) \geq f(0) - f(x), \quad (1)$$

and

$$f(x + 0) \leq f(0) + f(x) \implies f(0) \geq 0.$$

Since $f(0) \geq 0$, then from (1) we can conclude that $f(-x) \geq -f(x)$. Then we have

$$-f(x) \leq f(-x) \leq -f(x).$$

Hence $f(x)$ is an odd function.

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Problem of the Month

Ian VanderBurgh

Here is a neat problem that has an easy-to-understand and appreciate real-life context and leads us to a good discussion of two different methods of solution.

Problem (2010 AMC 8) Everyday at school, Jo climbs a flight of **6** stairs. Jo can take the stairs **1**, **2**, or **3** at a time. For example, Jo could climb **3**, then **2**, then **1**. In how many ways can Jo climb the stairs?

(A) 13 (B) 18 (C) 20 (D) 22 (E) 24

I can just picture hundreds of Grade 8 students trying this after writing this contest in 2010... There are actually quite a lot of ways of climbing the stairs! The first step (ahem, no pun intended) is to read and understand the problem. As part of this, we should fiddle to see what ways we can find. We are looking for ways of adding combinations of **1s**, **2s** and **3s** to get **6**. Try playing with this for a couple of minutes!

What combinations did you get? Some that you might get include $3 + 3$, $3 + 1 + 1 + 1$, and $2 + 3 + 1$. One question that we should immediately ask is whether re-arranging the order of a given sum makes a difference. Does it? Yes – for example, $2 + 3 + 1$ (Jo takes **2** steps, then **3** steps, then **1** step) is different from $2 + 1 + 3$ (**2** steps, then **1** step, then **3** steps) which are both different than $3 + 1 + 2$, and so on. Can you find more ways to re-arrange this particular sum? The sum $3 + 1 + 1 + 1$ can also be re-arranged in a number of ways. How many can you find?

So it looks as if there are now two sub-problems – finding the different combinations of **1**s, **2**s and **3**s that give **6**, and then figuring out the number of ways in which we can re-arrange each of these combinations. Let’s get a handle on the second sub-problem first.

To do this, we’ll consider a slightly different context: How many different “words” can be made from the letters of AAAAB, AAAC, AABB, and ABC? (By a “word” in this case, we mean a rearrangement of the letters; it doesn’t actually have to form a real word!) In each case, we could exhaustively list out the possibilities or look for a different approach:

- AAAAB: **5** words
List: AAAAB, AAABA, AABAA, ABAAA, BAAAA
Alternate approach: If we start with the four As (AAAA), there are then five possible positions for the B: either before the first A or after each of the four As. Thus, there are five words.
- AAAC: **4** words
List: AAAC, AACA, ACAA, CAAA
Alternate approach: Can you modify the previous argument to fit this case?
- AABB: **6** words
List: AABB, ABAB, ABBA, BAAB, BABA, BBAA
Alternate approach: While there are good ways to count the words in this case using more advanced mathematics like combinatorics, actually coming up with a simple explanation of why the answer is **6** without actually just doing it isn’t that easy. Here’s one try. Suppose that the word starts with A. Put in the A and the two Bs to get ABB; the remaining A can go in three places (right before the first B or after either B). So there are **3** words beginning with A. Can you see why there are also **3** words beginning with B?
- ABC: **6** words
List: ABC, ACB, BAC, BCA, CAB, CBA
Alternate approach: There are **3** possibilities for the first letter; for each of these, there are **2** possibilities for the second letter (all but the letter we already chose); the last letter is then completely determined. This tells us that there are $3 \times 2 \times 1 = 6$ possible words.

Now let’s combine this information about re-arrangements with a systematic way of finding the different combinations.

Solution 1. Let's find the possible combinations in an organized way. We'll start by assuming that the order of steps doesn't actually matter, and then incorporate the order at the end. Coming up with a good way to find all of the combinations might require a bit of fiddling. One good method to use is to organize these by the number of **1s**.

Can there be six **1s**? Yes: $1 + 1 + 1 + 1 + 1 + 1 = 6$.

Can there be five **1s**? No: if there are five **1s**, then Jo has only **1** step left for which she needs another **1**.

Can there be four **1s**? Yes: if there are four **1s**, then Jo has **2** steps left, which must be taken up by a **2**. (It can't be two **1s** since we're only allowed four **1s**.) This gives us $1 + 1 + 1 + 1 + 2$.

Can there be three **1s**? Yes: if there are three **1s**, then Jo has **3** steps left. We can't divide the **3** into two pieces without using a **1**, so the only way is $1 + 1 + 1 + 3$.

Can there be two **1s**? Yes: if there are two **1s**, then Jo has **4** steps left. To avoid using a **1**, this **4** must be $2 + 2$. This gives us $1 + 1 + 2 + 2$.

