

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3526. [2010 : 109, 111] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a , b , and c be positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{a}{\sqrt{a^2 + 2(b+c)^2}} \geq 1.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

By Hölder inequality we have

$$\left(\sum_{\text{cyclic}} \frac{a}{\sqrt{a^2 + 2(b+c)^2}} \right)^2 \left(\sum_{\text{cyclic}} a(a^2 + 2(b+c)^2) \right) \geq (a+b+c)^3.$$

Thus we only need to show that

$$(a+b+c)^3 \geq \sum_{\text{cyclic}} a(a^2 + 2(b+c)^2),$$

or equivalently that

$$a^2b + a^2c + ab^2 + ac^2 + b^2c + bc^2 \geq 6abc.$$

But this is immediate from the AM-GM inequality. This completes the proof.

Equality holds if and only if $a = b = c$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece (2 solutions); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3539. [2010 : 239, 241] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.*

Let \mathbf{A} and \mathbf{B} be 2×2 square matrices with real entries. Prove that the equations $\det(\mathbf{x}\mathbf{A} \pm \mathbf{B}) = 0$ have all of their roots real if and only if

$$[\text{trace}(\mathbf{AB}) - \text{trace}(\mathbf{A})\text{trace}(\mathbf{B})]^2 \geq 4 \det(\mathbf{A}) \det(\mathbf{B}).$$

Solution by the Henry Ricardo, Tappan, NY, USA.

The result is false without further conditions on the matrix \mathbf{A} . For example, letting $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, we find that $\det(\mathbf{x}\mathbf{A} \pm \mathbf{B}) = \det \begin{bmatrix} \mathbf{x} \pm 1 & \pm 1 \\ \mp 1 & 0 \end{bmatrix} = 1$, so that $\det(\mathbf{x}\mathbf{A} \pm \mathbf{B}) = 0$ has *no* roots despite the fact that $[\text{trace}(\mathbf{AB}) - \text{trace}(\mathbf{A})\text{trace}(\mathbf{B})]^2 = 0 = 4 \det(\mathbf{A}) \det(\mathbf{B})$.

If \mathbf{A} is a 2×2 matrix with real entries, we can use the easily proved result (see Fact 4.9.3 in *Matrix Mathematics (Second Edition)* by Dennis S. Bernstein, Princeton University Press, 2009) that

$$\det(\mathbf{A} + \mathbf{B}) - \det(\mathbf{A}) - \det(\mathbf{B}) = \text{trace}(\mathbf{A})\text{trace}(\mathbf{B}) - \text{trace}(\mathbf{AB}).$$

Thus, using basic properties of the determinant and trace,

$$\det(\mathbf{x}\mathbf{A} \pm \mathbf{B}) = \det(\mathbf{A})\mathbf{x}^2 \pm (\text{trace}(\mathbf{A})\text{trace}(\mathbf{B}) - \text{trace}(\mathbf{AB}))\mathbf{x} + \det(\mathbf{B}).$$

Now if \mathbf{A} is *nonsingular*, the equations $\det(\mathbf{x}\mathbf{A} \pm \mathbf{B}) = 0$ have all of their roots real if and only if $[\text{trace}(\mathbf{AB}) - \text{trace}(\mathbf{A})\text{trace}(\mathbf{B})]^2 \geq 4 \det(\mathbf{A}) \det(\mathbf{B})$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MURIEL BAKER, student, Southeast Missouri State University, Cape Girardeau, Missouri, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; HTET NAING LIN, Southeast Missouri State University, Cape Girardeau, Missouri, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JAMES MEYER, student, Southeast Missouri State University, Cape Girardeau, Missouri, USA; CRISTINEL MORTICI, Valahia University of Târgoviște, Romania; JOHN POSTL, St. Bonaventure University, St. Bonaventure, NY, USA; HANNAH PREST, student, Southeast Missouri State University, Cape Girardeau, Missouri, USA; JAMES REID, student, Angelo State University, San Angelo, TX, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; CULLAN SPRINGSTEAD, Southeast Missouri State University, Cape Girardeau, Missouri, USA; ALBERT STADLER, Herrliberg, Switzerland; ELIZABETH WAMSER, student, Southeast Missouri State University, Cape Girardeau, Missouri, USA; HAOHAO WANG and JERZY WOJDYŁO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; BRENT WESSEL, student, Southeast Missouri State University, Cape Girardeau, Missouri, USA; DANIEL WINGER, student, St. Bonaventure University, Allegany, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

As the matrices were all 2×2 almost all the other solutions were by direct computation. The proposer was the only other solver to use properties of determinants and the trace. The featured solution was the only solution to note the condition that \mathbf{A} needed to be nonsingular.

3540. [2010 : 239, 241] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Triangle ABC has semiperimeter s and area F . A square $PQRS$ with side length x is inscribed in ABC with P and Q on BC , R on AC , and S on AB . Similarly y and z are the sides of squares two vertices of which lie on AC and AB , respectively. Prove that

$$x^{-1} + y^{-1} + z^{-1} \leq \frac{s(2 + \sqrt{3})}{2F}.$$

Combination of solutions by John G. Heuver, Grande Prairie, AB and the proposer.

