THE OLYMPIAD CORNER

No. 294

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The problems from this issue come from the selection tests for the Balkan, Indian and Slovenian IMO teams and the Singapore Mathematical Olympiad. Our thanks go to Adrain Tang for sharing the material with the editor.

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **15 mars 2012**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

OC11. Etant donné deux sous-ensembles non vides $A, B \subseteq \mathbb{Z}$, on définit $A+B$ et $A-B$ par

$$A+B=\{a+b \mid a\in A, b\in B\}\,,\; A-B=\{a-b \mid a\in A, b\in B\}\,.$$

Dans ce qui suit, on travaille avec des sous-ensembles finis non vides de $\mathbb{Z}$. Montrer qu'on peut recouvrir B par au plus $\dfrac{|A+B|}{|A|}$ translatés de $A-A$, c.-à-d. qu'il existe $X \subseteq \mathbb{Z}$ avec $|X| \le \dfrac{|A+B|}{|A|}$ tel que

$$B \subseteq \bigcup_{x\in X} (x+(A-A)) = X+A-A\,.$$

OC12. Soit k un entier positif plus grand que 1. Montrer que pour tout entier non négatif m, il existe k entiers positifs $n_1, n_2, \ldots, n_k$, tels que

$$n_1^2+n_2^2+\cdots+n_k^2 = 5^{m+k}\,.$$

OC13. Soit ABC un triangle acutangle et soit D un point sur le côté AB. Le cercle circonscrit du triangle BCD coupe le côté AC en E. Le cercle circonscrit du triangle ADC coupe le côté BC en F. Soit O le centre de gravité du triangle CEF. Montrer que les points D et O et les centres de gravité des triangles ADE, ADC, DBF et DBC sont cocycliques et que la droite OD est perpendiculaire à AB.

OC14. Soit a_n, b_n, $n = 1, 2, \ldots$ deux suites d'entiers définis par $a_1 = 1$, $b_1 = 0$ et, pour $n \geq 1$,

$$\begin{aligned} a_{n+1} &= 7a_n + 12b_n + 6\,, \\ b_{n+1} &= 4a_n + 7b_n + 3\,. \end{aligned}$$

Montrer que a_n^2 est la différence de deux cubes consécutifs.

OC15. Une règle de longueur ℓ a $k \geq 2$ marques distantes de a_i unités d'une des extrémités avec $0 < a_1 < \cdots < a_k < \ell$. La règle est appelée *règle de Golomb* si les longueurs mesurables grâce aux marques de la règle sont des entiers consécutifs commençant avec 1, et telles que chaque longueur soit mesurable entre une unique paire de marques sur la règle. Trouver toutes les règles de Golomb.

OC16. Etant donné $a_1 \geq 1$ et $a_{k+1} \geq a_k + 1$ pour tout $k = 1, 2, \ldots, n$, montrer que

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq (a_1 + a_2 + \cdots + a_n)^2\,.$$

OC17. Montrer que les sommets d'un pentagone convexe $ABCDE$ sont cocycliques si et seulement si on a

$$d(E, AB) \cdot d(E, CD) = d(E, AC) \cdot d(E, BD) = d(E, AD) \cdot d(E, BC)\,.$$

OC18. Etant donné n nombres complexes $a_1, a_2, \ldots, a_n$, non nécessairement distincts, et des entiers positifs distincts k, l tels que $a_1^k, a_2^k, \ldots, a_n^k$ et $a_1^l, a_2^l, \ldots, a_n^l$ sont deux collections de nombres identiques, montrer que chaque a_j, $1 \leq j \leq n$, est une racine de l'unité.

OC19. Il y a eu 64 participants dans un tournoi d'échecs. Chaque paire a joué une partie qui s'est terminé soit par un gagnant ou par un match nul. Si une partie s'était terminée par un match nul, alors chacun des 62 participants restants gagnait contre au moins un des deux joueurs. Il y a eu au moins deux parties avec match nul dans ce tournoi. Montrer qu'on peut aligner tous les participants sur deux rangs de sorte que chacun d'eux a gagné contre celui qui se trouve juste derrière lui.

OC20. Etant donné un entier $n \geq 2$, trouver la valeur maximale que la somme $x_1 + x_2 + \cdots + x_n$ puisse atteindre lorsque les x_i prennent toutes les valeurs positives sujettes aux conditions $x_1 \leq x_2 \leq \cdots \leq x_n$ et $x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdots x_n$.

. . . .

OC11. For non-empty subsets $A, B \subseteq \mathbb{Z}$ define $A+B$ and $A-B$ by

$$A+B=\{a+b \mid a\in A, b\in B\}\,,\ A-B=\{a-b \mid a\in A, b\in B\}\,.$$

In the sequel we work with non-empty finite subsets of $\mathbb{Z}$.

Prove that we can cover B by at most $\dfrac{|A+B|}{|A|}$ translates of $A-A$, i.e. there exists $X\subseteq \mathbb{Z}$ with $|X|\le \dfrac{|A+B|}{|A|}$ such that

$$B\subseteq \bigcup_{x\in X}(x+(A-A)) = X+A-A\,.$$

OC12. Let k be a positive integer greater than 1. Prove that for every non-negative integer m there exist k positive integers $n_1, n_2, \ldots, n_k$, such that

$$n_1^2+n_2^2+\cdots+n_k^2=5^{m+k}\,.$$

OC13. Let ABC be an acute triangle and let D be a point on the side AB. The circumcircle of the triangle BCD intersects the side AC at E. The circumcircle of the triangle ADC intersects the side BC at F. Let O be the circumcentre of triangle CEF. Prove that the points D and O and the circumcentres of the triangles ADE, ADC, DBF and DBC are concyclic and the line OD is perpendicular to AB.

OC14. Let a_n, b_n, $n=1,2,\ldots$ be two sequences of integers defined by $a_1=1$, $b_1=0$ and for $n\ge 1$,

$$\begin{aligned} a_{n+1} &= 7a_n+12b_n+6\,,\\ b_{n+1} &= 4a_n+7b_n+3\,. \end{aligned}$$

Prove that a_n^2 is the difference of two consecutive cubes.

OC15. A ruler of length ℓ has $k\ge 2$ marks at positions a_i units from one of the ends with $0<a_1<\cdots<a_k<\ell$. The ruler is called a *Golomb ruler* if the lengths that can be measured using the marks on the ruler are consecutive integers starting with 1, and each such length be measurable between a unique pair of marks on the ruler. Find all Golomb rulers.

OC16. Given $a_1\ge 1$ and $a_{k+1}\ge a_k+1$ for all $k=1,2,\ldots,n$, show that

$$a_1^3+a_2^3+\cdots+a_n^3\ge (a_1+a_2+\cdots+a_n)^2\,.$$

OC17. Prove that the vertices of a convex pentagon $ABCDE$ are concyclic if and only if the following holds

$$d(E,AB)\cdot d(E,CD)=d(E,AC)\cdot d(E,BD)=d(E,AD)\cdot d(E,BC)\,.$$

OC18. If $a_1, a_2, \ldots, a_n$ are n non-zero complex numbers, not necessarily distinct, and k, l are distinct positive integers such that $a_1^k, a_2^k, \ldots, a_n^k$ and $a_1^l, a_2^l, \ldots, a_n^l$ are two identical collections of numbers. Prove that each a_j, $1 \leq j \leq n$, is a root of unity.

