EDITORIAL

Shawn Godin

Over the last few years the number of people subscribing to ***CRUX with MAYHEM*** has been declining while the publication and shipping costs have been increasing. As a result, the CMS is looking for ways to revitalize the journal and so we are conducting a reader survey. We want to hear from the readers about which parts of the journal they like, and which parts should we should consider retiring. Also, there are a number of new features that we are considering for which we need your input. It is vital that we hear from you, the readers, so that we can deliver a journal that best suits your needs. The survey should take about five minutes to complete, so please take some time to give us some feedback. You can access the survey through our Facebook page:

`www.facebook.com/pages/`
`Crux-Mathematicorum-with-Mathematical-Mayhem/152157028211955`,

or you can access it directly at:

`http://www.surveymonkey.com/s/W982CPC`

We have been working our way through the vast backlog of problem proposals. We have just started going through the proposals received in 2011. The hope is, that we will get through the most recent year within the next couple of months. That way, when a proposal gets submitted, the author will know its fate within a reasonable amount of time. We are also looking into an online system for problem proposal and solution submission that will streamline the process. After we get up to date on the current year, we will continue with the proposals from previous years.

A few years back, former editor-in-chief Václav (Vazz) Linek suggested the categorization of ***Crux*** problems. This will allow us to deliver a more diverse set of problems and to let the readers know what our needs are. We will group the problems into the following areas: Algebra and Number Theory, Logic, Calculus, Combinatorics, Inequalities (including Geometric Inequalities), Geometry, Probability, and Miscellaneous Problems. Currently we have a large number of inequalities, enough to last us for years to come. Historically, we do receive a good supply of geometry questions. We are in real need of good questions from the other areas. Please continue to send us your good problems, especially from those other areas.

SKOLIAD No. 133

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **February 15, 2012**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is selected problems from the 21st Transylvanian Hungarian Mathematical Competition, 9th Form, 2011. Our thanks go to Dr. Mihály Bencze, president of The Transylvanian Hungarian Competition, Brasov, Romania, for providing us with this contest and for permission to publish it.

La rédaction souhaite remercier Rolland Gaudet, de Collège universitaire de Saint-Boniface, Winnipeg, MB, d'avoir traduit ce concours.

La 21e compétition mathématique hongroise-transylvanienne, 2011
9e classe, Problèmes choisis

1. Démontrer que si a, b, c, et d sont des nombres réels, alors

$$a+b+c+d-a^2-b^2-c^2-d^2 \leq 1 .$$

2. Comparons les deux nombres suivants,

$$A = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{2011 \text{ copies de } 2} \quad \text{and} \quad B = \underbrace{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}_{2010 \text{ copies de } 3} ;$$

lequel est le plus élevé, A ou B ? (Noter que a^{b^c} égale $a^{(b^c)}$ et non $(a^b)^c$.)

3. Déterminer toutes les solutions en entiers naturels à chacune des équations.

a. $20x^2 + 11y^2 = 2011$.

b. $20x^2 - 11y^2 = 2011$.

4. Dans le parallelogramme $ABCD$, on a $\angle BAD = 45°$ et $\angle ABD = 30°$. Démontrer que la distance de B à la diagonale AC est $\frac{1}{2}|AD|$.

5. Quelle est la prochaine année avec quatre vendredi 13 ?

21st Transylvanian Hungarian Mathematical Competition, 2011
Selected problems for the 9th form

1. Prove that if a, b, c, and d are real numbers, then

$$a + b + c + d - a^2 - b^2 - c^2 - d^2 \leq 1 .$$

2. Compare the following two numbers,

$$A = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{2011 \text{ copies of } 2} \quad \text{and} \quad B = \underbrace{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}_{2010 \text{ copies of } 3} ;$$

which is larger, A or B? (Note that a^{b^c} equals $a^{(b^c)}$, not $(a^b)^c$.)

3. Find all natural number solutions to each equation:

a. $20x^2 + 11y^2 = 2011$.

b. $20x^2 - 11y^2 = 2011$.

4. In the parallelogram $ABCD$, $\angle BAD = 45^\circ$ and $\angle ABD = 30^\circ$. Show that the distance from B to the diagonal AC is $\frac{1}{2}|AD|$.

5. What is the next year with four Friday the 13ths?

Next follow solutions to the British Columbia Secondary School Mathematics Contest 2010, Junior Final Round, Part B, given in Skoliad 127 at [2010 : 353 – 354].

1a. Find the sum of all positive integers less than 2010 for which the ones digit is either a '3' or an '8'.

Solution by Szera Pinter, student, Moscrop Secondary School, Burnaby, BC.

The positive numbers with ones digit 3 or 8 form an arithmetic sequence with common difference 5, namely $3, 8, 13, 18, \ldots, 2008$. The formula for the n^{th} term of an arithmetic sequence with first term a and common difference d is $t_n = a + d(n-1)$. In the problem, $a = 3$, $d = 5$, and the last term is 2008, so $2008 = 3 + 5(n-1)$, thus $n = 402$, whence the sequence has 402 terms.

The formula for the sum of an arithmetic sequence is

$$S = \frac{(\text{first term}) + (\text{last term})}{2} \cdot \binom{\text{number}}{\text{of terms}} .$$

The sum in the problem is therefore $S = \dfrac{3 + 2008}{2} \cdot 402 = 404\,211$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; DAVID FAN, student, Campbell Collegiate, Regina, SK; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

Solvers who are not familiar with arithmetic sequences can easily find the sum using Gauss's trick as explained in the comments to the solution to Problem 7 in Skoliad 129 given at [2010 : 485].

1b. Two cans, X and Y, both contain some water. From X Tim pours as much water into Y as Y already contains. Then, from Y he pours as much water into X as X already contains. Finally, he pours from X into Y as much water as Y already contains. Each can now contains 24 units of water. Determine the number of units of water in each can at the beginning.

Solution by David Fan, student, Campbell Collegiate, Regina, SK.

Let x be the number of units of water originally contained in can X, and let y be the number of units in can Y. The following table then shows the contents in each can as water is poured back and forth:

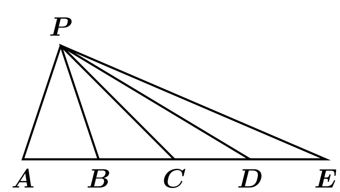
X	Y
x	y
$x-y$	$2y$
$2(x-y)=2x-2y$	$2y-(x-y)=3y-x$
$2x-2y-(3y-x)=3x-5y$	$2(3y-x)=6y-2x$

Thus $3x-5y=24$ and $6y-2x=24$. Solving these two simultaneous equations yields that $x=33$ and $y=15$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and JULIA PENG, student, Campbell Collegiate, Regina, SK.

Since the two cans together in the end hold 48 *units of water, you know from the outset that* $y=48-x$. *Whether using this fact saves effort is a matter of taste.*

2. The area of $\triangle APE$ shown in the diagram is 12. Given that $|AB| = |BC| = |CD| = |DE|$, determine the sum of the areas of all the triangles that appear in the diagram.



Solution by Janice Lew, student, École Alpha Secondary School, Burnaby, BC.

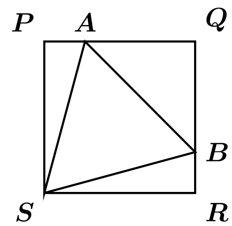
Since $|AB| = |BC| = |CD| = |DE|$, $\triangle APB$, $\triangle BPC$, $\triangle CPD$, and $\triangle DPE$ all have the same base and height, so they have the same area. Since $\triangle APE$ has area 12 each of $\triangle APB$, $\triangle BPC$, $\triangle CPD$, and $\triangle DPE$ have area 3.

