

# SKOLIAD No. 133

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **February 15, 2012**. A copy of *CRUX with Mayhem* will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is selected problems from the 21<sup>st</sup> Transylvanian Hungarian Mathematical Competition, 9<sup>th</sup> Form, 2011. Our thanks go to Dr. Mihály Bencze, president of The Transylvanian Hungarian Competition, Brasov, Romania, for providing us with this contest and for permission to publish it.

La rédaction souhaite remercier Rolland Gaudet, de Collège universitaire de Saint-Boniface, Winnipeg, MB, d'avoir traduit ce concours.

## La 21<sup>e</sup> compétition mathématique hongroise-transylvanienne, 2011 9<sup>e</sup> classe, Problèmes choisis

1. Démontrer que si  $a$ ,  $b$ ,  $c$ , et  $d$  sont des nombres réels, alors

$$a + b + c + d - a^2 - b^2 - c^2 - d^2 \leq 1.$$

2. Comparons les deux nombres suivants,

$$A = \underbrace{2^{2^{\dots^2}}}_{2011 \text{ copies de } 2} \quad \text{and} \quad B = \underbrace{3^{3^{\dots^3}}}_{2010 \text{ copies de } 3};$$

lequel est le plus élevé,  $A$  ou  $B$ ? (Noter que  $a^{b^c}$  égale  $a^{(b^c)}$  et non  $(a^b)^c$ .)

3. Déterminer toutes les solutions en entiers naturels à chacune des équations.

a.  $20x^2 + 11y^2 = 2011$ .

b.  $20x^2 - 11y^2 = 2011$ .

4. Dans le parallélogramme  $ABCD$ , on a  $\angle BAD = 45^\circ$  et  $\angle ABD = 30^\circ$ . Démontrer que la distance de  $B$  à la diagonale  $AC$  est  $\frac{1}{2}|AD|$ .

5. Quelle est la prochaine année avec quatre vendredi 13?

## 21st Transylvanian Hungarian Mathematical Competition, 2011

### Selected problems for the 9th form

1. Prove that if  $a$ ,  $b$ ,  $c$ , and  $d$  are real numbers, then

$$a + b + c + d - a^2 - b^2 - c^2 - d^2 \leq 1.$$

2. Compare the following two numbers,

$$A = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^2}}}}}_{\text{2011 copies of 2}} \quad \text{and} \quad B = \underbrace{3^{3^{\cdot^{\cdot^{\cdot^3}}}}}_{\text{2010 copies of 3}};$$

which is larger,  $A$  or  $B$ ? (Note that  $a^{b^c}$  equals  $a^{(b^c)}$ , not  $(a^b)^c$ .)

3. Find all natural number solutions to each equation:

a.  $20x^2 + 11y^2 = 2011$ .

b.  $20x^2 - 11y^2 = 2011$ .

4. In the parallelogram  $ABCD$ ,  $\angle BAD = 45^\circ$  and  $\angle ABD = 30^\circ$ . Show that the distance from  $B$  to the diagonal  $AC$  is  $\frac{1}{2}|AD|$ .

5. What is the next year with four Friday the 13ths?

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Next follow solutions to the British Columbia Secondary School Mathematics Contest 2010, Junior Final Round, Part B, given in Skoliad 127 at [2010 : 353 – 354].

- 1a.** Find the sum of all positive integers less than **2010** for which the ones digit is either a '3' or an '8'.

*Solution by Szera Pinter, student, Moscrop Secondary School, Burnaby, BC.*

The positive numbers with ones digit **3** or **8** form an arithmetic sequence with common difference **5**, namely **3, 8, 13, 18, ..., 2008**. The formula for the  $n^{\text{th}}$  term of an arithmetic sequence with first term  $a$  and common difference  $d$  is  $t_n = a + d(n - 1)$ . In the problem,  $a = 3$ ,  $d = 5$ , and the last term is **2008**, so  $2008 = 3 + 5(n - 1)$ , thus  $n = 402$ , whence the sequence has **402** terms.

The formula for the sum of an arithmetic sequence is

$$S = \frac{(\text{first term}) + (\text{last term})}{2} \cdot (\text{number of terms}).$$

The sum in the problem is therefore  $S = \frac{3 + 2008}{2} \cdot 402 = 404\,211$ .