OC19. There were 64 contestants at a chess tournament. Every pair played a game that ended either with one of them winning or in a draw. If a game ended in a draw, then each of the remaining 62 contestants won against at least one of these two contestants. There were at least two games ending in a draw at the tournament. Show that we can line up all the contestants so that each of them has won against the one standing right behind him.

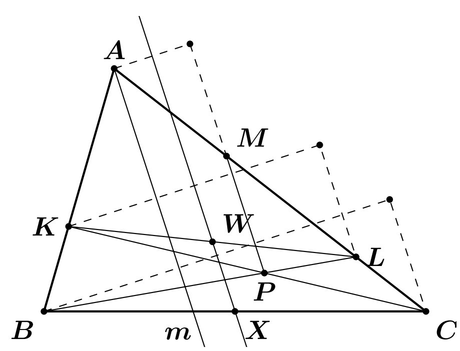
OC20. Given an integer $n \geq 2$, determine the maximum value the sum $x_1 + x_2 + \cdots + x_n$ may achieve, as the x_i run through the positive integers subject to $x_1 \leq x_2 \leq \cdots \leq x_n$ and $x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdots x_n$.

We now turn to solutions of the II International Zhautykov Olympiad in Mathematics given at [2009 : 376–377].

2. The points K and L lie on the sides AB and AC, respectively, of the triangle ABC such that $BK = CL$. Let P be the point of intersection of the segments BL and CK, and let M be an inner point of the segment AC such that the line MP is parallel to the bisector of the angle $\angle BAC$. Prove that $CM = AB$.

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comǎneşti, Romania. We give the solution of Bataille.

We introduce the glide-reflection (or reflection) $\mathbf{g}$ such that $\mathbf{g}(B) = C$ and $\mathbf{g}(K) = L$. The axis of $\mathbf{g}$ is parallel to the angle bisector m of $\angle BAC$ and passes through the midpoints X and W of BC and KL (see figure).



We observe that the line ALC is a transversal of ΔBKP (with A on BK, L on BP and C on KP). From a well-known theorem, the midpoints of the "diagonals" AP, LK, CB are collinear.

It follows that the midpoint of AP is on the axis XW of $\mathbf{g}$. Thus, the lines m and MP, which are parallel to the axis of $\mathbf{g}$, are equidistant of this axis. As a result, $\mathbf{g}(m) = MP$. Since in addition the image under $\mathbf{g}$ of the line $AB = BK$ is the line $AC = CL$, we obtain $\mathbf{g}(A) = M$. Recalling that $\mathbf{g}$ preserves distances, $AB = CM$ follows.

5. Let a, b, c, and d be real numbers such that $a+b+c+d=0$. Prove that

$$(ab+ac+ad+bc+bd+cd)^2+12 \;\geq\; 6(abc+abd+acd+bcd)\,.$$

Solved by Arkady Alt, San Jose, CA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.

Since $d=-a-b-c$, the given inequality is equivalent to

$$[ab+ac+bc-(a+b+c)^2]^2+12 \;\geq\; 6abc-6(a+b+c)(ab+bc+ca)$$
$$\Leftrightarrow (a^2+b^2+c^2+ab+bc+ca)^2+12$$
$$+\,6(a+b+c)(ab+bc+ca)-6abc \;\geq\; 0.$$

Because $(a+b+c)(ab+bc+ca)-abc=(a+b)(b+c)(c+a)$, we obtain

$$\frac{1}{4}[(a+b)^2+(b+c)^2+(c+a)^2]+12+6(a+b)(b+c)(c+a) \;\geq\; 0.$$

Denoting $z=\frac{a+b}{2}$, $x=\frac{b+c}{2}$, $y=\frac{c+a}{2}$, we have to prove that

$$(x^2+y^2+z^2)^2+24xyz+48 \;\geq\; 0.$$

By AM-GM Inequality, we have

$$(x^2+y^2+z^2)^2 \;\geq\; 9|xyz|^{4/3}$$

and because $24xyz \;\geq\; -24|xyz|$, it suffices to prove that

$$9t^4-24t^3+48 \;\geq\; 0, \qquad \text{where} \qquad t \;=\; |xyz|^{1/3}.$$

This is true, since $9t^4-24t^3+48=3(t-2)^2(3t^2+4t+4) \;\geq\; 0$.

Next we turn to problems of the 50^{th} Mathematical Olympiad of the Republic of Moldova given at [2009 : 377–378].

1. Let a, b, and c be the side lengths of a right triangle with hypotenuse of length c, and let h be the altitude from the right angle. Find the maximum value of $\dfrac{c+h}{a+b}$.

Solved by Arkady Alt, San Jose, CA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Bataille.

Let $Q=\frac{c+h}{a+b}$. We show that the maximum value of Q is $\frac{3\sqrt{2}}{4}$.
We have

$$Q \;=\; \frac{c^2+hc}{(a+b)c}=\frac{a^2+b^2+ab}{(a+b)\sqrt{a^2+b^2}}.$$

Clearly, $Q = \frac{3\sqrt{2}}{4}$ when $a = b$. To prove that $Q \le \frac{3\sqrt{2}}{4}$ in any case, we rewrite this inequality successively as

$$4(a^2 + b^2 + ab) \le 3\sqrt{2}(a+b)\sqrt{a^2+b^2}$$

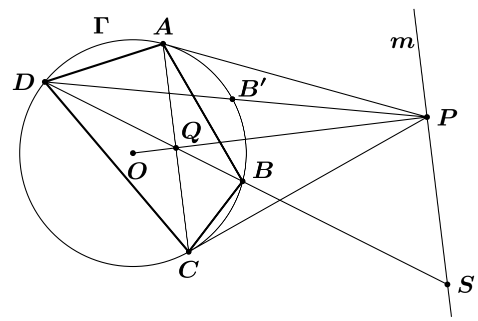
$$16(a^2+b^2)^2 + 16a^2b^2 + 32ab(a^2+b^2) \le 18(a^2+b^2)(a^2+b^2+2ab)$$

$$16a^2b^2 \le 2(a^2+b^2)^2 + 4ab(a^2+b^2). \qquad (1)$$

Now, $a^2+b^2 \ge 2ab > 0$, hence $2(a^2+b^2)^2 \ge 8a^2b^2$ and $4ab(a^2+b^2) \ge 8a^2b^2$ so that (1) certainly holds. This completes the proof.

3. The quadrilateral $ABCD$ is inscribed in a circle. The tangents to the circle at A and C intersect at a point P not on BD and such that $PA^2 = PB \cdot PD$. Prove that BD passes through the midpoint of AC.

Solution by Michel Bataille, Rouen, France.



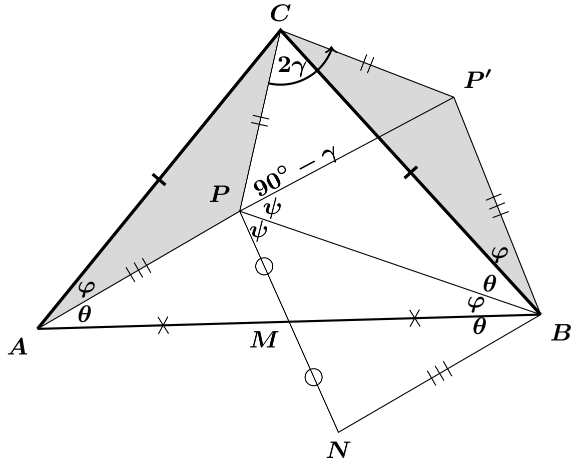
Let B' be the second point of intersection of PD with the circle $\Gamma = (ABCD)$. Note that $B' \neq B$ (since P is not on BD) and that $PB' \cdot PD$ is the power of P with respect to Γ. Thus $PB' \cdot PD = PA^2 = PB \cdot PD$, so that $PB' = PB$. If O is the centre of Γ, we also have $OB = OB'$, hence the line OP is the perpendicular bisector of BB'. It follows that OP is the bisector of the angle $\angle BPD$.