The following table now lists all the triangles in the diagram and their areas:

	Area	Total
$\triangle APB$, $\triangle BPC$, $\triangle CPD$, and $\triangle DPE$	3	12
$\triangle APC$, $\triangle BPD$, $\triangle CPE$	6	18
$\triangle APD$ and $\triangle BPE$	9	18
$\triangle APE$	12	12
Grand total		60

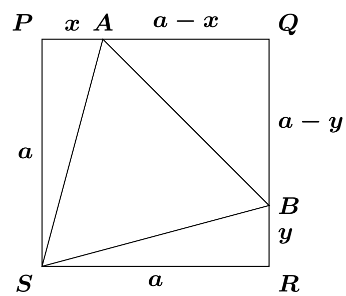
Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; DAVID FAN, student, Campbell Collegiate, Regina, SK; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC.

3. Given that $PQRS$ is a square and that ABS is an equilateral triangle (see the diagram), find the ratio of the area of $\triangle APS$ to the area of $\triangle ABQ$.



Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.

Let a denote the side length of the square, let x be $|PA|$, and let $y = |BR|$. By the Pythagorean Theorem, $|SA| = \sqrt{a^2+x^2}$ and $|SB| = \sqrt{a^2+y^2}$. Since $\triangle ABS$ is equilateral, $|SA| = |SB|$, so $a^2+x^2 = a^2+y^2$, so $x^2 = y^2$, so $x = y$.



Using the Pythagorean Theorem on $\triangle AQB$ yields that $|AB|^2 = (a-x)^2 + (a-y)^2$, but $|AB|^2 = |SA|^2 = a^2+x^2$ and $x = y$, so

$$a^2 + x^2 = 2(a-x)^2 = 2(a^2 - 2ax + x^2) = 2a^2 - 4ax + 2x^2\,.$$

Thus $2ax = a^2 - 2ax + x^2 = (a-x)^2$.

Now, the area of $\triangle APS$ is $\frac{1}{2}ax$, and the area of $\triangle ABQ$ is $\frac{1}{2}(a-x)(a-y) = \frac{1}{2}(a-x)^2 = \frac{1}{2}\cdot 2ax = ax$. Hence the ratio of the two areas is $\frac{1}{2}ax : ax = \frac{1}{2} : 1 = 1 : 2$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

4. Find the five distinct integers for which the sums of each distinct pair of integers are the numbers 0, 1, 2, 4, 7, 8, 9, 10, 11, and 12.

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

If the five numbers are a, b, c, d, and e, and $a < b < c < d < e$, then $a+b < a+c$, and all the other sums are larger. Thus $a+b = 0$ and $a+c = 1$, so $\boldsymbol{b = -a}$ and $\boldsymbol{c = b + 1}$. Moreover, $\boldsymbol{c + e < d + e}$ and all the other sums are smaller. Thus $\boldsymbol{c + e = 11}$ and $\boldsymbol{d + e = 12}$, so $\boldsymbol{d = c + 1 = b + 2}$. That is, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are consecutive integers. Since $\boldsymbol{c + e = 11}$, it follows that $\boldsymbol{b + e = 10}$.

Since $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are consecutive integers, so are $\boldsymbol{b + c}$, $\boldsymbol{b + d}$, and $\boldsymbol{c + d}$. Among the given sums, only $\mathbf{4}$, $\mathbf{7}$, $\mathbf{8}$, and $\mathbf{9}$ remain. Thus $\boldsymbol{b + c = 7}$, $\boldsymbol{b + d = 8}$, and $\boldsymbol{c + d = 9}$. Finally, $\boldsymbol{a + e = 4}$.

To summarise, $\boldsymbol{a + b = 0}$, $\boldsymbol{a + c = 1}$, $\boldsymbol{a + d = 2}$, $\boldsymbol{a + e = 4}$, $\boldsymbol{b + c = 7}$, $\boldsymbol{b + d = 8}$, $\boldsymbol{b + e = 10}$, $\boldsymbol{c + d = 9}$, $\boldsymbol{c + e = 11}$, and $\boldsymbol{d + e = 12}$.

Since $\boldsymbol{a + b = 0}$ and $\boldsymbol{a + c = 1}$, $\boldsymbol{2a + b + c = 1}$. But $\boldsymbol{b + c = 7}$, so $\boldsymbol{2a = -6}$, so $\boldsymbol{a = -3}$. Therefore $\boldsymbol{b = 3}$, $\boldsymbol{c = 4}$, $\boldsymbol{d = 5}$, and $\boldsymbol{e = 7}$.

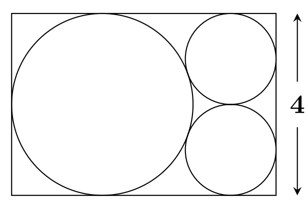
Also solved by DAVID FAN, student, Campbell Collegiate, Regina, SK; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and JULIA PENG, student, Campbell Collegiate, Regina, SK.

Placing the ten sums in a chart like this

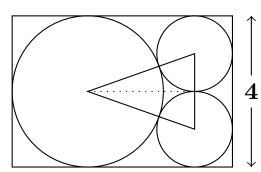
$\boldsymbol{a+b}$			
$\boldsymbol{a+c}$	$\boldsymbol{b+c}$		
$\boldsymbol{a+d}$	$\boldsymbol{b+d}$	$\boldsymbol{c+d}$	
$\boldsymbol{a+e}$	$\boldsymbol{b+e}$	$\boldsymbol{c+e}$	$\boldsymbol{d+e}$

can make the bookkeeping easier. Note that the sums increase as you move down or right in the diagram.

5. A rectangle contains three circles, as in the diagram, all tangent to the rectangle and to each other. The height of the rectangle is $\mathbf{4}$. Determine the width of the rectangle.



Solution by Julia Peng, student, Campbell Collegiate, Regina, SK.



Since the height of the rectangle is $\mathbf{4}$, the radius of the large circle is $\mathbf{2}$, and both the small circles have radius $\mathbf{1}$. Now connect the centres of the three circles. This forms an isosceles triangle with sides $\mathbf{3}$ and base $\mathbf{2}$. By the Pythagorean Theorem, the height of this triangle is $\boldsymbol{\sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}}$. The distance from the base of the triangle to the right-hand edge of the rectangle is $\mathbf{1}$, and the distance from the left vertex of the triangle to the left edge of the rectangle is $\mathbf{2}$, so the width of the rectangle is $\boldsymbol{2 + 2\sqrt{2} + 1 = 3 + 2\sqrt{2}}$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and JANICE LEW, student, École Alpha Secondary School, Burnaby, BC.

This issue's prize of one copy of ***Crux Mathematicorum*** for the best solutions goes to Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.

We wish our readers the best of luck solving our featured contest.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **15 février 2012**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

M488. *Proposé par l'Équipe de Mayhem.*

Un triangle a les sommets (x_1, y_1), (x_2, y_2) et (x_3, y_3).

(a) Si $x_1 < x_2 < x_3$ et $y_3 < y_1 < y_2$, déterminer la surface du triangle.

(b) Démontrer que si on laisse tomber les conditions sur x_1, x_2, x_3, y_1, y_2 et y_3, alors l'expression que vous avez fournie en (a) donne soit la surface, soit -1 fois la surface.

M489. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Démontrer que si m et n sont des entiers positifs relativement premiers tels que

$$m\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2010}\right) = n,$$

alors 2011 divise n.

M490. *Proposé par Johan Gunardi, étudiant, SMPK 4 BPK PENABUR, Jakarta, Indonésie.*

Pour n entier positif, soit $S(n)$ la somme des chiffres dans l'expression décimale (base 10) de n. Soit m un entier positif donné ; démontrer qu'il existe n entier positif tel que $m = \dfrac{S(n^2)}{S(n)}$.

M491. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Soient a, b et c des constantes, pas nécessairement distinctes. Résoudre l'équation ci-bas :

$$\frac{(x-a)^2}{(x-a)^2-(b-c)^2} + \frac{(x-b)^2}{(x-b)^2-(c-a)^2} + \frac{(x-c)^2}{(x-c)^2-(a-b)^2} = 1\,.$$

M492. *Proposé par Pedro Henrique O. Pantoja, étudiant, UFRN, Brésil.*

Démontrer que

$$\sum_{k=0}^{2009}(k+1)![6^k(6k+11)-k-1] = 2011!\left(6^{2010}-1\right)\,.$$

M493. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Déterminer tous les entiers positifs x qui satisfont à l'équation

$$\frac{x+\left[\sqrt{x}+\sqrt{x+1}\right]}{\left[\sqrt{4x+1}+4022\right]} + \frac{x}{\left[\sqrt{4x+2}\right]+4022} = 1\,,$$

où $[x]$ est la partie entière de x.