As usual, the side lengths of $\triangle ABC$ will be denoted by a, b , and c , and the altitudes by h_a, h_b , and h_c . By the similarity of triangles RAS and CAB we have $\frac{x}{a} = \frac{h_a - x}{h_a}$, so that

$$x = \frac{ah_a}{a + h_a} = \frac{2F}{a + h_a}, \text{ or } x^{-1} = \frac{a + h_a}{2F}.$$

Similarly,

$$y^{-1} = \frac{b + h_b}{2F} \quad \text{and} \quad z^{-1} = \frac{c + h_c}{2F}.$$

This allows us to deduce that

$$x^{-1} + y^{-1} + z^{-1} = \frac{2s + h_a + h_b + h_c}{2F}.$$

The desired conclusion follows from the familiar inequality

$$h_a + h_b + h_c \leq s\sqrt{3};$$

see, for example, [1] page 60, formula 6.1 or 6.2. Equality holds if and only if $\triangle ABC$ is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD EDEN, student, Purdue University, West Lafayette, IN, USA; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comănești, Romania.

Heuver used the Cauchy-Schwarz Inequality to deduce that $h_a + h_b + h_c \leq \sqrt{3}\sqrt{h_a^2 + h_b^2 + h_c^2}$, and then applied $\sqrt{h_a^2 + h_b^2 + h_c^2} \leq s^2$, which is (9.8) on page 201 of [2]. Zvonaru finished his solution with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{\sqrt{3}}{2r}$, which is item 5.22 on page 54 of [1].

References

- [1] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969
- [2] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989

3541. [2010 : 240, 242] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Triangle ABC has circumcentre O , circumradius R , orthocentre H , side lengths a, b, c , and altitudes AD, BE, CF , where points D, E, F lie on the sides BC, AC, AB , respectively. The Euler line of triangle ABC intersects BC in P and AC in Q , and the quadrilateral $ABPQ$ is cyclic.

Show that $a^2 + b^2 = 6R^2$, and express the length of PQ in terms of a, b, c .

Solution by the proposer.

Because $ABPQ$ is cyclic, $\angle CPQ = \angle BAQ = \angle BAC$. Since DE joins the feet of two altitudes, we also have $\angle BAC = \angle CDE$, whence $PQ \parallel DE$. OH is (by assumption) the same line as PQ . Because P lies on BC , we have $\angle ACH = \angle OCP$ so that ACF and PCO are similar right triangles; that is, $CO \perp PQ$. In the right triangle HOC ,

$$CH^2 = OH^2 + OC^2 = OH^2 + R^2. \quad (1)$$

Formula 5.8(1) on page 50 of O. Bottema et al., *Geometric Inequalities*, Groningen, 1969 says that

$$OH^2 = 9R^2 - (a^2 + b^2 + c^2). \quad (2)$$

Moreover, in any triangle $CH = 2R \cos C$ (see, for example Roger A. Johnson *Advanced Euclidean Geometry*, Paragraph 252(e), page 163), so that equations (1) and (2) gives us

$$9R^2 - (a^2 + b^2 + c^2) + R^2 = 4R^2 \cos^2 C.$$

Using the sine law, we replace c by $2R \sin C$ in the last equation to get

$$a^2 + b^2 = 6R^2,$$

as desired.

To determine the length of PQ we first recall that $\angle CPQ = \angle BAC$, which implies that the triangles PQC and ABC are similar. Because CO and CF are corresponding altitudes, we deduce that $\frac{PQ}{CO} = \frac{AB}{CF}$, or (using the definition of sine, the sine law, and the equation from the previous paragraph),

$$PQ = \frac{Rc}{b \sin A} = \frac{2R^2 c}{b \cdot 2R \sin A} = \frac{2R^2 c}{ba} = \frac{c(a^2 + b^2)}{3ab}.$$

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Although the proposer stated his problem correctly, the published statement of the problem [2010 : 240, 242] contained two errors. Both Geupel and Woo reported the errors, but Woo figured out how to correct them, and he provided a complete solution. One of the editor's errors is worth describing: The proposer correctly called the quadrilateral $ABPQ$ inscriptible to mean that it can be inscribed in a circle. Unfortunately the word is ambiguous in English, as is its more common version inscribable—it can mean “permitting something to be inscribed in it” as in, “This book is inscribable.” (that is, it is possible to write an inscription in the book), as well as meaning capable of being inscribed in something, as in, “A rectangle is the only parallelogram that is inscribable in a circle.” One should restrict the use of the word to situations where the

context is clear (that is, when it is clear which object is being inscribed where); otherwise it is preferable to use the word cyclic (a cyclic polygon) or concyclic (a concyclic set of points). A similar error in *CRUX with MAYHEM* a few years ago brought forth a similar editorial comment [1997 : 530-531].

3542★. [2010 : 240, 242] *Proposed by Cosmin Pohoăță, Tudor Vianu National College, Bucharest, Romania.*

The mixtilinear incircles of a triangle ABC are the three circles each tangent to two sides and to the circumcircle internally. Let Γ be the circle tangent to each of these three circles internally. Prove that Γ is orthogonal to the circle passing through the incentre and the isodynamic points of the triangle ABC .