Now, let m be the perpendicular to OP at P. The lines OP and m are the two bisectors of the lines BP, BD, hence (PD, PO, PB, m) is a harmonic pencil of lines and PO and m meet BD at points Q and S which are conjugate with respect to Γ. As a result, m is the polar of Q since it passes through S and is perpendicular to OQ. Finally, Q is on the polar AC of P and so is the common point of OP and AC, that is, the midpoint of AC. This completes the proof.

6. Triangle ABC is isosceles with $AC = BC$ and P is a point inside the triangle such that $\angle PAB = \angle PBC$. If M is the midpoint of AB, prove that $\angle APM + \angle BPC = 180^\circ$.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Let $\theta = \angle PAB = \angle PBC$, $\varphi = \angle PAC = \angle PBA$, $\psi = \angle MPB$, $2\gamma = \angle ACB$. Then $\theta + \varphi + \gamma = 90^\circ$.



Rotate $\triangle ACP$ about C counterclockwise through an angle 2γ to the position BCP' and draw PP'. We have $CP = CP'$, $PA = P'B$, $\angle P'BC = \varphi$, and $\angle CPP' = 90^\circ - \gamma$.

Extend PM past M its own length to a point N and draw BN. Then $\triangle AMP \cong \triangle BMN$, so $NB = PA$ and $\angle NBM = \theta$. It now follows that $\triangle P'BP \cong \triangle NBP$, so $\angle P'PB = \psi$. Consequently,

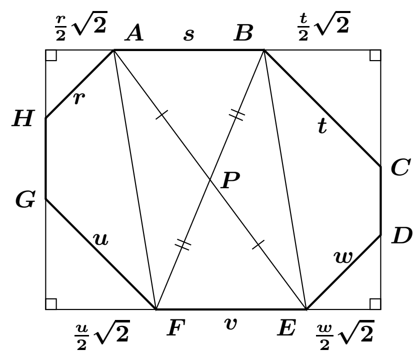
$$\angle APM + \angle BPC \;=\; [180^\circ - (\theta + \varphi) - \psi] + [(90^\circ - \gamma) + \psi] \;=\; 180^\circ\,.$$

7. The interior angles of a convex octagon are all equal and all side lengths are rational numbers. Prove that the octagon has a centre of symmetry.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Our notation is evident from the figure.

The interior angle sum of the octagon $ABCDEFGH$ is 1080°, hence each interior angle is 135° and each exterior angle 45°. Thus, the octagon can be enclosed in a rectangle, as shown. Consequently, $s + \frac{r+t}{2}\sqrt{2} = v + \frac{u+w}{2}\sqrt{2}$, that is $s + a\sqrt{2} = v + b\sqrt{2}$, where s, v, a, b are rational numbers.



If $a \neq b$, then $\sqrt{2}\frac{v-s}{a-b}$ would be rational, which is not so. Thus, $a = b$, hence $s = v$. Therefore, $ABEF$ is a parallelogram and its diagonals AE and BF bisect each other at P, which is the centre of symmetry of $ABEF$.

Similarly, this same point P is the centre of symmetry of parallelogram $BCFG$, and so on around the octagon. Thus, P is the centre of symmetry of the octagon.

8. Let $M = \{x^2 + x \mid x \text{ is a positive integer}\}$. For each integer $k \geq 2$ prove that there exist a_1, a_2, $\ldots$, a_k, b_k in M such that $a_1 + a_2 + \cdots + a_k = b_k$.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

The proof is by induction on k.

Note that $x^2 + x = x(x+1)$, so each element of M is even.

Since $12 + 30 = 42$ and $12 = 3 \cdot 4$, $30 = 5 \cdot 6$, $42 = 6 \cdot 7 \in M$, the desired result holds for $k = 2$.

Now the inductive step. Suppose that for some $k \geq 2$ we have $a_1 + a_2 + \cdots + a_k = b_k$, where $a_1, a_2, \ldots, a_k$, $b_k \in M$. Since b_k is even, we have $b_k = 2c$ for some positive integer c. Moreover, $b_k \geq a_1 + a_2 \geq 4$, so $c \geq 2$. Let $a_{k+1} = (c-1)c \in M$. Then

$$a_1 + a_2 + \cdots + a_k + a_{k+1} \;=\; 2c + (c-1)c \;=\; c(c+1) \in M,$$

and the induction is complete.

Next we turn to solutions from our readers to problems of the Republic of Moldova Mathematical Olympiad Second and Third Team Selection Tests given at [2009 : 378–379].

3. Let a, b, c be the side lengths of a triangle and let s be the semiperimeter. Prove that

$$a\sqrt{\frac{(s-b)(s-c)}{bc}} \;+\; b\sqrt{\frac{(s-c)(s-a)}{ac}} \;+\; c\sqrt{\frac{(s-a)(s-b)}{ab}} \;\geq\; s\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

Let $x := s-a, y := s-b, z := s-c$ then $x, y, z > 0$, $a = y+z$, $b = z+x$, $c = x+y$, $s = x+y+z$ and the original inequality becomes

$$\sum_{cyc} (y+z)\sqrt{\frac{yz}{(x+y)(z+x)}} \geq x+y+z,$$

where $x, y, z > 0.$

Since

$$\sum_{cyc} (y+z)\sqrt{\frac{yz}{(x+y)(z+x)}} \;=\; \sum_{cyc} \frac{(y+z)\sqrt{yz\,(x+y)(z+x)}}{(x+y)(z+x)}$$

and by Cauchy and AM-GM Inequalities

$$\begin{aligned}(y+z)\sqrt{yz(x+y)(z+x)} &\geq (y+z)\sqrt{yz\left(x+\sqrt{yz}\right)^2}\\ &= (y+z)\sqrt{yz}\left(x+\sqrt{yz}\right)\\ &= x(y+z)\sqrt{yz}+(y+z)yz\\ &\geq 2x\sqrt{yz}\sqrt{yz}+(x+y)yz\\ &= 2xyz+(y+z)yz\\ &= yz((x+y)+(x+z))\end{aligned}$$

then

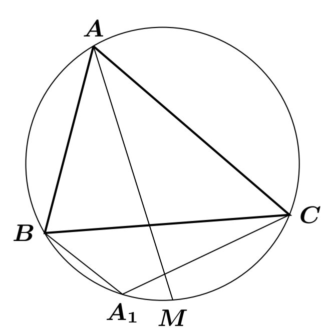
$$\begin{aligned}\sum_{cyc}\frac{(y+z)\sqrt{yz(x+y)(z+x)}}{(x+y)(z+x)} &\geq \sum_{cyc}\frac{yz((x+y)+(x+z))}{(x+y)(z+x)}\\ &= \sum_{cyc}\left(\frac{yz}{z+x}+\frac{yz}{x+y}\right)\\ &= \sum_{cyc}\frac{yz}{z+x}+\sum_{cyc}\frac{yz}{x+y}\\ &= \sum_{cyc}\frac{zx}{x+y}+\sum_{cyc}\frac{yz}{x+y}=\sum_{cyc}\frac{zx+yz}{x+y}\\ &= \sum_{cyc}\frac{z(x+y)}{x+y}=x+y+z.\end{aligned}$$

5. The point P is in the interior of triangle ABC. The rays AP, BP, and CP cut the circumcircle of the triangle at the points A_1, B_1, and C_1, respectively. Prove that the sum of the areas of the triangles A_1BC, B_1AC, and C_1AB does not exceed $s(R-r)$, where s, R, and r are the semiperimeter, the circumradius, and the inradius of triangle ABC, respectively.