M494. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit z un nombre complexe tel que $|z| = 2$. Déterminer la valeur minimum de $\left|z - \dfrac{1}{z}\right|$.

. . . .

M488. *Proposed by the Mayhem Staff.*

A triangle has vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3).

(a) If $x_1 < x_2 < x_3$ and $y_3 < y_1 < y_2$, determine the area of the triangle.

(b) Show that, if the conditions on x_1, x_2, x_3, y_1, y_2, and y_3 are dropped, the expression from (a) gives either the area or -1 times the area.

M489. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Prove that if m and n are relatively prime positive integers such that

$$m\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2010}\right)=n,$$

then 2011 divides n.

M490. *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

For any positive integer n, let $S(n)$ denote the sum of the digits of n (in base 10). Given a positive integer m, prove that there exists a positive integer n such that $m=\dfrac{S(n^2)}{S(n)}$.

M491. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let a, b, and c be given constants, not necessarily distinct. Solve the equation below:

$$\frac{(x-a)^2}{(x-a)^2-(b-c)^2}+\frac{(x-b)^2}{(x-b)^2-(c-a)^2}+\frac{(x-c)^2}{(x-c)^2-(a-b)^2}=1\,.$$

M492. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Prove that

$$\sum_{k=0}^{2009}(k+1)![6^k(6k+11)-k-1]=2011!\left(6^{2010}-1\right)\,.$$

M493. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Find all positive integers x that satisfy the equation

$$\frac{x+\left[\sqrt{x}+\sqrt{x+1}\right]}{\left[\sqrt{4x+1}+4022\right]}+\frac{x}{\left[\sqrt{4x+2}\right]+4022}=1\,,$$

where $[x]$ is the integer part of x.

M494. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

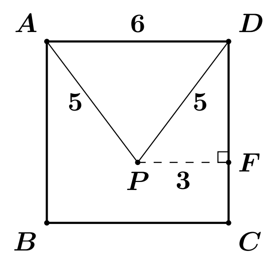
Let z be a complex number such that $|z|=2$. Find the minimum value of $\left|z-\dfrac{1}{z}\right|$.

Mayhem Solutions

M451. *Proposed by the Mayhem Staff.*

Square $ABCD$ has side length 6. Point P is inside the square so that $AP = DP = 5$. Determine the length of PC.

Solution by Florencio Cano Vargas, Inca, Spain.



We draw the line parallel to BC that passes through P. Let F be the intersection point of this line with DC. Since $PA = PD$, the point P lies on the perpendicular bisector of AD and then $PF = \frac{AD}{2} = 3$. Triangle DPF is a right-angled triangle, so:

$$DF = \sqrt{DP^2 + PF^2} = 4\,.$$

Moreover, triangle CFP is also right angled, hence:

$$PC = \sqrt{PF^2 + CF^2} = \sqrt{3^2 + 2^2} = \sqrt{13}\,.$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JORGE ARMERO JIMÉNEZ, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; SCOTT BROWN, Auburn University, Montgomery, AL, USA; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; GERHARDT HINKLE, Student, Central High School, Springfield, MO, USA; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; HEEYOON KIM, Conyers, GA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; JAGDISH MADNANI, Bangalore, India; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

M452. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

(a) Suppose that $x = \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}$ for some real number $a > 0$. Prove that $x^2 - a = x$.

(b) Determine the integer equal to

$$\frac{\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}}{\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}} - \sqrt{42 + \sqrt{42 + \sqrt{42 + \cdots}}}.$$

Solution by Pedro Henrique O. Pantoja, student, UFRN, Brazil.

(a) By squaring both sides of the equation we have,

$$x^2 = a + \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}} \Rightarrow x^2 - a = x\,.$$

(b) From part (a) we have that $\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}$ is the positive root of the equation $x^2 - x - 30 = 0$. We can factor the equation to get $(x-6)(x+5) = 0$, so $x = -5$ or $x = 6$. Since x is positive, then

$$\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}} = 6\,.$$

Similarly, we get

$$\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}} = 3 \text{ and } \sqrt{42 + \sqrt{42 + \sqrt{42 + \cdots}}} = 7\,.$$

Hence

$$\frac{\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}}{\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}} - \sqrt{42 + \sqrt{42 + \sqrt{42 + \cdots}}} = \frac{6}{3} - 7 = -5\,.$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SCOTT BROWN, Auburn University, Montgomery, AL, USA; FLORENCIO CANO VARGAS, Inca, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; MARC FOSTER, student, Angelo State University, San Angelo, TX, USA; G.C. GREUBEL, Newport News, VA, USA; GERHARDT HINKLE, Student, Central High School, Springfield, MO, USA; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; HEEYOON KIM, Conyers, GA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ANDRÉS PLANELLS CÁRCEL, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer. One incorrect solution was received.

M453. *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

Let $ABCD$ be a parallelogram. Sides AB and AD are extended to points E and F so that E, C, and F lie on a straight line. In problem M447, we saw that $BE \cdot DF = AB \cdot AD$. Prove that

$$\sqrt{AE + AF} \;\geq\; \sqrt{AB} + \sqrt{AD}\,.$$

Solution by George Apostolopoulos, Messolonghi, Greece.

We have

$$\begin{aligned}
(BE-DF)^2 \geq 0 &\iff BE^2+DF^2-2BE\cdot DF \geq 0\\
&\iff BE^2+DF^2+2BE\cdot DF-4BE\cdot DF \geq 0\\
&\iff (BE+DF)^2 \geq 4BE\cdot DF\\
&\iff (BE+DF)^2 \geq 4AB\cdot AD\\
&\iff BE+DF \geq 2\sqrt{AB\cdot AD}\\
&\iff (AE-AB)+(AF-AD) \geq 2\sqrt{AB\cdot AD}\\
&\iff AE+AF \geq AB+AD+2\sqrt{AB\cdot AD}\\
&\iff AE+AF \geq \sqrt{AB}^2+\sqrt{AD}^2+2\sqrt{AB\cdot AD}\\
&\iff (\sqrt{AE+AF})^2 \geq (\sqrt{AB}+\sqrt{AD})^2
\end{aligned}$$

So $\sqrt{AE+AF} \geq \sqrt{AB}+\sqrt{AD}$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FLORENCIO CANO VARGAS, Inca, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; HEEYOON KIM, Conyers, GA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; JORGE SEVILLA LACRUZ, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.

M454. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Determine all real numbers x with $16^x+1=2^x+8^x$.

Solution by Marc Foster and Travis B. Little, students, Angelo State University, San Angelo, TX, USA.

The equation may be re-written in the following ways:

$$\begin{aligned}
16^x+1 &= 8^x+2^x\,,\\
2^x8^x+1 &= 8^x+2^x\,,\\
(2^x-1)(8^x-1) &= 0\,.
\end{aligned}$$

This implies that either $2^x=1$ or $8^x=1$. In either case, the solution is $x=0$, which is the solution to the original equation.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; HEEYOON KIM, Conyers, GA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, *Indonesia; LUIZ ERNESTO LEITÃO, Pará, Brazil; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; ARTURO PARDO PÉREZ, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.*

Most of the other solvers converted everything to base $\mathbf{2}$ *and solved the related equation* $y^4+1=y+y^3$.

M455. *Proposed by Gheorghe Ghiţă, M. Eminescu National College, Buzău, Romania.*

Suppose that n is a positive integer.

(a) If the positive integer d is a divisor of each of the integers n^2+n+1 and $2n^3+3n^2+3n-1$, prove that d is also a divisor of n^2+n-1.

(b) Prove that the fraction $\dfrac{n^2+n+1}{2n^3+3n^2+3n-1}$ is irreducible.

Solution by Florencio Cano Vargas, Inca, Spain.