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; DAVID FAN, student, Campbell Collegiate, Regina, SK; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

Solvers who are not familiar with arithmetic sequences can easily find the sum using Gauss's trick as explained in the comments to the solution to Problem 7 in Skoliad 129 given at [2010 : 485].

**1b.** Two cans,  $X$  and  $Y$ , both contain some water. From  $X$  Tim pours as much water into  $Y$  as  $Y$  already contains. Then, from  $Y$  he pours as much water into  $X$  as  $X$  already contains. Finally, he pours from  $X$  into  $Y$  as much water as  $Y$  already contains. Each can now contains **24** units of water. Determine the number of units of water in each can at the beginning.

*Solution by David Fan, student, Campbell Collegiate, Regina, SK.*

Let  $x$  be the number of units of water originally contained in can  $X$ , and let  $y$  be the number of units in can  $Y$ . The following table then shows the contents in each can as water is poured back and forth:

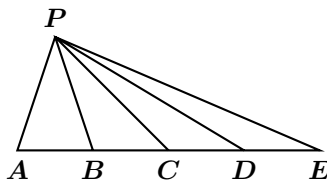
$X$	$Y$
$x$	$y$
$x - y$	$2y$
$2(x - y) = 2x - 2y$	$2y - (x - y) = 3y - x$
$2x - 2y - (3y - x) = 3x - 5y$	$2(3y - x) = 6y - 2x$

Thus  $3x - 5y = 24$  and  $6y - 2x = 24$ . Solving these two simultaneous equations yields that  $x = 33$  and  $y = 15$ .

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and JULIA PENG, student, Campbell Collegiate, Regina, SK.

Since the two cans together in the end hold **48** units of water, you know from the outset that  $y = 48 - x$ . Whether using this fact saves effort is a matter of taste.

**2.** The area of  $\triangle APE$  shown in the diagram is **12**. Given that  $|AB| = |BC| = |CD| = |DE|$ , determine the sum of the areas of all the triangles that appear in the diagram.



*Solution by Janice Lew, student, École Alpha Secondary School, Burnaby, BC.*

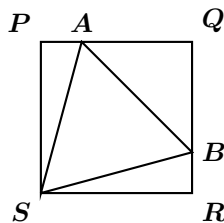
Since  $|AB| = |BC| = |CD| = |DE|$ ,  $\triangle APB$ ,  $\triangle BPC$ ,  $\triangle CPD$ , and  $\triangle DPE$  all have the same base and height, so they have the same area. Since  $\triangle APE$  has area **12** each of  $\triangle APB$ ,  $\triangle BPC$ ,  $\triangle CPD$ , and  $\triangle DPE$  have area **3**.

The following table now lists all the triangles in the diagram and their areas:

	Area	Total
$\triangle APB$ , $\triangle BPC$ , $\triangle CPD$ , and $\triangle DPE$	3	12
$\triangle APC$ , $\triangle BPD$ , $\triangle CPE$	6	18
$\triangle APD$ and $\triangle BPE$	9	18
$\triangle APE$	12	12
Grand total		60

Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; DAVID FAN, student, Campbell Collegiate, Regina, SK; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC.

**3.** Given that  $PQRS$  is a square and that  $ABS$  is an equilateral triangle (see the diagram), find the ratio of the area of  $\triangle APS$  to the area of  $\triangle ABQ$ .



*Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.*

Let  $a$  denote the side length of the square, let  $x$  be  $|PA|$ , and let  $y = |BR|$ . By the Pythagorean Theorem,  $|SA| = \sqrt{a^2 + x^2}$  and  $|SB| = \sqrt{a^2 + y^2}$ . Since  $\triangle ABS$  is equilateral,  $|SA| = |SB|$ , so  $a^2 + x^2 = a^2 + y^2$ , so  $x^2 = y^2$ , so  $x = y$ .