Solution by Titu Zvonaru, Cománeşti, Romania.

We will prove the statement of the problem for the points A_1, B_1, C_1 such that A_1 belongs to arc BC which does not contain point A, and similarly for B_1 and C_1.

Let $[XYZ]$ be the area of $\triangle XYZ$. We denote $a=BC$, $b=CA$, $c=AB$. Let M be the mid-point of arc BC which contains the point A_1 (which does not contain the point A). It is easy to see that



$$[A_1BC] \leq [MBC]. \qquad (1)$$

We have

$$\angle MBC = \angle MCB = \angle MAC = \frac{A}{2}.$$

By the Law of Sines, we obtain $BM = 2R\sin\frac{A}{2}$. It follows that

$$[BMC] = \frac{BM \cdot BC \cdot \sin\angle MBC}{2} = aR\sin^2\frac{A}{2} \qquad (2)$$

By (1) and (2), it follows that

$$\begin{aligned}
&[A_1BC] + [B_1AC] + [C_1AB] \\
&\quad \le aR\sin^2\frac{A}{2} + bR\sin^2\frac{B}{2} + cR\sin^2\frac{C}{2} \\
&\quad = \frac{aR(1-\cos A) + bR(1-\cos B) + cR(1-\cos C)}{2} \\
&\quad = \frac{1}{2}R(a+b+c) - \frac{R}{2}(a\cos A + b\cos B + c\cos C) \\
&\quad = sR - \frac{R^2}{2}(2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C) \\
&\quad = sR - \frac{R^2}{2}(\sin 2A + \sin 2B + \sin 2C) \\
&\quad = sR - \frac{R^2}{2}\cdot 4\sin A\sin B\sin C = sR - \frac{R^2}{2}\cdot 4\cdot\frac{abc}{8R^3} \\
&\quad = sR - \frac{abc}{4R} = sR - [ABC] = sR - sr = s(R-r)\,.
\end{aligned}$$

The equality holds if and only if AA_1, BB_1, CC_1 are the bisectors of $\triangle ABC$.

7. Let a, b, and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a+3}{(a+1)^2} + \frac{b+3}{(b+1)^2} + \frac{c+3}{(c+1)^2} \ge 3\,.$$

Solved by Arkady Alt, San Jose, CA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Zvonaru.

We have

$$\begin{aligned}
\frac{2(a+3)}{(a+1)^2} - 2 &= \frac{2a + 6 - 2a^2 - 4a - 2}{(1+a)^2} \\
&= \frac{1 - 2a + a^2 + 3(1-a^2)}{(1+a)^2} = \left(\frac{1-a}{1+a}\right)^2 + 3\cdot\frac{1-a}{1+a}.
\end{aligned}$$

Denoting $x = \frac{1-a}{1+a}$, $y = \frac{1-b}{1+b}$, and $z = \frac{1-c}{1+c}$, it results that $x, y, z \in [-1, 1]$ and we have to prove that

$$x^2 + y^2 + z^2 + 3(x+y+z) \ge 0. \qquad (1)$$

Since $abc = 1$, we obtain

$$x+y+z = \frac{(1-a)(1+b+c+bc)+(1-b)(1+a+c+ac)}{(1+a)(1+b)(1+c)} + \frac{(1-c)(1+a+b+ab)}{(1+a)(1+b)(1+c)}$$
$$= \frac{a+b+c-ab-bc-ca}{(1+a)(1+b)(1+c)}$$
$$= \frac{-1+a+b(1-a)+c(1-a)-bc(1-a)}{(1+a)(1+b)(1+c)}$$
$$= -\frac{(1-a)(1-b)(1-c)}{(1+a)(1+b)(1+c)} = -xyz.$$

Thus, the inequality (1) is equivalent to

$$x^2+y^2+z^2 \;\geq\; 3xyz,$$

which is true because by AM-GM Inequality and since $x, y, z \in [-1, 1]$ we have

$$x^2+y^2+z^2 \;\geq\; 3\sqrt[3]{x^2y^2z^2} \geq 3xyz.$$

Next we move to the Russian Mathematical Olympiad 2007, 11$^{\text{th}}$ Grade, given at [2010: 151–152].

1. *(N. Agakhanov)* The product $f(x) = \cos x \cos 2x \cos 3x \ldots \cos 2^k x$ is written on the blackboard $(k \geq 10)$. Prove that it is possible to replace one "**cos**" by "**sin**" such that the product obtained $f_1(x)$ satisfies the inequality $|f_1(x)| \leq 3 \cdot 2^{-1-k}$ for all real k.

Solved by Oliver Geupel, Brühl, NRW, Germany.

We prove that it suffices to replace the factor "$\cos 3x$" by "$\sin 3x$" whenever $k \geq 2$.

We start with the identity

$$\sin x \cdot \cos x \cos 2x \cos 4x \cdots \cos 2^k x = \frac{\sin 2^{k+1}x}{2^{k+1}} \qquad (k \geq 0). \qquad (1)$$

We prove (1) by induction. It is obvious for $k = 0$. Assume that it holds for some fixed integer $k \geq 0$. Then

$$\sin x \cdot \cos x \cos 2x \cos 4x \cdots \cos 2^k x \cos 2^{k+1}x = \frac{\sin 2^{k+1}x}{2^{k+1}} \cdot \cos 2^{k+1}x$$
$$= \frac{\sin 2^{k+2}x}{2^{k+2}},$$

which completes the induction and therefore the proof of (1).

Moreover, we have that

$$\begin{aligned}\sin 3x &= \sin x\cos 2x + \cos x\sin 2x\\ &= \sin x(1-2\sin^2 x) + 2\sin x(1-\sin^2 x)\\ &= 3\sin x - 4\sin^3 x.\end{aligned}$$

Putting this all together, we conclude that

$$\begin{aligned}|f_1(x)| &\le |3\sin x - 4\sin^3 x|\cdot|\cos x\cos 2x\cos 4x\cdots\cos 2^k x|\\ &= |3-4\sin^2 x|\cdot|\sin x|\cdot|\cos x\cos 2x\cos 4x\cdots\cos 2^k x|\\ &\le 3\cdot\left|\frac{\sin 2^{k+1}x}{2^{k+1}}\right| \le 3\cdot 2^{-1-k}.\end{aligned}$$

2. *(A. Polyansky)* The incircle of a triangle ABC touches sides BC, AC, AB at points A_1, B_1, C_1, respectively. Segment AA_1 intersects the incircle again at point Q. Line ℓ is parallel to BC and passes through A. Lines A_1C_1 and A_1B_1 intersect ℓ at points P and R, respectively. Prove that $\angle PQR = \angle B_1QC_1$.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give Kandall's presentation.