(a) We can write:

$$n^2+n-1=(2n^3+3n^2+3n-1)-2n(n^2+n+1)$$

Then if d divides $2n^3+3n^2+3n-1$ and n^2+n+1, then it also divides n^2+n-1.

(b) Let d be a positive common divisor of the numerator and denominator. From (a) we know that d also divides n^2+n-1. But n^2+n+1 and n^2+n-1 only differ by two units, so then $d\le 2$. To discard the possibility $d=2$, we note that $n^2+n+1=n(n+1)+1$ is always odd since either n or $n+1$ is even. Then the only possibility that remains is $d=1$ and the fraction is irreductible.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; JACLYN CHANG, student, University of Calgary, Calgary, AB (part (a) only); SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; LEI WANG, Missouri State University, Springfield, MO, USA; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer. One incorrect solution was received.

M456. *Proposed by Mihály Bencze, Brasov, Romania.*

Let f and g be real-valued functions with g an odd function, $f(x)\le g(x)$ for all real numbers x, and $f(x+y)\le f(x)+f(y)$ for all real numbers x and y. Prove that f is an odd function.

Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

From $f(x) \leq g(x)$ for any x, and from the oddness of $g(x)$ we get

$$f(-x) \leq g(-x) = -g(x) \leq -f(x) \, .$$

Moreover, we have

$$f(0) = f(x + (-x)) \leq f(x) + f(-x) \Longrightarrow f(-x) \geq f(0) - f(x) \, , \qquad (1)$$

and

$$f(x + 0) \leq f(0) + f(x) \Longrightarrow f(0) \geq 0 \, .$$

Since $f(0) \geq 0$, then from (1) we can conclude that $f(-x) \geq -f(x)$. Then we have

$$-f(x) \leq f(-x) \leq -f(x) \, .$$

Hence $f(x)$ is an odd function.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; JAVIER GARCÍA CAVERO, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; LEI WANG, Missouri State University, Springfield, MO, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.

Problem of the Month

Ian VanderBurgh

Here is a neat problem that has an easy-to-understand and appreciate real-life context and leads us to a good discussion of two different methods of solution.

Problem (2010 AMC 8) Everyday at school, Jo climbs a flight of **6** stairs. Jo can take the stairs **1**, **2**, or **3** at a time. For example, Jo could climb **3**, then **2**, then **1**. In how many ways can Jo climb the stairs?
(A) 13 (B) 18 (C) 20 (D) 22 (E) 24

I can just picture hundreds of Grade 8 students trying this after writing this contest in 2010... There are actually quite a lot of ways of climbing the stairs! The first step (ahem, no pun intended) is to read and understand the problem. As part of this, we should fiddle to see what ways we can find. We are looking for ways of adding combinations of **1**s, **2**s and **3**s to get **6**. Try playing with this for a couple of minutes!

What combinations did you get? Some that you might get include $\mathbf{3+3}$, $\mathbf{3+1+1+1}$, and $\mathbf{2+3+1}$. One question that we should immediately ask is whether re-arranging the order of a given sum makes a difference. Does it? Yes – for example, $\mathbf{2+3+1}$ (Jo takes **2** steps, then **3** steps, then **1** step) is different from $\mathbf{2+1+3}$ (**2** steps, then **1** step, then **3** steps) which are both different than $\mathbf{3+1+2}$, and so on. Can you find more ways to re-arrange this particular sum? The sum $\mathbf{3+1+1+1}$ can also be re-arranged in a number of ways. How many can you find?

So it looks as if there are now two sub-problems – finding the different combinations of **1**s, **2**s and **3**s that give **6**, and then figuring out the number of ways in which we can re-arrange each of these combinations. Let's get a handle on the second sub-problem first.

To do this, we'll consider a slightly different context: How many different "words" can be made from the letters of AAAAB, AAAC, AABB, and ABC? (By a "word" in this case, we mean a rearrangement of the letters; it doesn't actually have to form a real word!) In each case, we could exhaustively list out the possibilities or look for a different approach:

- AAAAB: **5** words
  List: AAAAB, AAABA, AABAA, ABAAA, BAAAA
  Alternate approach: If we start with the four As (AAAA), there are then five possible positions for the B: either before the first A or after each of the four As. Thus, there are five words.

- AAAC: **4** words
  List: AAAC, AACA, ACAA, CAAA
  Alternate approach: Can you modify the previous argument to fit this case?

- AABB: **6** words
  List: AABB, ABAB, ABBA, BAAB, BABA, BBAA
  Alternate approach: While there are good ways to count the words in this case using more advanced mathematics like combinatorics, actually coming up with a simple explanation of why the answer is **6** without actually just doing it isn't that easy. Here's one try. Suppose that the word starts with A. Put in the A and the two Bs to get ABB; the remaining A can go in three places (right before the first B or after either B). So there are **3** words beginning with A. Can you see why there are also **3** words beginning with B?

- ABC: **6** words
  List: ABC, ACB, BAC, BCA, CAB, CBA
  Alternate approach: There are **3** possibilities for the first letter; for each of these, there are **2** possibilities for the second letter (all but the letter we already chose); the last letter is then completely determined. This tells us that there are $\mathbf{3\times 2\times 1=6}$ possible words.

Now let's combine this information about re-arrangements with a systematic way of finding the different combinations.

Solution 1. Let's find the possible combinations in an organized way. We'll start by assuming that the order of steps doesn't actually matter, and then incorporate the order at the end. Coming up with a good way to find all of the combinations might require a bit of fiddling. One good method to use is to organize these by the number of **1**s.

Can there be six **1**s? Yes: $\mathbf{1+1+1+1+1+1=6}$.

Can there be five **1**s? No: if there are five **1**s, then Jo has only **1** step left for which she needs another **1**.

Can there be four **1**s? Yes: if there are four **1**s, then Jo has **2** steps left, which must be taken up by a **2**. (It can't be two **1**s since we're only allowed four **1**s.) This gives us $\mathbf{1+1+1+1+2}$.

Can there be three **1**s? Yes: if there are three **1**s, then Jo has **3** steps left. We can't divide the **3** into two pieces without using a **1**, so the only way is $\mathbf{1+1+1+3}$.

Can there be two **1**s? Yes: if there are two **1**s, then Jo has **4** steps left. To avoid using a **1**, this **4** must be $\mathbf{2+2}$. This gives us $\mathbf{1+1+2+2}$.

Can there be one **1**? Yes: with **5** steps left and not using a **1**, the remaining **5** must be $\mathbf{2+3}$. This gives us $\mathbf{1+2+3}$.

Can there be zero **1**s? Yes. If there are no **2**s, then there are only **3**s, so we have $\mathbf{3+3}$. If there is a **2**, then Jo has **4** steps left, which must be $\mathbf{2+2}$ since no **1**s are used. In this case, we have $\mathbf{3+3}$ or $\mathbf{2+2+2}$.

So ignoring order, the possibilities are (i) $\mathbf{1+1+1+1+1+1}$, (ii) $\mathbf{1+1+1+1+2}$, (iii) $\mathbf{1+1+1+3}$, (iv) $\mathbf{1+1+2+2}$, (v) $\mathbf{1+2+3}$, (vi) $\mathbf{3+3}$, and (vii) $\mathbf{2+2}$. The combinations in (i), (vi) and (vii) can't be re-arranged in any other order. That gives us **1** way in each case.

How can we re-arrange the sums in (ii), (iii), (iv), and (v)? There are **5** ways of arranging the sum in (ii). This is because this sum can be related to the word AAAAB from before, with each A representing a **1** and B representing the **2**. Each re-arrangement of the sum in (ii) is the same as one of the words that we talked about earlier. Since there were **5** words, then there are **5** ways of arranging the sum.

Can you see how to relate the sums in (iii), (iv) and (v) to the words earlier? Try this out! You'll find that the sum in (iii) can be arranged in **4** ways, and the sum in each of (iv) and (v) can be arranged in **6** ways.