[*Ed.*: Let Γ_A be the circle passing through A and the intersection points of the internal and external angle bisectors at A with the line BC . The isodynamic points are the two points that Γ_A , Γ_B , and Γ_C have in common.]

No solutions to this problem were submitted. Problem 3542 therefore remains open.

3543. [2010 : 240, 242] *Proposed by Mehmet Mehmet Şahin, Ankara, Turkey.*

Triangle ABC has inradius r , circumradius R , and angle bisectors $[AD]$, $[BE]$, $[CF]$, where points D , E , F lie on the sides BC , AC , AB , respectively. Let R' be the circumradius of triangle DEF . Prove that

$$R' \leq \frac{R^4}{16r^3}.$$

A combination of solutions by Arkady Alt, San Jose, CA, USA and Michel Bataille, Rouen, France.

We will prove the inequality $R' \leq \frac{R}{2}$, which implies the required inequality, since $\frac{R}{2} \leq \frac{R^4}{16r^3}$ is equivalent to Euler's inequality, $2r \leq R$.

Let $a = BC$, $b = CA$, $c = AB$, $s = \frac{1}{2}(a + b + c)$, and let $[\cdot]$ denote the area of the enclosed figure.

Since $\frac{BD}{c} = \frac{DC}{b} = \frac{a}{c+b}$, we have $BD = \frac{ca}{b+c}$ and $CD = \frac{ab}{b+c}$. Similarly, $AE = \frac{bc}{a+c}$, $CE = \frac{ab}{a+c}$, $AF = \frac{bc}{a+b}$, $BF = \frac{ca}{a+b}$, and it follows that

$$\begin{aligned}
[BDF] &= \frac{1}{2} \cdot \frac{ca}{b+c} \cdot \frac{ca}{a+b} \cdot \sin B = \frac{a^2bc^2}{4R(b+c)(a+b)}, \\
[CED] &= \frac{a^2b^2c}{4R(b+c)(a+c)}, \\
[AFE] &= \frac{ab^2c^2}{4R(a+b)(a+c)}.
\end{aligned}$$

Using $[ABC] = \frac{abc}{4R}$, a straightforward calculation yields

$$[EDF] = [ABC] - [BDF] - [CED] - [AFE] = \frac{(abc)^2}{2R(a+b)(b+c)(c+a)}.$$

From the Law of Cosines,

$$EF^2 = \frac{(bc)^2}{(a+c)^2} + \frac{(bc)^2}{(a+b)^2} - \frac{2(bc)^2}{(a+b)(a+c)} \cdot \cos A = \frac{(abc)^2}{(a+b)^2(a+c)^2} \cdot K_a,$$

where $K_a = \frac{1}{a^2} [(a+c)^2 + (a+b)^2 - 2(a+b)(a+c) \cos A]$. Now,

$$\begin{aligned}
K_a &= \frac{1}{a^2} [2a^2 + b^2 + c^2 + 2ab + 2ac \\
&\quad - 2a^2 \cos A - 2ab \cos A - 2ac \cos A - 2bc \cos A] \\
&= \frac{1}{a^2} [a^2 + 2a(1 - \cos A)(a+b+c)] \\
&= 1 + \frac{8s}{a} \sin^2(A/2) \\
&= 1 + \frac{8s(s-b)(s-c)}{abc} \\
&= 1 + \frac{8}{4Rrs} \cdot \frac{r^2s^2}{s-a} = 1 + \frac{2rs}{R(s-a)}.
\end{aligned}$$

As a result, $EF = \frac{abc}{(a+b)(a+c)} \sqrt{K_a}$, and similarly

$$FD = \frac{abc}{(a+b)(b+c)} \sqrt{K_b}, \quad DE = \frac{abc}{(a+c)(b+c)} \sqrt{K_c},$$

where $K_b = 1 + \frac{2rs}{R(s-b)}$ and $K_c = 1 + \frac{2rs}{R(s-c)}$.

From these results, we obtain

$$R' = \frac{EF \cdot FD \cdot DE}{4[EDF]} = \frac{Rabc}{2(a+b)(b+c)(c+a)} \sqrt{K_a K_b K_c}.$$

Using $abc = 4Rrs$ and $(a+b)(b+c)(c+a) = 2s(s^2 + r^2 + 2rR)$, it readily follows that

$$R' = \frac{rR^2}{s^2 + r^2 + 2rR} \sqrt{K_a K_b K_c},$$

and we will have proved $R' \leq \frac{R}{2}$ if we can show that

$$\sqrt{K_a K_b K_c} \leq \frac{R}{2} \cdot \frac{s^2 + r^2 + 2rR}{rR^2} = 1 + \frac{s^2 + r^2}{2rR}. \quad (1)$$

With this aim, we compute $K_a K_b K_c$ as follows

$$K_a K_b K_c = 1 + \left(\frac{2rs}{R}\right) S + \left(\frac{4r^2 s^2}{R^2}\right) T + \frac{8r^3 s^3}{R^3(s-a)(s-b)(s-c)}.$$

where

$$S = \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} = \frac{r+4R}{rs},$$

$$T = \frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)} = \frac{1}{r^2},$$

$$r^2 s = (s-a)(s-b)(s-c).$$

This yields

$$K_a K_b K_c = 9 + \frac{2r}{R} + \frac{4s^2}{R^2} + \frac{8rs^2}{R^3}.$$

Now, from the well-known inequalities $2r \leq R$ and $2s \leq 3R\sqrt{3}$, we obtain $K_a K_b K_c \leq 9 + 1 + 27 + 27 = 64$, and so

$$\sqrt{K_a K_b K_c} \leq 8. \quad (2)$$

But we also have $s \geq 3r\sqrt{3}$, hence

$$1 + \frac{s^2 + r^2}{2rR} \geq 1 + \frac{27r^2 + r^2}{2r \cdot 2r} = 8, \quad (3)$$

and (1) directly follows from (2) and (3).