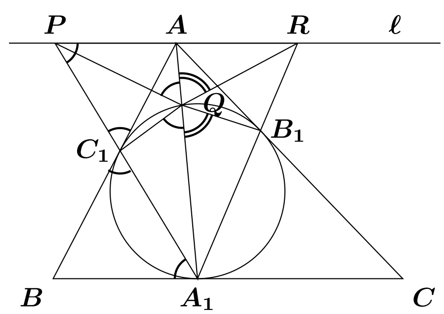
Observe that

$$\begin{aligned}\angle A_1QC_1 &= \frac{1}{2}\widehat{A_1C_1} = \angle BA_1C_1 = \angle BC_1A_1,\\ \angle BA_1C_1 &= \angle APC_1, \quad \angle BC_1A_1 = \angle PC_1A\end{aligned}$$

so $\angle APC_1 = \angle PC_1A$. Therefore, $\angle APC_1$ is supplementary to $\angle AQC_1$, so quadrilateral $PAQC_1$ is cyclic. Consequently,

$$\angle PQA = \angle PC_1A = \angle A_1QC_1$$

and similarly, $\angle RQA = \angle A_1QB_1$. Thus, by angle addition, $\angle PQR = \angle B_1QC_1$.



Next we look at solutions to problems of the XV Olympíada Matemática Rioplatense, Nivel 2, given at [2010; 214] that we started last issue.

4. Let $a_1, a_2, \dots, a_n$ be positive numbers. The sum of all the products a_ia_j with $i < j$ is equal to 1. Show that there is a number among them such that the sum of the remaining numbers is less than $\sqrt{2}$.

Solved by Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Cománeşti, Romania. We give Zvonaru's write-up.

We prove by induction that

$$2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i a_j \;=\; \sum_{i=1}^{n}\left(a_i \sum_{\substack{j=1\\ j\neq i}}^{n} a_j\right). \tag{1}$$

For $n = 2$ we have $2a_1a_2 = a_1a_2 + a_2a_1$, which is true.

For $n = 3$ we have $2(a_1a_2 + a_1a_3 + a_2a_3) = a_1(a_2 + a_3) + a_2(a_1 + a_3) + a_3(a_1 + a_2)$, which is also true.

Suppose that (1) is true for some $n = k$, $k > 1$. We have to prove that

$$2\sum_{i=1}^{k}\sum_{j=i+1}^{k+1} a_i a_j \;=\; \sum_{i=1}^{k+1}\left(a_i \sum_{\substack{j=1\\ j\neq i}}^{k+1} a_j\right). \tag{2}$$

We have

$$\begin{aligned}
2\sum_{i=1}^{k}\sum_{j=i+1}^{k+1} a_i a_j &= 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} a_i a_j + 2\sum_{i=1}^{k} a_i a_{k+1} \\
&= \sum_{i=1}^{k}\left(a_i \sum_{\substack{j=1\\ j\neq i}}^{k} a_j\right) + \sum_{i=1}^{k} a_i a_{k+1} + a_{k+1}\sum_{j=1}^{k} a_j \\
&= \sum_{i=1}^{k}\left(a_i\left(\sum_{\substack{j=1\\ j\neq i}}^{k} a_j + a_{k+1}\right)\right) + a_{k+1}\sum_{\substack{j=1\\ j\neq k+1}}^{k+1} a_j \\
&= \sum_{i=1}^{k}\left(a_i \sum_{\substack{j=1\\ j\neq i}}^{k+1} a_j\right) + a_{k+1}\sum_{\substack{j=1\\ j\neq k+1}}^{k+1} a_j \\
&= \sum_{i=1}^{k+1}\left(a_i \sum_{\substack{j=1\\ j\neq i}}^{k+1} a_j\right).
\end{aligned}$$

Denoting $s = a_1 + a_2 + \cdots + a_n$ and using (1) we obtain

$$2 \;=\; a_1(s - a_1) + a_2(s - a_2) + \cdots + a_n(s - a_n). \tag{3}$$

If there is i such that $s - a_i < \sqrt{2}$, we are done.

Suppose that for $i = 1, 2, \ldots, n$, $s - a_i \geq \sqrt{2}$. Using (3) we deduce that

$$\begin{aligned} 2 &= a_1(s - a_1) + a_2(s - a_2) + \cdots + a_n(s - a_n) \\ &\geq \sqrt{2}(a_1 + a_2 + \cdots + a_n), \end{aligned}$$

hence $a_1 + a_2 + \cdots + a_n \leq \sqrt{2}$. It follows that, for example, $a_1 + a_2 + \cdots + a_{n-1} < \sqrt{2}$, a contradiction with the inequality $s - a_n \geq \sqrt{2}$.

Next we move to solutions to problems of the XV Olympíada Matemática Rioplatense 2006, Nivel 3, given at [2010; 215–216].

1. (a) For each $k \geq 3$, find a positive integer n that can be represented as the sum of exactly k mutually distinct positive divisors of n.

(b) Suppose that n can be expressed as the sum of exactly k mutualy distinct positive divisors of n for some $k \geq 3$. Let p be the smallest prime divisor of n.

Show that
$$\frac{1}{p} + \frac{1}{p+1} + \cdots + \frac{1}{p+k-1} \geq 1\,.$$

Solved by Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give Geupel's write-up.

(a) Consider $n = 2^{k-2} \cdot 3$ with the divisors $d_j = 2^{k-2-j} \cdot 3 \quad (1 \leq j \leq k-2)$, $d_{k-1} = 2$, $d_k = 1$. We have

$$\sum_{j=1}^{k} d_j = 3 \sum_{j=1}^{k-2} 2^{k-2-j} + 3 = 3(2^{k-2} - 1) + 3 = 2^{k-2} \cdot 3 = n.$$

(b) Let $d_1 > d_2 > \ldots > d_k$ be divisors of the integer n such that $\sum_{j=1}^{k} d_j = n$.

By $d_1 < n$ we have
$$\frac{n}{d_1} \geq p.$$

Since $\frac{n}{d_1},\ \frac{n}{d_2},\ \ldots,\ \frac{n}{d_k}$ is an increasing sequence of integers, we also have

$$\frac{n}{d_2} \geq p+1,\ \frac{n}{d_3} \geq p+2,\ \ldots,\ \frac{n}{d_k} \geq p+k-1.$$

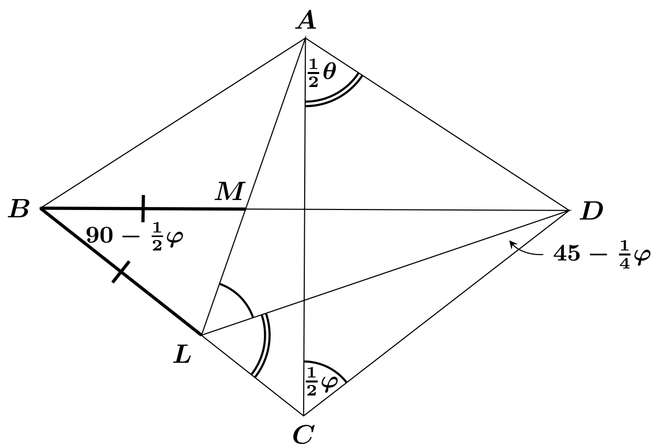
Consequently,

$$\frac{1}{p} + \frac{1}{p+1} + \cdots + \frac{1}{p+k-1} \geq \frac{1}{n} \sum_{j=1}^{k} d_j = 1,$$

which completes the proof.

2. Let $ABCD$ be a convex quadrilateral such that $AB = AD$ and $CB = CD$. The bisector of $\angle BDC$ cuts BC at L, and AL cuts BD at M, and it is known that $BL = BM$. Determine the value of $2\angle BAD + 3\angle BCD$.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Kandall's solution.