Therefore, there are $\mathbf{1+5+4+6+6+1+1=24}$ ways that Jo can climb the stairs. □

While there was a fair bit of work required to actually make that solution work, we didn't have to do anything really hard. But, we had to be very, very careful. Also, this method might not "scale up" very well to a larger number of steps, because of the number of cases that we had to consider.

Let's switch gears. Sometimes looking at smaller cases helps in one of two ways: either by showing us a pattern that might continue or more directly by allowing us to capitalize on these smaller cases.

What do smaller cases look like here? They are cases with fewer stairs. Let's try a few:

- With **1** stair, there is only **1** way for Jo to climb.
- With **2** stairs, there are **2** ways: $\mathbf{1+1}$ and **2**.
- With **3** stairs, there are **4** ways: $\mathbf{1+1+1}$, $\mathbf{1+2}$, $\mathbf{2+1}$, and **3**.
- With **4** stairs, there are **7** ways: $\mathbf{1+1+1+1}$, $\mathbf{1+1+2}$, $\mathbf{1+2+1}$, $\mathbf{2+1+1}$, $\mathbf{1+3}$, $\mathbf{3+1}$, and $\mathbf{2+2}$.

Do you notice anything about the number of ways in these four cases? Do you think that it is a coincidence that $\mathbf{1+2+4=7}$? In other words, is it a coincidence that the sum of the numbers of ways for **1**, **2** and **3** stairs gives us the number of ways for **4** stairs?

Solution 2. We have seen that with **1**, **2** and **3** stairs, there are **1**, **2** and **4** ways, respectively.

If Jo is to climb **4** stairs, then she starts by climbing **1** stair (leaving **3**) or by climbing **2** stairs (leaving **2**) or by climbing **3** stairs (leaving **1**).

If she starts by climbing **1** stair, then the number of ways that she can finish climbing is the number of ways to climb the remaining **3** stairs. In other words, the number of ways that she can climb the stairs staring with **1** stair is equal to the number of ways in which she can climb **3** stairs. (There are **4** ways to do this.)

If she starts by climbing **2** stairs, then the number of ways that she can finish climbing is the number of ways to climb the remaining **2** stairs. In other words, the number of ways that she can climb the stairs staring with **2** stairs is equal to the number of ways in which she can climb **2** stairs. (There are **2** ways.)

Similarly, the number of ways of climbing starting with **3** stairs is equal to the number of ways of climbing the remaining **1** stair. (There is **1** way.)

Therefore, the number of ways of climbing **4** stairs equals the sum of the number of ways of climbing **3**, **2** and **1** stairs, or $\mathbf{4+2+1=7}$.

What happens with **5** stairs? In this case, Jo starts with **1**, **2** or **3** stairs, leaving **4**, **3** or **2** stairs. Using a similar argument, the total number of ways of climbing **5** stairs equals the sum of the number of ways of climbing **4**, **3** and **2** stairs, or $\mathbf{7+4+2=13}$.

Continuing along these lines, for **6** stairs, the total number of ways will equal the sum of the number of ways of climbing **5**, **4** and **3** stairs, or $\mathbf{13+7+4=24}$. □

We've just used a method called *recursion* in this second solution. This can be a very powerful approach in cases where it will work. Recursion is particularly useful in fields like computer science.

I'll leave you with a challenge. Can you determine the number of ways that Jo could go up the stairs if there were **10** stairs? Which approach do think that you'd want to use?

THE OLYMPIAD CORNER

No. 294

R.E. Woodrow and Nicolae Strugaru

The problems from this issue come from the selection tests for the Balkan, Indian and Slovenian IMO teams and the Singapore Mathematical Olympiad. Our thanks go to Adrain Tang for sharing the material with the editor.

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **15 mars 2012**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

OC11. Etant donné deux sous-ensembles non vides $A, B \subseteq \mathbb{Z}$, on définit $A + B$ et $A - B$ par

$$A+B=\{a+b \mid a\in A, b\in B\}\,,\ A-B=\{a-b \mid a\in A, b\in B\}\,.$$

Dans ce qui suit, on travaille avec des sous-ensembles finis non vides de $\mathbb{Z}$. Montrer qu'on peut recouvrir B par au plus $\dfrac{|A+B|}{|A|}$ translatés de $A - A$, c.-à-d. qu'il existe $X \subseteq \mathbb{Z}$ avec $|X| \le \dfrac{|A+B|}{|A|}$ tel que

$$B \subseteq \bigcup_{x\in X} (x+(A-A)) = X+A-A\,.$$

OC12. Soit k un entier positif plus grand que 1. Montrer que pour tout entier non négatif m, il existe k entiers positifs $n_1, n_2, \ldots, n_k$, tels que

$$n_1^2+n_2^2+\cdots+n_k^2 = 5^{m+k}\,.$$

OC13. Soit ABC un triangle acutangle et soit D un point sur le côté AB. Le cercle circonscrit du triangle BCD coupe le côté AC en E. Le cercle circonscrit du triangle ADC coupe le côté BC en F. Soit O le centre de gravité du triangle CEF. Montrer que les points D et O et les centres de gravité des triangles ADE, ADC, DBF et DBC sont cocycliques et que la droite OD est perpendiculaire à AB.

OC14. Soit a_n, b_n, $n = 1, 2, \ldots$ deux suites d'entiers définis par $a_1 = 1$, $b_1 = 0$ et, pour $n \geq 1$,

$$a_{n+1} = 7a_n + 12b_n + 6\,,$$
$$b_{n+1} = 4a_n + 7b_n + 3\,.$$

Montrer que a_n^2 est la différence de deux cubes consécutifs.

OC15. Une règle de longueur ℓ a $k \geq 2$ marques distantes de a_i unités d'une des extrémités avec $0 < a_1 < \cdots < a_k < \ell$. La règle est appelée *règle de Golomb* si les longueurs mesurables grâce aux marques de la règle sont des entiers consécutifs commençant avec 1, et telles que chaque longueur soit mesurable entre une unique paire de marques sur la règle. Trouver toutes les règles de Golomb.

OC16. Etant donné $a_1 \geq 1$ et $a_{k+1} \geq a_k + 1$ pour tout $k = 1, 2, \ldots, n$, montrer que

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq (a_1 + a_2 + \cdots + a_n)^2\,.$$

OC17. Montrer que les sommets d'un pentagone convexe $ABCDE$ sont cocycliques si et seulement si on a

$$d(E, AB) \cdot d(E, CD) = d(E, AC) \cdot d(E, BD) = d(E, AD) \cdot d(E, BC)\,.$$

OC18. Etant donné n nombres complexes $a_1, a_2, \ldots, a_n$, non nécessairement distincts, et des entiers positifs distincts k, l tels que $a_1^k, a_2^k, \ldots, a_n^k$ et $a_1^l, a_2^l, \ldots, a_n^l$ sont deux collections de nombres identiques, montrer que chaque a_j, $1 \leq j \leq n$, est une racine de l'unité.

OC19. Il y a eu 64 participants dans un tournoi d'échecs. Chaque paire a joué une partie qui s'est terminé soit par un gagnant ou par un match nul. Si une partie s'était terminée par un match nul, alors chacun des 62 participants restants gagnait contre au moins un des deux joueurs. Il y a eu au moins deux parties avec match nul dans ce tournoi. Montrer qu'on peut aligner tous les participants sur deux rangs de sorte que chacun d'eux a gagné contre celui qui se trouve juste derrière lui.

OC20. Etant donné un entier $n \geq 2$, trouver la valeur maximale que la somme $x_1 + x_2 + \cdots + x_n$ puisse atteindre lorsque les x_i prennent toutes les valeurs positives sujettes aux conditions $x_1 \leq x_2 \leq \cdots \leq x_n$ et $x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdots x_n$.

. . . .

OC11. For non-empty subsets $A, B \subseteq \mathbb{Z}$ define $A + B$ and $A - B$ by

$$A + B = \{a + b \mid a \in A, b \in B\}, \; A - B = \{a - b \mid a \in A, b \in B\}.$$

In the sequel we work with non-empty finite subsets of $\mathbb{Z}$.