Also solved by OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

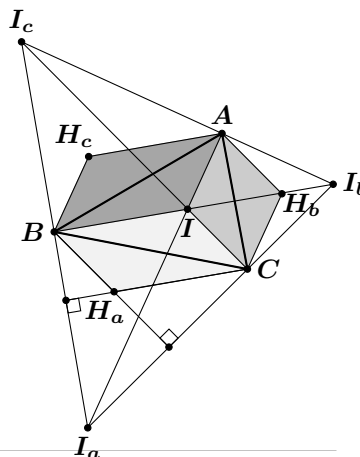
3544. [2010 : 240, 242] *Proposed by Mehmet Şahin, Ankara, Turkey.*

Triangle ABC has excentres I_a, I_b, I_c and H_a, H_b, H_c are the orthocentres of triangles I_aBC, I_bCA, I_cAB , respectively. Prove that

$$\text{Area}(H_aCH_bAH_cB) = 2\text{Area}(ABC).$$

Similar approaches by the solvers listed below with an asterisk.

Let I be the incenter of $\triangle ABC$. Since H_aB and IC are each perpendicular to I_aC , $H_aB \parallel IC$. Similarly, $H_aC \parallel IB$, so H_aBIC is a parallelogram. Similarly, H_bCIA and H_cAIB are parallelograms. Since $\angle BI_aC = 90^\circ - \frac{1}{2}\angle BAC$, $\angle I_aBC = 90^\circ - \frac{1}{2}\angle ABC$, and $\angle I_aCB = 90^\circ - \frac{1}{2}\angle ACB$, $\triangle I_aBC$ is acute, so H_a is inside $\triangle I_aBC$. Similarly, H_b is inside $\triangle I_bCA$ and H_c is inside $\triangle I_cAB$. Since A, B and C lie on segments I_bI_c, I_cI_a and I_aI_b respectively and I is inside $\triangle ABC$, $H_aCH_bAH_cB$ is convex and I is in its interior. Therefore,



$$\begin{aligned} \text{Area}(H_aCH_bAH_cB) &= \text{Area}(H_aBIC) + \text{Area}(H_bCIA) + \text{Area}(H_cAIB) \\ &= 2\text{Area}(BIC) + 2\text{Area}(CIA) + 2\text{Area}(AIB) = 2\text{Area}(ABC). \end{aligned}$$

*Solved by *ARKADY ALT, San Jose, CA, USA; *GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; *EMMANUEL LANCE CHRISTOPHER, Ateneo de Manila University, The Philippines; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; *RICHARD EDEN, student, Purdue University, West Lafayette, IN, USA; OLEH FAYNSHTEYN, Leipzig, Germany; *OLIVER GEUPEL, Brühl, NRW, Germany; *JOEL SCHLOSBERG, Bayside, NY, USA; *PETER Y. WOO, Biola University, La Mirada, CA, USA; *TITU ZVONARU, Comănești, Romania; and the proposer.*

3545. [2010 : 240, 242] *Proposed by Michel Bataille, Rouen, France.*

Given a line ℓ and points A and B with $A \notin \ell$ and $B \in \ell$, find the locus of points P in their plane such that $PA + QB = PQ$ for a unique point Q of ℓ .

Solution by Oliver Geupel, Brühl, NRW, Germany.

Proposition 1 *If $\ell \perp AB$, then \mathcal{L} consists of the line parallel to ℓ through A with the exception of the point A , and the mid-perpendicular line of the segment AB .*

Proof: We consider Cartesian (x, y) -coordinates such that, without loss of generality, $A = (2, 0)$, $B = (0, 0)$, and ℓ is the line $x = 0$.

For any points $P = (p, r)$, $Q = (0, y)$, the condition

$$PA + QB = PQ \quad (1)$$

is equivalent to

$$\sqrt{(p-2)^2 + r^2} + |y| = \sqrt{p^2 + (r-y)^2}.$$

After squaring both sides and erasing equal terms on both sides, this successively becomes equivalent to

$$(p-2)^2 + r^2 + y^2 + 2|y|\sqrt{(p-2)^2 + r^2} = p^2 + (r-y)^2,$$

and

$$[y < 0 \wedge (r-s)y = 2(p-1)] \vee [y \geq 0 \wedge (r+s)y = 2(p-1)], \quad (2)$$

where $s = \sqrt{(p-2)^2 + r^2}$.

We consider the three cases $p = 1$, $p = 2$, and $p \notin \{1, 2\}$ in succession.