Let $\angle BAD = \theta$ and $\angle BCD = \varphi$ (in degrees). We want to determine $2\theta + 3\varphi$.

The line AC bisects $\angle BAD$ and $\angle BCD$; $\angle CBD = 90 - \frac{1}{2}\varphi$, $\angle CDL = \frac{1}{2}(90 - \frac{1}{2}\varphi) = 45 - \frac{1}{4}\varphi$. From $\triangle BLM$, $\angle BLM = \frac{1}{2}(90 + \frac{1}{2}\varphi) = 45 + \frac{1}{4}\varphi$; from $\triangle DLC$, $\angle DLC = 180 - \varphi - (45 - \frac{1}{4}\varphi) = 135 - \frac{3}{4}\varphi$. Consequently, $\angle ALD = 180 - (45 + \frac{1}{4}\varphi) - (135 - \frac{3}{4}\varphi) = \frac{1}{2}\varphi = \angle ACD$.

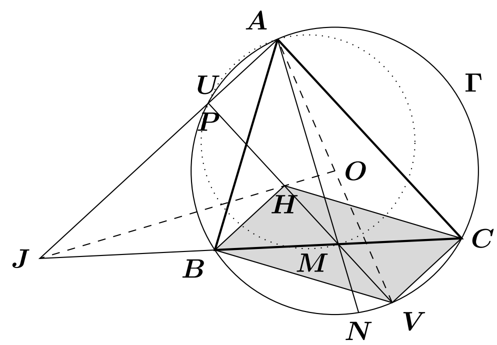
Therefore, quadrilateral $ALCD$ is cyclic, so $\angle DLC = \angle DAC$, that is, $135 - \frac{3}{4}\varphi = \frac{1}{2}\theta$. It follows easily that $2\theta + 3\varphi = 540$.

4. The acute triangle ABC with $(AB \neq AC)$ has circumcircle Γ, circumcentre O and orthocentre H. The midpoint of BC is M and the extension of the median AM intersects Γ at N. The circle of diameter AM intersects again Γ at A and P.

Show that the lines AP, GC, and OH are concurrent if and only if $AH = HN$.

Solved by Michel Bataille, Rouen, France; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Bataille's solution.

Let V denote the point diametrically opposite to A on Γ. Then, $VC \perp CA$, hence $BH \parallel VC$. Similarly $CH \parallel VB$ and it follows that $BHCV$ is a parallelogram with centre M. As a result, the line $HM = HV$ meets Γ again at U such that $AU \perp UM$. Therefore U is also on the circle with diameter AM and $U = P$.

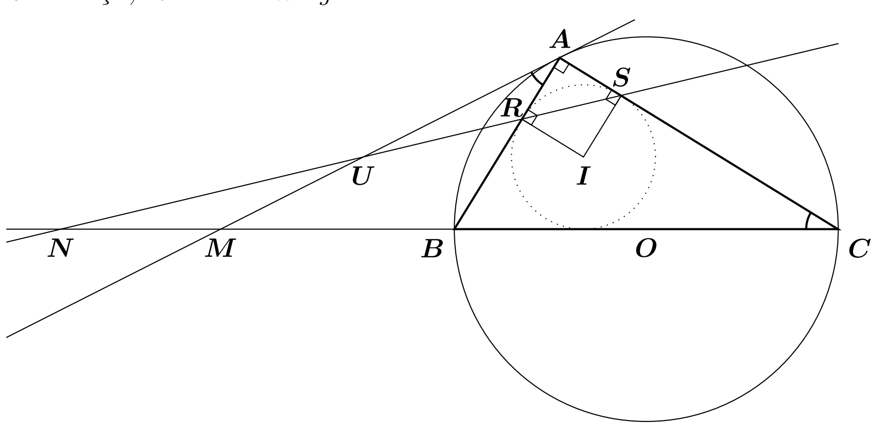


Now, let $\boldsymbol{AP}$ intersect the line $\boldsymbol{BC}$ at $\boldsymbol{J}$. The point $\boldsymbol{H}$ is also the orthocentre of $\boldsymbol{\Delta AJM}$ (since lines $\boldsymbol{AH}$ and $\boldsymbol{MP}$ are altitudes). Thus, the line $\boldsymbol{OH}$ passes through $\boldsymbol{J}$ if and only if $\boldsymbol{OH \perp AN}$. But $\boldsymbol{OA = ON}$, hence $\boldsymbol{OH}$ is perpendicular to $\boldsymbol{AN}$ if and only if $\boldsymbol{OH}$ is the perpendicular bisector of the line segment $\boldsymbol{AN}$ that is, if and only if $\boldsymbol{HA = HN}$. The result follows.

Next we turn to solutions from our readers to problems of the 21 Olimpiada Iberoamericana de Matematico, Guayoquil, given at [2010; 216–217].

1. In the scalene triangle $\boldsymbol{ABC}$, with $\boldsymbol{\angle BAC = 90^\circ}$, the tangent line to the circumcircle at $\boldsymbol{A}$ intersects the line $\boldsymbol{BC}$ at $\boldsymbol{M}$. Let $\boldsymbol{S}$ and $\boldsymbol{R}$ be the points where the incircle of $\boldsymbol{ABC}$ touches $\boldsymbol{AC}$ and $\boldsymbol{AB}$ respectively. The line $\boldsymbol{RS}$ intersects the line $\boldsymbol{BC}$ at $\boldsymbol{N}$. The lines $\boldsymbol{AM}$ and $\boldsymbol{SR}$ meet at $\boldsymbol{U}$. Show that triangle $\boldsymbol{UMN}$ is isosceles.

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Zelator; and Titu Zvonaru, Cománeşti, Romania. We give Bataille's solution.



Let $\boldsymbol{I}$ denote the incentre of $\boldsymbol{\Delta ABC}$. The quadrilateral $\boldsymbol{ARIS}$ is a rectangle (because $\boldsymbol{\angle ARI = \angle ASI = \angle RAI = 90^\circ}$), even a square since in addition $\boldsymbol{IR = IS}$. It follows that $\boldsymbol{\angle ARS = 45^\circ}$.

Observing that $\gamma = \angle ACB$ and $\angle UAB$ subtend the same arc of the circumcircle, we have

$$\angle NUM = \angle AUR = 180^\circ - \gamma - 135^\circ = 45^\circ - \gamma. \qquad (1)$$

From

$$\begin{aligned}\angle UMN = 180^\circ - \angle AMB &= \angle MAB + \angle MBA \\ &= \gamma + (180^\circ - (90^\circ - \gamma)) = 90^\circ + 2\gamma\end{aligned}$$

we deduce

$$\angle UNM = 180^\circ - (45^\circ - \gamma) - (90^\circ + 2\gamma) = 45^\circ - \gamma. \qquad (2)$$

From (1) and (2), $\angle NUM = \angle UMN$ and so ΔUMN is isosceles with $MN = MU$.

2. Let $a_1, a_2, \ldots, a_n$ be real numbers. Let d be the difference between the smallest and the largest of them, and let $s = \sum_{i<j} |a_i - a_j|$. Show that

$$(n-1)d \;\le\; s \;\le\; \frac{n^2 d}{4}$$

and determine the conditions under which equality holds in each equality.

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give Zvonaru's version.

Without loss of generality we may assume that $a_1 \ge a_2 \ge \cdots \ge a_n$, and we denote $d(a_1, \ldots, a_n) = a_1 - a_n$ and $s(a_1, \ldots, a_n) = \sum_{i<j} |a_i - a_j|$.