Prove that we can cover B by at most $\dfrac{|A+B|}{|A|}$ translates of $A - A$, i.e. there exists $X \subseteq \mathbb{Z}$ with $|X| \leq \dfrac{|A+B|}{|A|}$ such that

$$B \subseteq \bigcup_{x \in X} (x + (A - A)) = X + A - A.$$

OC12. Let k be a positive integer greater than 1. Prove that for every non-negative integer m there exist k positive integers $n_1, n_2, \ldots, n_k$, such that

$$n_1^2 + n_2^2 + \cdots + n_k^2 = 5^{m+k}.$$

OC13. Let ABC be an acute triangle and let D be a point on the side AB. The circumcircle of the triangle BCD intersects the side AC at E. The circumcircle of the triangle ADC intersects the side BC at F. Let O be the circumcentre of triangle CEF. Prove that the points D and O and the circumcentres of the triangles ADE, ADC, DBF and DBC are concyclic and the line OD is perpendicular to AB.

OC14. Let a_n, b_n, $n = 1, 2, \ldots$ be two sequences of integers defined by $a_1 = 1$, $b_1 = 0$ and for $n \geq 1$,

$$a_{n+1} = 7a_n + 12b_n + 6,$$
$$b_{n+1} = 4a_n + 7b_n + 3.$$

Prove that a_n^2 is the difference of two consecutive cubes.

OC15. A ruler of length ℓ has $k \geq 2$ marks at positions a_i units from one of the ends with $0 < a_1 < \cdots < a_k < \ell$. The ruler is called a *Golomb ruler* if the lengths that can be measured using the marks on the ruler are consecutive integers starting with 1, and each such length be measurable between a unique pair of marks on the ruler. Find all Golomb rulers.

OC16. Given $a_1 \geq 1$ and $a_{k+1} \geq a_k + 1$ for all $k = 1, 2, \ldots, n$, show that

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq (a_1 + a_2 + \cdots + a_n)^2.$$

OC17. Prove that the vertices of a convex pentagon $ABCDE$ are concyclic if and only if the following holds

$$d(E, AB) \cdot d(E, CD) = d(E, AC) \cdot d(E, BD) = d(E, AD) \cdot d(E, BC).$$

OC18. If $a_1, a_2, \ldots, a_n$ are n non-zero complex numbers, not necessarily distinct, and k, l are distinct positive integers such that $a_1^k, a_2^k, \ldots, a_n^k$ and $a_1^l, a_2^l, \ldots, a_n^l$ are two identical collections of numbers. Prove that each a_j, $1 \le j \le n$, is a root of unity.

OC19. There were 64 contestants at a chess tournament. Every pair played a game that ended either with one of them winning or in a draw. If a game ended in a draw, then each of the remaining 62 contestants won against at least one of these two contestants. There were at least two games ending in a draw at the tournament. Show that we can line up all the contestants so that each of them has won against the one standing right behind him.

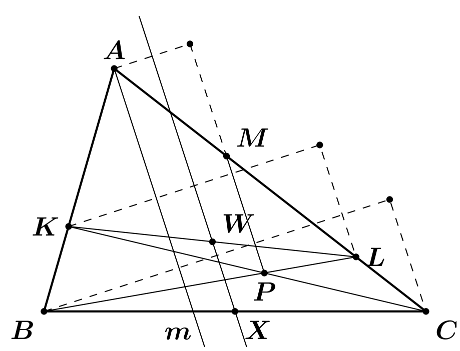
OC20. Given an integer $n \ge 2$, determine the maximum value the sum $x_1 + x_2 + \cdots + x_n$ may achieve, as the x_i run through the positive integers subject to $x_1 \le x_2 \le \cdots \le x_n$ and $x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdots x_n$.

We now turn to solutions of the II International Zhautykov Olympiad in Mathematics given at [2009 : 376–377].

2. The points K and L lie on the sides AB and AC, respectively, of the triangle ABC such that $BK = CL$. Let P be the point of intersection of the segments BL and CK, and let M be an inner point of the segment AC such that the line MP is parallel to the bisector of the angle $\angle BAC$. Prove that $CM = AB$.

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Bataille.

We introduce the glide-reflection (or reflection) $\mathbf{g}$ such that $\mathbf{g}(B) = C$ and $\mathbf{g}(K) = L$. The axis of $\mathbf{g}$ is parallel to the angle bisector m of $\angle BAC$ and passes through the midpoints X and W of BC and KL (see figure).



We observe that the line ALC is a transversal of ΔBKP (with A on BK, L on BP and C on KP). From a well-known theorem, the midpoints of the "diagonals" AP, LK, CB are collinear.

It follows that the midpoint of AP is on the axis XW of $\mathbf{g}$. Thus, the lines m and MP, which are parallel to the axis of $\mathbf{g}$, are equidistant of this axis. As a result, $\mathbf{g}(m) = MP$. Since in addition the image under $\mathbf{g}$ of the line $AB = BK$ is the line $AC = CL$, we obtain $\mathbf{g}(A) = M$. Recalling that $\mathbf{g}$ preserves distances, $AB = CM$ follows.

5. Let a, b, c, and d be real numbers such that $a+b+c+d=0$. Prove that

$$(ab+ac+ad+bc+bd+cd)^2+12 \;\geq\; 6(abc+abd+acd+bcd)\,.$$

Solved by Arkady Alt, San Jose, CA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.

Since $d=-a-b-c$, the given inequality is equivalent to

$$[ab+ac+bc-(a+b+c)^2]^2+12 \;\geq\; 6abc-6(a+b+c)(ab+bc+ca)$$
$$\Leftrightarrow (a^2+b^2+c^2+ab+bc+ca)^2+12 + 6(a+b+c)(ab+bc+ca)-6abc \;\geq\; 0.$$

Because $(a+b+c)(ab+bc+ca)-abc=(a+b)(b+c)(c+a)$, we obtain

$$\frac{1}{4}[(a+b)^2+(b+c)^2+(c+a)^2]+12+6(a+b)(b+c)(c+a) \;\geq\; 0.$$

Denoting $z=\frac{a+b}{2}$, $x=\frac{b+c}{2}$, $y=\frac{c+a}{2}$, we have to prove that

$$(x^2+y^2+z^2)^2+24xyz+48 \;\geq\; 0.$$

By AM-GM Inequality, we have

$$(x^2+y^2+z^2)^2 \;\geq\; 9|xyz|^{4/3}$$

and because $24xyz \;\geq\; -24|xyz|$, it suffices to prove that

$$9t^4-24t^3+48 \;\geq\; 0, \qquad \text{where} \qquad t \;=\; |xyz|^{1/3}.$$

This is true, since $9t^4-24t^3+48=3(t-2)^2(3t^2+4t+4) \;\geq\; 0$.

Next we turn to problems of the 50th Mathematical Olympiad of the Republic of Moldova given at [2009 : 377–378].

1. Let a, b, and c be the side lengths of a right triangle with hypotenuse of length c, and let h be the altitude from the right angle. Find the maximum value of $\dfrac{c+h}{a+b}$.

Solved by Arkady Alt, San Jose, CA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Bataille.

Let $Q=\frac{c+h}{a+b}$. We show that the maximum value of Q is $\frac{3\sqrt{2}}{4}$.
We have

$$Q \;=\; \frac{c^2+hc}{(a+b)c}=\frac{a^2+b^2+ab}{(a+b)\sqrt{a^2+b^2}}.$$

Clearly, $Q = \frac{3\sqrt{2}}{4}$ when $a = b$. To prove that $Q \le \frac{3\sqrt{2}}{4}$ in any case, we rewrite this inequality successively as

$$4(a^2 + b^2 + ab) \le 3\sqrt{2}(a + b)\sqrt{a^2 + b^2}$$

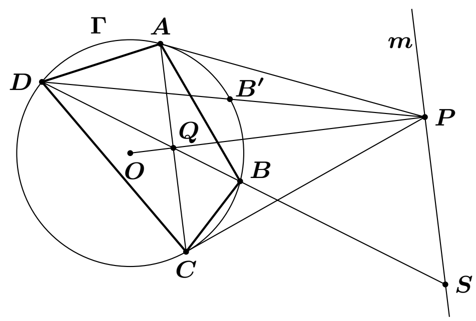
$$16(a^2 + b^2)^2 + 16a^2b^2 + 32ab(a^2 + b^2) \le 18(a^2 + b^2)(a^2 + b^2 + 2ab)$$

$$16a^2b^2 \le 2(a^2 + b^2)^2 + 4ab(a^2 + b^2). \qquad (1)$$

Now, $a^2+b^2 \ge 2ab > 0$, hence $2(a^2+b^2)^2 \ge 8a^2b^2$ and $4ab(a^2+b^2) \ge 8a^2b^2$ so that (1) certainly holds. This completes the proof.