If $p = 1$, then, by the definition of s , $r - s$ and $r + s$ are both nonzero. Hence, $y = 0$ is the unique solution of the logical expression (2), that is, the mid-perpendicular of AB belongs to the locus \mathcal{L} .

Let $p = 2$. If $r < 0$, then the first alternative of the condition (2) yields the solution $y = 1/r$, while the second alternative is contradictory. Similarly, if $r > 0$, there is the unique solution $y = 1/r$. If $r = 0$ then the condition (2) is contradictory. Therefore, the parallel to ℓ through A with the exception of the point A is included in \mathcal{L} .

Finally, let $p \notin \{1, 2\}$. By $(r-s)(r+s) = r^2 - s^2 < 0$, so numbers $2(p-1)/(r-s)$ and $2(p-1)/(r+s)$ have opposite signs. Therefore there cannot be exactly one value of y for which (2) is true. Thus, the number of points Q with the property (1) is either 0 or 2, that is, $P \notin \mathcal{L}$. \square

Proposition 2 *Assume that ℓ and AB are not perpendicular. Let g be the perpendicular to ℓ through B . Let π denote the parabola with directrix g and focus A . Let \mathcal{R} be the region that is bounded by π and that contains the point A , where the boundary π belongs to \mathcal{R} . Let m be the mid-perpendicular of the segment AB . Then m is tangential to π and the locus \mathcal{L} consists of the region \mathcal{R} and the line m with the exception of their tangential point.*

Proof: We consider Cartesian (x, y) -coordinates such that, without loss of generality, $A = (a, 2)$, $a > 0$, $B = (0, 0)$, and ℓ is the line $x = 0$. For any points $P = (p, r)$, $Q = (0, y)$, the condition (1) is equivalent to

$$\sqrt{(p-a)^2 + (r-2)^2} + |y| = \sqrt{p^2 + (y-r)^2}.$$

After squaring both sides, this becomes equivalent to

$$[y < 0 \wedge (r-s)y = t] \vee [y \geq 0 \wedge (r+s)y = t], \quad (3)$$

where $s = \sqrt{(p-a)^2 + (r-2)^2}$ and $t = pa + 2r - 2 - a^2/2$. The parabola π is given by the equation $4(\mathbf{y} - 1) = (\mathbf{x} - a)^2$. We have $t = \vec{P} \cdot \vec{A} - \frac{1}{2}\vec{A}^2 = \left(\mathbf{P} - \frac{1}{2}\vec{A}\right) \cdot \vec{A}$. Hence, the condition $t = 0$ holds if and only if \mathbf{P} is on \mathbf{m} . It is easy to check that \mathbf{m} touches π at the point $(0, a^2/4 + 1)$ on the line ℓ .

We consider in succession the three cases where \mathbf{P} is above π , on π , and below π .

If \mathbf{P} is above π , then $(r-s)(r+s) = r^2 - s^2 = [Pg]^2 - [PA]^2 > 0$. Hence the numbers $t/(r-s)$ and $t/(r+s)$ have the same sign. Thus, exactly one of the two alternatives (3) is satisfiable, and the solution is unique. Therefore, the interior of \mathcal{R} is included in \mathcal{L} .

Let \mathbf{P} be on π . Then $r-s = [Pg] - [PA] = 0$, $r+s > 0$, $r = (p-a)^2/4 + 1$, $t = p^2/2$. If $p \neq 0$ then $t > 0$ holds. Hence $t/(r+s) > 0$. Thus, the first alternative in (3) is contradictory, while the second alternative yields a unique solution. Otherwise, if $p = 0$, then each $\mathbf{y} < 0$ satisfies the condition (3). Therefore, the parabola π with the exception of its tangential point with \mathbf{m} belongs to \mathcal{L} .

Finally, let \mathbf{P} be below π . We consider the situations for $t \neq 0$ and $t = 0$ in succession.

Firstly, let $t \neq 0$. By $(r-s)(r+s) = r^2 - s^2 = [Pg]^2 - [PA]^2 < 0$, the numbers $t/(r-s)$ and $t/(r+s)$ have opposite signs. Therefore, either both alternatives in (3) are satisfiable or both are contradictory. Thus, the number of points \mathbf{Q} with the property (1) is either 0 or 2, that is, $\mathbf{P} \notin \mathcal{L}$.

Finally, let $t = 0$. Then \mathbf{P} is on \mathbf{m} so $PA = PB$. The condition (1) is therefore equivalent to $PB + BQ = PQ$, which is satisfied if and only if the points \mathbf{B} , \mathbf{P} , and \mathbf{Q} are collinear where \mathbf{B} belongs to the line segment PQ . The unique point \mathbf{Q} with this property is $\mathbf{Q} = \mathbf{B}$. We have shown that the line \mathbf{m} with the exception of its tangential point with the parabola π is included in the locus \mathcal{L} . \square

Also incompletely solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and the proposer whose solutions were relatively short and more geometric.

3546. [2010 : 240, 242] *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer. Prove that

$$0 < \sum_{k=0}^{\binom{n}{2}} \frac{(-1)^k}{n+k} \binom{\binom{n}{2}}{k} \leq \frac{1}{n^n}.$$

Solution by the proposer.