For $n = 2$, we have $s = d$ and both inequalities are equalities.

For $n = 3$, we have $s = 2d$; the left inequality is equality and the right inequality is true.

Assume that $n \ge 4$. We have

$$\begin{aligned} s(a_1, \ldots, a_n) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_i - a_j) \\ &= a_1 - a_2 + a_2 - a_n + \cdots + a_1 - a_{n-1} + a_{n-1} - a_n \\ &\quad + a_1 - a_n + \sum_{i=2}^{n-2} \sum_{j=i+1}^{n-1} (a_i - a_j) \\ &= (n-1)d(a_1, \ldots a_n) + s(a_2, \ldots, a_{n-1}). \end{aligned}$$

Since $s(a_2, \ldots a_{k-1}) \ge 0$, it results that the left inequality is true. The equality holds if and only if $n = 2$ or $n = 3$ or $n \ge 4$ and $a_2 = \cdots = a_{n-1}$ (that is, $s(a_2, \ldots, a_{n-1}) = 0$).

For the right inequality we proceed by induction. If $n = 2, 3$, the inequality is true. Suppose that the inequality is true for any $K \le n$ and we have to prove that

$$s(a_1, \ldots, a_{n+1}) \;\le\; \frac{(n+1)^2}{4} d(a_1, \ldots, a_{n+1}).$$

Since it is obvious that $d(a_2,\ldots,a_n) \le d(a_1,\ldots,a_{n+1})$, we obtain

$$\begin{aligned} s(a_1,\ldots,a_{n+1}) &= nd(a_1,\ldots,a_{n+1}) + s(a_2,\ldots,a_n) \\ &\le nd(a_1,\ldots,a_{n+1}) + \frac{(n-1)^2}{4} d(a_2,\ldots,a_n) \\ &\le nd(a_1,\ldots,a_{n+1}) + \frac{(n-1)^2}{4} d(a_1,\ldots,a_{n+1}) \\ &= \frac{4n+(n-1)^2}{4} d(a_1,\ldots,a_{k+1}) \\ &= \frac{(n+1)^2}{n} d(a_1,\ldots,a_{n+1}) \end{aligned}$$

and the induction is complete.

The equality holds if and only if $n = 2$ or $a_1 = a_2 = \cdots = a_n$.

3. The numbers $1, 2, 3, \ldots, n^2$ are placed in the cells of an $n \times n$ board, one number per cell. A coin is initially placed in the cell containing the number n^2. The coin can move to any of the cells which share a side with the cell it currently occupies.

First, the coin travels from the cell containing the number 1 to the cell containing the number n^2, using the smallest possible number of moves. Then the coin travels from the cell containing the number 1 to the cell containing the number 2 using the smallest number of moves, and then from the cell containing the number 3, and continuing until the coin returns to the initial cell, taking a shortest route each time it travels. The complete trip takes N steps. Determine the smallest and largest possible values of N.

Solved by Oliver Geupel, Brühl, NRW, Germany.

According to `http://www.imomath.com/othercomp/Ib/IbMO06.pdf` the trip should be $n^2 \to 1 \to 2 \to \ldots \to n^2-1 \to n^2$. We will show that $\min N$ is n^2 if n is even and n^2+1 if n is odd. Moreover, we prove that $\max N$ is n^3-2 for even n and n^3-n for odd n.

Since we must enter each of the n^2 cells, we have $\min N \ge n^2$. By colouring the cells alternately black and white like a chess board, we see that we can return to a cell of the colour of the initial cell only after an even number of steps; hence $\min N \ge n^2+1$ if n is odd. Examples of trips that reach these bounds are given in Figures 1 and 2, for even and odd n, respectively.

In each step one edge is passed. Thus, the total number of steps is equal to the number of passed edges, counted with the multiplicity of crossings. Consider a $k \times n$ rectangle consisting of the k leftmost or rightmost columns or of the k uppermost or lowermost rows of the board, where $1 \le k \le n/2$. The interior edge of this rectangle can be passed not more than $2kn$ times, specifically, twice for each cell, that is on entering and on leaving the rectangle.

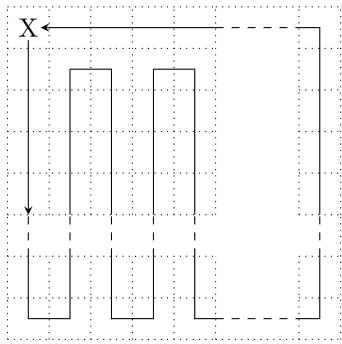


Figure 1

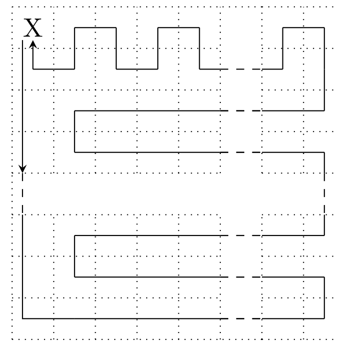


Figure 2

If n is odd, then

$$N \leq 4 \sum_{k=1}^{(n-1)/2} 2kn = (n-1)n(n+1) = n^3 - n.$$

If n is even, the board consists of four quadrants, where we can alternate between the upper-left and the lower-right quadrant as well as between the upper-right and the lower-left quadrant. We must at least twice change the pair of quadrants in order to return to the inital cell thus losing two crossings. Therefore,

$$N \leq 4 \sum_{k=1}^{(n-2)/2} 2kn + 2n^2 - 2 = n^3 - 2.$$

Examples of arrangements that reach these bounds are given in Figures 3 and 4, for $n = 2m$ and $n = 2m + 1$, respectively.

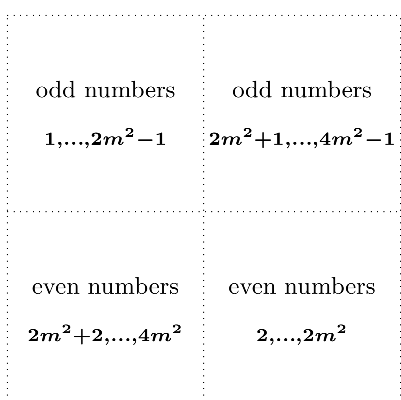


Figure 3

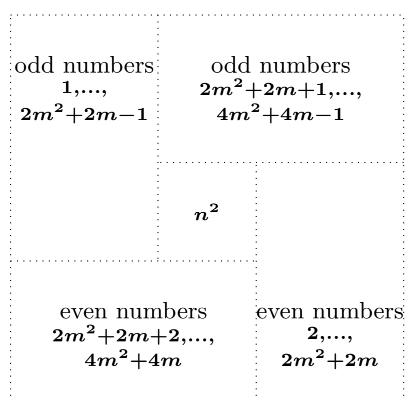


Figure 4

4. Determine all pairs (a, b) of positive integers such that $2a+1$ and $2b-1$ are relatively prime and $a+b$ divides $4ab+1$.

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give Zelator's write-up.

Suppose that a and b are positive integers such that $a+b \mid 4ab+1$ and $\gcd(2a+1, 2b-1) = 1$.