3. The quadrilateral $ABCD$ is inscribed in a circle. The tangents to the circle at A and C intersect at a point P not on BD and such that $PA^2 = PB \cdot PD$. Prove that BD passes through the midpoint of AC.

Solution by Michel Bataille, Rouen, France.



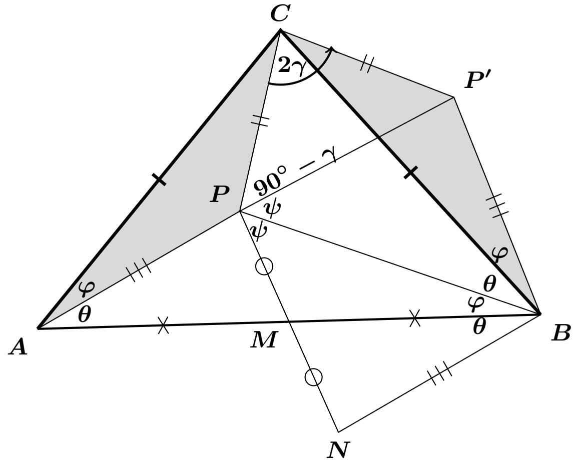
Let B' be the second point of intersection of PD with the circle $\Gamma = (ABCD)$. Note that $B' \ne B$ (since P is not on BD) and that $PB' \cdot PD$ is the power of P with respect to Γ. Thus $PB' \cdot PD = PA^2 = PB \cdot PD$, so that $PB' = PB$. If O is the centre of Γ, we also have $OB = OB'$, hence the line OP is the perpendicular bisector of BB'. It follows that OP is the bisector of the angle $\angle BPD$.

Now, let m be the perpendicular to OP at P. The lines OP and m are the two bisectors of the lines BP, BD, hence (PD, PO, PB, m) is a harmonic pencil of lines and PO and m meet BD at points Q and S which are conjugate with respect to Γ. As a result, m is the polar of Q since it passes through S and is perpendicular to OQ. Finally, Q is on the polar AC of P and so is the common point of OP and AC, that is, the midpoint of AC. This completes the proof.

6. Triangle ABC is isosceles with $AC = BC$ and P is a point inside the triangle such that $\angle PAB = \angle PBC$. If M is the midpoint of AB, prove that $\angle APM + \angle BPC = 180°$.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Let $\theta = \angle PAB = \angle PBC$, $\varphi = \angle PAC = \angle PBA$, $\psi = \angle MPB$, $2\gamma = \angle ACB$. Then $\theta + \varphi + \gamma = 90^\circ$.



Rotate $\triangle ACP$ about C counterclockwise through an angle 2γ to the position BCP' and draw PP'. We have $CP = CP'$, $PA = P'B$, $\angle P'BC = \varphi$, and $\angle CPP' = 90^\circ - \gamma$.

Extend PM past M its own length to a point N and draw BN. Then $\triangle AMP \cong \triangle BMN$, so $NB = PA$ and $\angle NBM = \theta$. It now follows that $\triangle P'BP \cong \triangle NBP$, so $\angle P'PB = \psi$. Consequently,

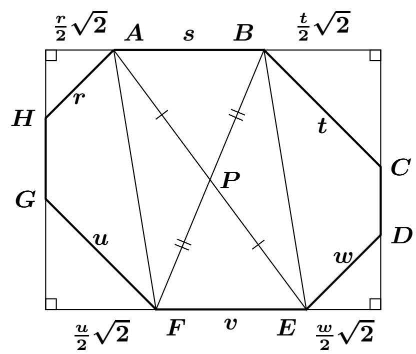
$$\angle APM + \angle BPC \;=\; [180^\circ - (\theta + \varphi) - \psi] + [(90^\circ - \gamma) + \psi] \;=\; 180^\circ\,.$$

7. The interior angles of a convex octagon are all equal and all side lengths are rational numbers. Prove that the octagon has a centre of symmetry.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Our notation is evident from the figure.

The interior angle sum of the octagon $ABCDEFGH$ is 1080°, hence each interior angle is 135° and each exterior angle 45°. Thus, the octagon can be enclosed in a rectangle, as shown. Consequently, $s + \frac{r+t}{2}\sqrt{2} = v + \frac{u+w}{2}\sqrt{2}$, that is $s + a\sqrt{2} = v + b\sqrt{2}$, where s, v, a, b are rational numbers.



If $a \neq b$, then $\sqrt{2}\frac{v-s}{a-b}$ would be rational, which is not so. Thus, $a = b$, hence $s = v$. Therefore, $ABEF$ is a parallelogram and its diagonals AE and BF bisect each other at P, which is the centre of symmetry of $ABEF$.

Similarly, this same point P is the centre of symmetry of parallelogram $BCFG$, and so on around the octagon. Thus, P is the centre of symmetry of the octagon.

8. Let $M = \{x^2 + x \mid x \text{ is a positive integer}\}$. For each integer $k \geq 2$ prove that there exist a_1, a_2, $\dots$, a_k, b_k in M such that $a_1 + a_2 + \cdots + a_k = b_k$.
Solution by Geoffrey A. Kandall, Hamden, CT, USA.

The proof is by induction on k.

Note that $x^2 + x = x(x+1)$, so each element of M is even.

Since $12 + 30 = 42$ and $12 = 3 \cdot 4$, $30 = 5 \cdot 6$, $42 = 6 \cdot 7 \in M$, the desired result holds for $k = 2$.

Now the inductive step. Suppose that for some $k \geq 2$ we have $a_1 + a_2 + \cdots + a_k = b_k$, where $a_1, a_2, \dots, a_k$, $b_k \in M$. Since b_k is even, we have $b_k = 2c$ for some positive integer c. Moreover, $b_k \geq a_1 + a_2 \geq 4$, so $c \geq 2$. Let $a_{k+1} = (c-1)c \in M$. Then

$$a_1 + a_2 + \cdots + a_k + a_{k+1} \;=\; 2c + (c-1)c \;=\; c(c+1) \in M,$$

and the induction is complete.

Next we turn to solutions from our readers to problems of the Republic of Moldova Mathematical Olympiad Second and Third Team Selection Tests given at [2009 : 378–379].