The inequality holds for $n = 1$, so we can assume that $n \geq 2$. Let $N = \binom{n}{2}$

and let S denote the central sum,

$$\begin{aligned} S &= \sum_{k=0}^N (-1)^k \binom{N}{k} \int_0^1 x^{n+k-1} dx = \int_0^1 x^{n-1} \sum_{k=0}^N \binom{N}{k} (-x)^k dx \\ &= \int_0^1 x^{n-1} (1-x)^N dx. \end{aligned}$$

It follows that $S > 0$. For $k = 1, 2, \dots, n-1$ and by the AM-GM inequality,

$$\begin{aligned} (1-x)x^k &= k^k \left((1-x) \cdot \frac{x}{k} \cdot \frac{x}{k} \cdots \frac{x}{k} \right) \\ &\leq k^k \left(\frac{(1-x) + k(x/k)}{k+1} \right)^{k+1} = \frac{k^k}{(k+1)^{k+1}}. \end{aligned}$$

Since $N = 1 + 2 + \cdots + (n-1)$,

$$\begin{aligned} (1-x)^{n-1} x^N &= [(1-x)x] \cdot [(1-x)x^2] \cdots [(1-x)x^{n-1}] \\ &\leq \frac{1^1}{2^2} \cdot \frac{2^2}{3^3} \cdots \frac{(n-1)^{n-1}}{n^n} = \frac{1}{n^n}. \end{aligned}$$

Thus,

$$S = \int_0^1 x^N (1-x)^{n-1} dx \leq \frac{1}{n^n},$$

which completes the proof.

Also solved by Oliver Geupel, Brühl, NRW, Germany; and Albert Stadler, Herrliberg, Switzerland.

3547. [2010 : 241, 243] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Triangle ABC has perimeter equal to 1 , inradius r , circumradius R , and side lengths a, b, c . Prove that

$$\frac{a}{\sqrt{1-a}} + \frac{b}{\sqrt{1-b}} + \frac{c}{\sqrt{1-c}} \geq \sqrt{\frac{2}{1+4r(r+4R)}}.$$

Solution by Richard Eden, student, Purdue University, West Lafayette, IN, USA.

The relations $R = \frac{abc}{4rs}$ and $r = \sqrt{(s-a)(s-b)(s-c)/s}$, where s is the semiperimeter, are well-known. Since $s = 1/2$, then $16Rr = 8abc$ and $4r^2 = 8(1/2-a)(1/2-b)(1/2-c) = (1-2a)(1-2b)(1-2c)$. Therefore,

$$\begin{aligned} 1 + 4r(r + 4R) &= 1 + (1 - 2a)(1 - 2b)(1 - 2c) + 8abc \\ &= 4(ab + bc + ca) - 2(a + b + c) + 2 \\ &= 4(ab + bc + ca) \end{aligned}$$

The inequality to prove is then

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{a+c}} + \frac{c}{\sqrt{a+b}} \geq \frac{1}{\sqrt{2(ab+bc+ca)}}$$

The function $f(x) = 1/\sqrt{x}$ is convex for $x > 0$. Since $a + b + c = 1$, we can think of a, b, c as weights and apply Jensen's inequality,

$$\begin{aligned} \frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{a+c}} + \frac{c}{\sqrt{a+b}} &= af(b+c) + bf(a+c) + cf(a+b) \\ &\geq f(a(b+c) + b(a+c) + c(b+c)) \\ &= f(2(ab+bc+ca)) \\ &= \frac{1}{\sqrt{2(ab+bc+ca)}}, \end{aligned}$$

which completes the proof.

Also solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Oliver Geupel, Brühl, NRW, Germany; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland; Peter Y. Woo, Biola University, La Mirada, CA, USA; and the proposer.

3548. [2010 : 241, 243] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let $x, y,$ and z be nonnegative real numbers. Prove that

$$\sum_{\text{cyclic}} \sqrt{x^2 - xy + y^2} \leq x + y + z + \sqrt{x^2 + y^2 + z^2 - xy - yz - zx}.$$

I. *Solution by the proposer, modified slightly by the editor.*

Due to complete symmetry we may assume without loss of generality that $x = \min\{x, y, z\}$. Then

$$\begin{aligned} \sqrt{x^2 - xy + y^2} &= \sqrt{x(x-y) + y^2} \leq y && \text{and} \\ \sqrt{x^2 - xz + z^2} &= \sqrt{x(x-z) + z^2} \leq z. \end{aligned}$$

Hence it suffices to show that

$$\sqrt{y^2 - yz + z^2} - x \leq \sqrt{x^2 + y^2 + z^2 - xy - yz - zx}. \quad (1)$$

Since $\sqrt{y^2 - yz + z^2} - x = \sqrt{(y-z)^2 + yz} - x \geq \sqrt{x^2} - x = 0$, we may square both sides of (1) to obtain

$$x^2 + y^2 + z^2 - yz - 2x\sqrt{y^2 - yz + z^2} \leq x^2 + y^2 + z^2 - xy - yz - zx$$

or

$$x(y+z) \leq 2x\sqrt{y^2 - yz + z^2}. \quad (2)$$

Squaring both sides, (2) is equivalent, in succession, to

$$\begin{aligned} x^2(y^2 + 2yz + z^2) &\leq 4x^2(y^2 - yz + z^2) \\ 3x^2y^2 - 6x^2yz + 3x^2z^2 &\geq 0 \\ 3x^2(y-z)^2 &\geq 0. \end{aligned}$$

The proof is complete.