Then

$$4ab+1 \;=\; k \cdot (a+b)\,, \tag{1}$$

for some positive integer k. From $(2a+1)(2b+1) = 4ab + 2(a+b) + 1$ and (1), we obtain

$$(2a+1)(2b+1) \;=\; (k+2) \cdot (a+b)\,. \tag{2}$$

Observe that $a+b$ is relatively prime to $2a+1$. Indeed, if d is the greatest common divisor of $a+b$ and $2a+1$, then d must be a divisor of any linear combination (with integer coefficients) of $a+b$ and $2a+1$. In particular, d must divide $2(a+b) - (2a+1) = 2b-1$. Thus $d \mid 2a+1$ and $d \mid 2b-1$ and so, by the coprimeness condition in (1), it follows that $d = 1$.

We have shown that,

$$\gcd(a+b, 2a+1) \;=\; 1. \tag{3}$$

Euclid's lemma in number theory postulates that if an integer divides the product of two other integers, and it is relatively prime to one of those (two) integers, then it must divide the other one.

Clearly then, by Euclid's lemma, (3), and (2), it follows that $a+b$ must divide $2b+1$. Thus there exists a positive integer, m, such that

$$2b+1 \;=\; m \cdot (a+b)\,. \tag{4}$$

We rewrite (4) in the form,

$$m \cdot a + (m-2) \cdot b \;=\; 1\,. \tag{5}$$

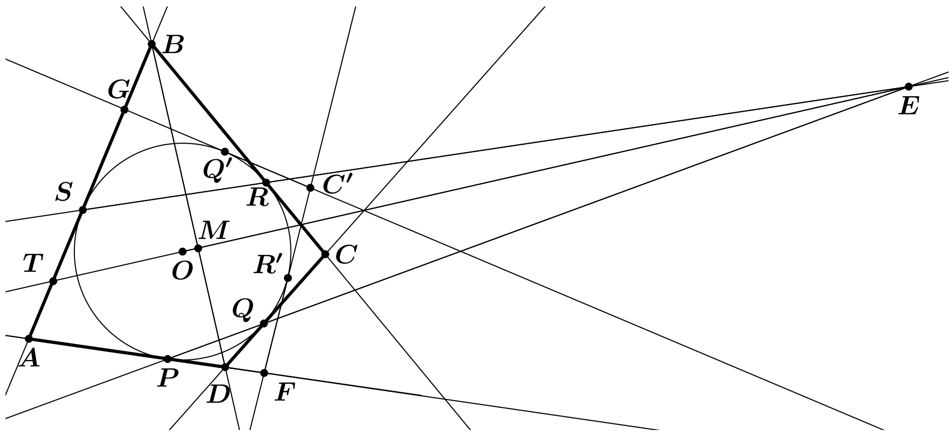
Clearly, if $m \geq 2$, the lefthand side of (5) is greater than 1, since a and b are positive integers. Therefore $m = 1$, which yields, $a - b \;=\; 1$ and $a \;=\; b+1$.

We have proven that if two positive integers a and b satisfy conditions (1), then $a \;=\; b+1$. The converse is also true. Indeed if $a \;=\; b+1$, then $4ab+1 = 4(b+1)b+1 = (2b+1)^2 = (a+b)^2$, which shows that $a+b$ divides $4ab+1$. Moreover, we have $2a+1 \;=\; 2(b+1)+1 \;=\; 2b+3$, which is relatively prime to $2b-1$, since if $D = \gcd(2b-1, 2b+3)$. Then $D \mid [(2b+3) - (2b-1)] = 4$. But D is odd, since $2b-1$ and $2b+3$ are. Thus $D = 1$.

Conclusion: The pairs $(a, b) = (b+1, b)$, where b can be any positive integer, are the solution to this problem.

5. The circle Γ is inscribed in quadrilateral $ABCD$ with AD and CD tangent to Γ, at P and Q, respectively. If BD intersects Γ at X and Y and M is the midpoint of XY, prove that $\angle AMP = \angle CMQ$.

Solved by Oliver Geupel, Brühl, NRW, Germany.



We argue in terms of projective geometry, assuming that parallel lines meet at a point of infinity. Let O be the midpoint of Γ, let E be the pole of BD with respect to Γ, and let BC and AB be tangent to Γ at points R and S, respectively. We assume without loss of generality that the points C and E are on the same side of the line BD.

Lemma 1. The lines AC, PQ, and RS meet at E. Moreover, $BD \perp EM$, and the points E, M, and O are collinear.

Proof. Let PQ meet AC at E_1, and let RS and AC meet at E_2. By $PD = PQ$ and Menelaus' Theorem, for $\triangle ACD$ and the line PQ it holds

$$\frac{E_1A}{E_1C} \cdot \frac{QC}{PA} = \frac{E_1A}{E_1C} \cdot \frac{QC}{QD} \cdot \frac{PD}{PA} = 1$$

Analogously,

$$\frac{E_2A}{E_2C} \cdot \frac{RC}{SA} = 1.$$

Since $QC = RC$ and $PA = SA$, we obtain

$$\frac{E_1A}{E_1C} = \frac{E_2A}{E_2C}$$

and consequently $E_1 = E_2$. The point E_1 lies on the polar PQ of point D. Hence, D is on the polar of E_1. Similarly, B is also on the polar of E_1. Thus, BD is the polar of E_1, that is, E_1 coincides with the pole E of BD, and $BD \perp EM$. Thus, E, M, and O are collinear. □

Let P', Q', R', and S' be the reflections of P, Q, R, and S in the line EM.

Lemma 2. The lines PQ', $P'Q$, RS', and $R'S$ meet at M.

Proof. Let U and V be the intersection of BD and PQ and the intersection of BD and $P'Q'$, respectively. Let the line UV meet PQ' at point W. By Menelaus' Theorem, for $\triangle EUV$ and the line PQ' it holds

$$\frac{PE}{PU} \cdot \frac{WU}{WV} \cdot \frac{Q'V}{Q'E} = 1;$$

hence, by $Q'V = QU$ and $Q'E = QE$,

$$\frac{PE}{PU} : \frac{QE}{QU} = \frac{WV}{WU}.$$

On the other hand, since E and U are harmonic conjugates with respect to the points P and Q, we have

$$\frac{PE}{PU} : \frac{QE}{QU} = 1.$$

Consequently, $W = M$, that is, M lies on PQ'. Analogously, M lies on $P'Q$, RS', and $R'S$. $\square$

We are now prepared to prove that $\angle AMP = \angle CMQ$.

Since EM bisects $\angle PMP'$, the orthogonal line BD bisects the complementary angle PMQ, that is,

$$\angle DMP = \angle DMQ. \tag{1}$$

Let C' be the reflection of C in the line EM. Since the tangents of Γ in Q and R meet at C, we see that the tangents in Q' and R' meet at C'. Let the tangents to Γ in P and R' meet at point F, and let the tangents in S and Q' meet at point G.

Consider the circumscribed hexagon $APFC'Q'G$. By Brianchon's Thereom, the lines AC', PQ', and FG are concurrent. Next, consider the circumscribed hexagon $FR'C'GSA$. By Brianchon's Theorem, the lines AC', $R'S$, and FG are concurrent. Therefore, the four lines AC', FG, PQ', and $R'S$ are concurrent. By Lemma 2, their intersection is the point M. We deduce that the points A, C', and M are collinear. Let T be the intersection of AB and EM. Then,

$$\angle AMT = \angle C'ME = \angle CME\,. \tag{2}$$

By (1) and (2), we obtain

$$\angle AMP = 90^\circ - \angle DMP - \angle AMT = 90^\circ - \angle DMQ - \angle CME = \angle CMQ,$$

which completes the proof.

That completes the material for this number of the *Corner*.