Can there be one **1**? Yes: with **5** steps left and not using a **1**, the remaining **5** must be $2 + 3$. This gives us $1 + 2 + 3$.

Can there be zero **1s**? Yes. If there are no **2s**, then there are only **3s**, so we have $3 + 3$. If there is a **2**, then Jo has **4** steps left, which must be $2 + 2$ since no **1s** are used. In this case, we have $3 + 3$ or $2 + 2 + 2$.

So ignoring order, the possibilities are (i) $1 + 1 + 1 + 1 + 1 + 1$, (ii) $1 + 1 + 1 + 1 + 2$, (iii) $1 + 1 + 1 + 3$, (iv) $1 + 1 + 2 + 2$, (v) $1 + 2 + 3$, (vi) $3 + 3$, and (vii) $2 + 2$. The combinations in (i), (vi) and (vii) can't be re-arranged in any other order. That gives us **1** way in each case.

How can we re-arrange the sums in (ii), (iii), (iv), and (v)? There are **5** ways of arranging the sum in (ii). This is because this sum can be related to the word AAAAB from before, with each A representing a **1** and B representing the **2**. Each re-arrangement of the sum in (ii) is the same as one of the words that we talked about earlier. Since there were **5** words, then there are **5** ways of arranging the sum.

Can you see how to relate the sums in (iii), (iv) and (v) to the words earlier? Try this out! You'll find that the sum in (iii) can be arranged in **4** ways, and the sum in each of (iv) and (v) can be arranged in **6** ways.

Therefore, there are $1 + 5 + 4 + 6 + 6 + 1 + 1 = 24$ ways that Jo can climb the stairs. \square

While there was a fair bit of work required to actually make that solution work, we didn't have to do anything really hard. But, we had to be very, very careful. Also, this method might not "scale up" very well to a larger number of steps, because of the number of cases that we had to consider.

Let's switch gears. Sometimes looking at smaller cases helps in one of two ways: either by showing us a pattern that might continue or more directly by allowing us to capitalize on these smaller cases.

What do smaller cases look like here? They are cases with fewer stairs. Let's try a few:

- With **1** stair, there is only **1** way for Jo to climb.
- With **2** stairs, there are **2** ways: **1 + 1** and **2**.
- With **3** stairs, there are **4** ways: **1 + 1 + 1**, **1 + 2**, **2 + 1**, and **3**.
- With **4** stairs, there are **7** ways: **1 + 1 + 1 + 1**, **1 + 1 + 2**, **1 + 2 + 1**, **2 + 1 + 1**, **1 + 3**, **3 + 1**, and **2 + 2**.

Do you notice anything about the number of ways in these four cases? Do you think that it is a coincidence that $1+2+4 = 7$? In other words, is it a coincidence that the sum of the numbers of ways for **1**, **2** and **3** stairs gives us the number of ways for **4** stairs?

Solution 2. We have seen that with **1**, **2** and **3** stairs, there are **1**, **2** and **4** ways, respectively.

If Jo is to climb **4** stairs, then she starts by climbing **1** stair (leaving **3**) or by climbing **2** stairs (leaving **2**) or by climbing **3** stairs (leaving **1**).

If she starts by climbing **1** stair, then the number of ways that she can finish climbing is the number of ways to climb the remaining **3** stairs. In other words, the number of ways that she can climb the stairs starting with **1** stair is equal to the number of ways in which she can climb **3** stairs. (There are **4** ways to do this.)

If she starts by climbing **2** stairs, then the number of ways that she can finish climbing is the number of ways to climb the remaining **2** stairs. In other words, the number of ways that she can climb the stairs starting with **2** stairs is equal to the number of ways in which she can climb **2** stairs. (There are **2** ways.)

Similarly, the number of ways of climbing starting with **3** stairs is equal to the number of ways of climbing the remaining **1** stair. (There is **1** way.)

Therefore, the number of ways of climbing **4** stairs equals the sum of the number of ways of climbing **3**, **2** and **1** stairs, or $4 + 2 + 1 = 7$.

What happens with **5** stairs? In this case, Jo starts with **1**, **2** or **3** stairs, leaving **4**, **3** or **2** stairs. Using a similar argument, the total number of ways of climbing **5** stairs equals the sum of the number of ways of climbing **4**, **3** and **2** stairs, or $7 + 4 + 2 = 13$.

Continuing along these lines, for **6** stairs, the total number of ways will equal the sum of the number of ways of climbing **5**, **4** and **3** stairs, or $13 + 7 + 4 = 24$.
□

We've just used a method called *recursion* in this second solution. This can be a very powerful approach in cases where it will work. Recursion is particularly useful in fields like computer science.

I'll leave you with a challenge. Can you determine the number of ways that Jo could go up the stairs if there were **10** stairs? Which approach do you think that you'd want to use?