Using the Pythagorean Theorem on  $\triangle AQB$  yields that  $|AB|^2 = (a - x)^2 + (a - y)^2$ , but  $|AB|^2 = |SA|^2 = a^2 + x^2$  and  $x = y$ , so

$$a^2 + x^2 = 2(a - x)^2 = 2(a^2 - 2ax + x^2) = 2a^2 - 4ax + 2x^2.$$

Thus  $2ax = a^2 - 2ax + x^2 = (a - x)^2$ .

Now, the area of  $\triangle APS$  is  $\frac{1}{2}ax$ , and the area of  $\triangle ABQ$  is  $\frac{1}{2}(a - x)(a - y) = \frac{1}{2}(a - x)^2 = \frac{1}{2} \cdot 2ax = ax$ . Hence the ratio of the two areas is  $\frac{1}{2}ax : ax = \frac{1}{2} : 1 = 1 : 2$ .

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

**4.** Find the five distinct integers for which the sums of each distinct pair of integers are the numbers **0, 1, 2, 4, 7, 8, 9, 10, 11, and 12**.

*Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.*

If the five numbers are  $a, b, c, d$ , and  $e$ , and  $a < b < c < d < e$ , then  $a + b < a + c$ , and all the other sums are larger. Thus  $a + b = 0$  and  $a + c = 1$ ,

so  $b = -a$  and  $c = b + 1$ . Moreover,  $c + e < d + e$  and all the other sums are smaller. Thus  $c + e = 11$  and  $d + e = 12$ , so  $d = c + 1 = b + 2$ . That is,  $b$ ,  $c$ , and  $d$  are consecutive integers. Since  $c + e = 11$ , it follows that  $b + e = 10$ .

Since  $b$ ,  $c$ , and  $d$  are consecutive integers, so are  $b + c$ ,  $b + d$ , and  $c + d$ . Among the given sums, only **4**, **7**, **8**, and **9** remain. Thus  $b + c = 7$ ,  $b + d = 8$ , and  $c + d = 9$ . Finally,  $a + e = 4$ .

To summarise,  $a + b = 0$ ,  $a + c = 1$ ,  $a + d = 2$ ,  $a + e = 4$ ,  $b + c = 7$ ,  $b + d = 8$ ,  $b + e = 10$ ,  $c + d = 9$ ,  $c + e = 11$ , and  $d + e = 12$ .

Since  $a + b = 0$  and  $a + c = 1$ ,  $2a + b + c = 1$ . But  $b + c = 7$ , so  $2a = -6$ , so  $a = -3$ . Therefore  $b = 3$ ,  $c = 4$ ,  $d = 5$ , and  $e = 7$ .

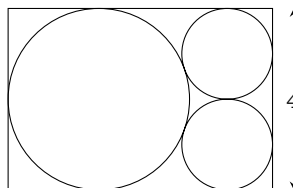
Also solved by DAVID FAN, student, Campbell Collegiate, Regina, SK; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and JULIA PENG, student, Campbell Collegiate, Regina, SK.

Placing the ten sums in a chart like this

$$\begin{array}{cccc} a + b & & & \\ a + c & b + c & & \\ a + d & b + d & c + d & \\ a + e & b + e & c + e & d + e \end{array}$$

can make the bookkeeping easier. Note that the sums increase as you move down or right in the diagram.

**5.** A rectangle contains three circles, as in the diagram, all tangent to the rectangle and to each other. The height of the rectangle is **4**. Determine the width of the rectangle.



*Solution by Julia Peng, student, Campbell Collegiate, Regina, SK.*

Since the height of the rectangle is **4**, the radius of the large circle is **2**, and both the small circles have radius **1**. Now connect the centres of the three circles. This forms an isosceles triangle with sides **3** and base **2**. By the Pythagorean Theorem, the height of this triangle is  $\sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$ . The distance from the base of the triangle to the right-hand edge of the rectangle is **1**, and the distance from the left vertex of the triangle to the left edge of the rectangle is **2**, so the width of the rectangle is  $2 + 2\sqrt{2} + 1 = 3 + 2\sqrt{2}$ .

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and JANICE LEW, student, École Alpha Secondary School, Burnaby, BC.

This issue's prize of one copy of *CruX Mathematicorum* for the best solutions goes to Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.

We wish our readers the best of luck solving our featured contest.