II. *Solution by Albert Stadler, Herrliberg, Switzerland, expanded by the editor.*

We show that the proposed inequality follows from the result below, known as Hlawka Inequality [see D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993; p.521]:

If \mathbf{V} is an inner product space, then for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}$,

$$\|\mathbf{a} + \mathbf{b}\| + \|\mathbf{b} + \mathbf{c}\| + \|\mathbf{c} + \mathbf{a}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| + \|\mathbf{c}\| + \|\mathbf{a} + \mathbf{b} + \mathbf{c}\|,$$

where $\|\cdot\|$ denotes the norm induced by the inner product.

If we let $\mathbf{V} = \mathbb{R}^2$ and set $\mathbf{a} = x(1, 0)$, $\mathbf{b} = y\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $\mathbf{c} = z\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, then $\|\mathbf{a}\| = \sqrt{x^2} = x$, $\|\mathbf{b}\| = \sqrt{y^2} = y$, $\|\mathbf{c}\| = \sqrt{z^2} = z$. Furthermore,

$$\begin{aligned} \|\mathbf{a} + \mathbf{b}\| &= \left\| \left(x - \frac{1}{2}y, \frac{\sqrt{3}}{2}y \right) \right\| = \sqrt{\left(x - \frac{1}{2}y \right)^2 + \frac{3}{4}y^2} = \sqrt{x^2 - xy + y^2}, \\ \|\mathbf{b} + \mathbf{c}\| &= \left\| \left(-\frac{1}{2}(y+z), \frac{\sqrt{3}}{2}(y-z) \right) \right\| \\ &= \sqrt{\frac{1}{4}(y+z)^2 + \frac{3}{4}(y-z)^2} = \sqrt{y^2 - yz + z^2}, \\ \|\mathbf{c} + \mathbf{a}\| &= \left\| \left(x - \frac{1}{2}z, -\frac{\sqrt{3}}{2}z \right) \right\| = \sqrt{\left(x - \frac{1}{2}z \right)^2 + \frac{3}{4}z^2} = \sqrt{x^2 - zx + z^2}, \end{aligned}$$

and

$$\begin{aligned} \|\mathbf{a} + \mathbf{b} + \mathbf{c}\| &= \left\| \left(x - \frac{1}{2}(y+z), \frac{\sqrt{3}}{2}(y-z) \right) \right\| \\ &= \sqrt{\left(x - \frac{1}{2}(y+z) \right)^2 + \frac{3}{4}(y-z)^2} \\ &= \sqrt{x^2 + y^2 + z^2 - xy - yz - zx}. \end{aligned}$$

The result follows.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; RICHARD EDEN, student, Purdue University, West Lafayette, IN, USA; OLEH FAYNSHTEYN, Leipzig, Germany; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and PETER Y. WOO, Biola University, La Mirada, CA, USA. As usual, Stan Wagon verified the validity of the inequality by using “FindInstance” (in 13 seconds).

3549. [2010 : 241, 243] Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.

Let a , b , and c be nonnegative real numbers such that $a + b + c = 3$. Prove that $(1 + a^2b)(1 + b^2c)(1 + c^2a) \leq 5 + 3abc$.

Solution by Arkady Alt, San Jose, CA, USA.

Note first that the given inequality is equivalent to

$$a^2b + b^2c + c^2a + abc(ab^2 + bc^2 + ca^2) + a^3b^3c^3 \leq 4 + 3abc. \quad (1)$$

We first establish the following lemma:

Lemma If a , b , and c are nonnegative real numbers, then

$$a^2b + b^2c + c^2a + abc \leq \frac{4}{27}(a + b + c)^3. \quad (2)$$

Proof: Due to the cyclic symmetry of a , b , and c in (2) we may assume, without loss of generality, that $c = \min\{a, b, c\}$. We consider two cases separately:

Case (i). Suppose $b \leq a$. By the AM-GM Inequality, we have

$$\frac{4}{27}(a + b + c)^3 = \frac{1}{2} \left(\frac{2b + 2(a + c)}{3} \right)^3 \geq \frac{1}{2} (2b(a + c))^2 = b(a + c)^2. \quad (3)$$

Since

$$\begin{aligned} b(a + c)^2 - (a^2b + b^2c + c^2a + abc) &= abc + bc^2 - b^2c - c^2a \\ &= c(ab + bc - b^2 - ca) \\ &= c(a - b)(b - c) \geq 0 \end{aligned}$$

we have

$$a^2b + b^2c + c^2a + abc \leq b(a + c)^2 \quad (4)$$

and (2) follows from (3) and (4).

Case (ii). Suppose $b > a$. We have

$$\begin{aligned} 2(a^2b + b^2c + c^2a + abc) &= \sum_{cyclic} (a^2b + ab^2) + 2abc + \sum_{cyclic} (a^2b - ab^2) \\ &= (a + b)(b + c)(c + a) - (a - b)(b - c)(c - a). \quad (5) \end{aligned}$$

By the AM-GM Inequality we have

$$\begin{aligned}(a+b)(b+c)(c+a) &\leq \left(\frac{(a+b) + (b+c) + (c+a)}{3} \right)^3 \\ &= \frac{8}{27}(a+b+c)^3\end{aligned}$$

so

$$\begin{aligned}\frac{4}{27}(a+b+c)^3 &\geq \frac{1}{2}(a+b)(b+c)(c+a) \\ &= a^2b + b^2c + c^2a + abc + \frac{1}{2}(a-b)(b-c)(c-a) \\ &\geq a^2b + b^2c + c^2a + abc\end{aligned}$$

since $(a-b)(b-c)(c-a) \geq 0$.

This completes the proof of the lemma.

Since $a+b+c=3$, (2) becomes $a^2b + b^2c + c^2a \leq 4 - abc$ and since $4 - abc$ is invariant under the interchanging of a and b , we have $\max\{a^2b + b^2c + c^2a, ab^2 + bc^2 + ca^2\} \leq 4 - abc$. Therefore,

$$\sum_{cyclic} a^2b + abc \sum_{cyclic} ab^2 + a^3b^3c^3 \leq (1+abc)(4-abc) + a^3b^3c^3.$$

Finally, since $abc \leq \left(\frac{a+b+c}{3}\right)^3 = 1$ we have

$$\begin{aligned}4 + 3abc - \left(\sum_{cyclic} a^2b + abc \sum_{cyclic} ab^2 + a^3b^3c^3 \right) \\ \geq 4 + 3abc - (1+abc)(4-abc) - a^3b^3c^3 \\ = a^2b^2c^2 - a^3b^3c^3 = a^2b^2c^2(1-abc) \geq 0\end{aligned}$$

which establishes (1) and completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer. Stan Wagon gave his usual verification using "FindInstances".

3550. [2010 : 241, 243] Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.

Find the sum

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{n+m} \left(\ln 2 - \sum_{i=1}^{n+m} \frac{1}{n+m+i} \right).$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let

$$a_m = \ln 2 - \sum_{i=1}^m \frac{1}{m+i},$$

and

$$b_n = \sum_{m=1}^{\infty} (-1)^{m+n} a_{m+n}.$$

We have

$$\begin{aligned} a_m &= \ln 2 - \sum_{i=1}^{2m} \frac{(-1)^{i-1}}{i} \\ &= \int_0^1 \frac{1}{1+x} dx - \sum_{i=1}^{2m} \int_0^1 (-1)^{i-1} x^{i-1} dx \\ &= \int_0^1 \frac{1}{1+x} dx - \int_0^1 \frac{1-x^{2m}}{1+x} dx = \int_0^1 \frac{x^{2m}}{1+x} dx; \end{aligned}$$

hence

$$\begin{aligned} \sum_{i=1}^m (-1)^{n+1} a_{n+1} &= \int_0^1 \frac{1}{1+x} \sum_{i=1}^m (-1)^{n+i} x^{2(n+i)} dx \\ &= \int_0^1 \frac{(-x^2)^{n+1}}{1+x} \cdot \frac{1 - (-x^2)^m}{1+x^2} dx \end{aligned}$$

Let $f_m(x) = \frac{(-x^2)^{n+1}}{1+x} \cdot \frac{1 - (-x^2)^m}{1+x^2}$ and $f(x) = \frac{(-x^2)^{n+1}}{(1+x)(1+x^2)}$.

Then for all $0 \leq x \leq 1$ we have $0 \leq |f_m(x)| \leq 2f(x)$. Also, on $[0, 1)$ $f_m(x) \rightarrow f(x)$ pointwise. Then, by the Lebesgue Dominated Convergence Theorem we have

$$b_n = \lim_{m \rightarrow \infty} \int_0^1 \frac{(-x^2)^{n+1}}{1+x} \cdot \frac{1 - (-x^2)^m}{1+x^2} dx = \int_0^1 \frac{(-x^2)^{n+1}}{(1+x)(1+x^2)} dx.$$

It follows that

$$\begin{aligned} \sum_{i=1}^N b_n &= \int_0^1 \frac{1}{(1+x)(1+x^2)} \sum_{n=1}^N (-x^2)^{n+1} dx \\ &= \int_0^1 \frac{x^4}{(1+x)(1+x^2)} \cdot \frac{1 - (-x^2)^N}{1+x^2} dx \end{aligned}$$

Applying again the Lebesgue Dominated Convergence Theorem we get:

$$\begin{aligned} \sum_{i=1}^{\infty} b_n &= \int_0^1 \frac{x^4}{(1+x)(1+x^2)^2} \\ &= \frac{1}{8} \left[\frac{2(1+x)}{1+x^2} + 3 \ln(1+x^2) + 2 \ln(1+x) - 4 \arctan(x) \right]_0^1 \\ &= \frac{5 \ln 2 - \pi}{8} \end{aligned}$$

Thus, the sum is $\frac{5 \ln 2 - \pi}{8}$.

Also solved by PAUL BRACKEN, University of Texas, Edinburg, TX, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; and the proposer .

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