3. Let a, b, c be the side lengths of a triangle and let s be the semiperimeter. Prove that

$$a\sqrt{\frac{(s-b)(s-c)}{bc}} \;+\; b\sqrt{\frac{(s-c)(s-a)}{ac}} \;+\; c\sqrt{\frac{(s-a)(s-b)}{ab}} \;\geq\; s\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

Let $x := s-a, y := s-b, z := s-c$ then $x, y, z > 0$, $a = y+z$, $b = z+x$, $c = x+y$, $s = x+y+z$ and the original inequality becomes

$$\sum_{cyc} (y+z)\sqrt{\frac{yz}{(x+y)(z+x)}} \geq x+y+z,$$

where $x, y, z > 0.$

Since

$$\sum_{cyc} (y+z)\sqrt{\frac{yz}{(x+y)(z+x)}} \;=\; \sum_{cyc} \frac{(y+z)\sqrt{yz(x+y)(z+x)}}{(x+y)(z+x)}$$

and by Cauchy and AM-GM Inequalities

$$\begin{aligned}(y+z)\sqrt{yz(x+y)(z+x)} &\geq (y+z)\sqrt{yz\left(x+\sqrt{yz}\right)^2}\\ &= (y+z)\sqrt{yz}\left(x+\sqrt{yz}\right)\\ &= x(y+z)\sqrt{yz}+(y+z)yz\\ &\geq 2x\sqrt{yz}\sqrt{yz}+(x+y)yz\\ &= 2xyz+(y+z)yz\\ &= yz\left((x+y)+(x+z)\right)\end{aligned}$$

then

$$\begin{aligned}\sum_{cyc}\frac{(y+z)\sqrt{yz(x+y)(z+x)}}{(x+y)(z+x)} &\geq \sum_{cyc}\frac{yz\left((x+y)+(x+z)\right)}{(x+y)(z+x)}\\ &= \sum_{cyc}\left(\frac{yz}{z+x}+\frac{yz}{x+y}\right)\\ &= \sum_{cyc}\frac{yz}{z+x}+\sum_{cyc}\frac{yz}{x+y}\\ &= \sum_{cyc}\frac{zx}{x+y}+\sum_{cyc}\frac{yz}{x+y}=\sum_{cyc}\frac{zx+yz}{x+y}\\ &= \sum_{cyc}\frac{z(x+y)}{x+y}=x+y+z.\end{aligned}$$

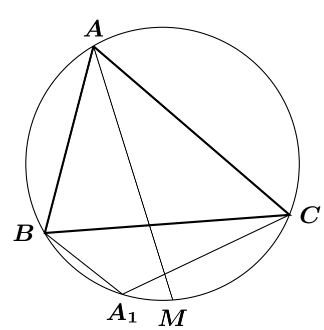
5. The point P is in the interior of triangle ABC. The rays AP, BP, and CP cut the circumcircle of the triangle at the points A_1, B_1, and C_1, respectively. Prove that the sum of the areas of the triangles A_1BC, B_1AC, and C_1AB does not exceed $s(R-r)$, where s, R, and r are the semiperimeter, the circumradius, and the inradius of triangle ABC, respectively.

Solution by Titu Zvonaru, Cománeşti, Romania.

We will prove the statement of the problem for the points A_1, B_1, C_1 such that A_1 belongs to arc BC which does not contain point A, and similarly for B_1 and C_1.

Let $[XYZ]$ be the area of $\triangle XYZ$. We denote $a=BC$, $b=CA$, $c=AB$. Let M be the midpoint of arc BC which contains the point A_1 (which does not contain the point A). It is easy to see that

$$[A_1BC] \leq [MBC]. \qquad (1)$$



We have

$$\angle MBC = \angle MCB = \angle MAC = \frac{A}{2}.$$

By the Law of Sines, we obtain $BM = 2R\sin\frac{A}{2}$. It follows that

$$[BMC] = \frac{BM\cdot BC\cdot\sin\angle MBC}{2} = aR\sin^2\frac{A}{2} \tag{2}$$

By (1) and (2), it follows that

$$\begin{aligned}
&[A_1BC]+[B_1AC]+[C_1AB]\\
&\quad\le aR\sin^2\frac{A}{2}+bR\sin^2\frac{B}{2}+cR\sin^2\frac{C}{2}\\
&\quad=\frac{aR(1-\cos A)+bR(1-\cos B)+cR(1-\cos C)}{2}\\
&\quad=\frac{1}{2}R(a+b+c)-\frac{R}{2}(a\cos A+b\cos B+c\cos C)\\
&\quad=sR-\frac{R^2}{2}(2\sin A\cos A+2\sin B\cos B+2\sin C\cos C)\\
&\quad=sR-\frac{R^2}{2}(\sin 2A+\sin 2B+\sin 2C)\\
&\quad=sR-\frac{R^2}{2}\cdot 4\sin A\sin B\sin C=sR-\frac{R^2}{2}\cdot 4\cdot\frac{abc}{8R^3}\\
&\quad= sR-\frac{abc}{4R}=sR-[ABC]=sR-sr = s(R-r)\,.
\end{aligned}$$

The equality holds if and only if AA_1, BB_1, CC_1 are the bisectors of $\triangle ABC$.

7. Let a, b, and c be positive real numbers such that $abc=1$. Prove that

$$\frac{a+3}{(a+1)^2}+\frac{b+3}{(b+1)^2}+\frac{c+3}{(c+1)^2}\ \ge\ 3\,.$$

Solved by Arkady Alt, San Jose, CA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Zvonaru.

We have

$$\begin{aligned}
\frac{2(a+3)}{(a+1)^2}-2 &= \frac{2a+6-2a^2-4a-2}{(1+a)^2}\\
&= \frac{1-2a+a^2+3(1-a^2)}{(1+a)^2} = \left(\frac{1-a}{1+a}\right)^2+3\cdot\frac{1-a}{1+a}.
\end{aligned}$$

Denoting $x=\frac{1-a}{1+a}$, $y=\frac{1-b}{1+b}$, and $z=\frac{1-c}{1+c}$, it results that $x,y,z\in[-1,1]$ and we have to prove that

$$x^2+y^2+z^2+3(x+y+z)\ \ge\ 0. \tag{1}$$

Since $abc = 1$, we obtain

$$\begin{aligned} x+y+z &= \frac{(1-a)(1+b+c+bc)+(1-b)(1+a+c+ac)}{(1+a)(1+b)(1+c)} \\ &\qquad + \frac{(1-c)(1+a+b+ab)}{(1+a)(1+b)(1+c)} \\ &= \frac{a+b+c-ab-bc-ca}{(1+a)(1+b)(1+c)} \\ &= \frac{-1+a+b(1-a)+c(1-a)-bc(1-a)}{(1+a)(1+b)(1+c)} \\ &= -\frac{(1-a)(1-b)(1-c)}{(1+a)(1+b)(1+c)} = -xyz. \end{aligned}$$

Thus, the inequality (1) is equivalent to

$$x^2+y^2+z^2 \ \geq\ 3xyz,$$

which is true because by AM-GM Inequality and since $x, y, z \in [-1, 1]$ we have

$$x^2+y^2+z^2 \ \geq\ 3\sqrt[3]{x^2y^2z^2} \geq 3xyz.$$

Next we move to the Russian Mathematical Olympiad 2007, 11^{th} Grade, given at [2010: 151–152].

1. *(N. Agakhanov)* The product $f(x) = \cos x \cos 2x \cos 3x \ldots \cos 2^k x$ is written on the blackboard $(k \geq 10)$. Prove that it is possible to replace one "**cos**" by "**sin**" such that the product obtained $f_1(x)$ satisfies the inequality $|f_1(x)| \leq 3 \cdot 2^{-1-k}$ for all real k.

Solved by Oliver Geupel, Brühl, NRW, Germany.

We prove that it suffices to replace the factor "$\cos 3x$" by "$\sin 3x$" whenever $k \geq 2$.

We start with the identity

$$\sin x \cdot \cos x \cos 2x \cos 4x \cdots \cos 2^k x = \frac{\sin 2^{k+1}x}{2^{k+1}} \qquad (k \geq 0). \tag{1}$$

We prove (1) by induction. It is obvious for $k = 0$. Assume that it holds for some fixed integer $k \geq 0$. Then

$$\begin{aligned} \sin x \cdot \cos x \cos 2x \cos 4x \cdots \cos 2^k x \cos 2^{k+1}x &= \frac{\sin 2^{k+1}x}{2^{k+1}} \cdot \cos 2^{k+1}x \\ &= \frac{\sin 2^{k+2}x}{2^{k+2}}, \end{aligned}$$

which completes the induction and therefore the proof of (1).

Moreover, we have that

$$\begin{aligned}\sin 3x &= \sin x\cos 2x + \cos x\sin 2x\\ &= \sin x(1-2\sin^2 x) + 2\sin x(1-\sin^2 x)\\ &= 3\sin x - 4\sin^3 x.\end{aligned}$$

Putting this all together, we conclude that

$$\begin{aligned}|f_1(x)| &\le |3\sin x - 4\sin^3 x|\cdot|\cos x\cos 2x\cos 4x\cdots\cos 2^k x|\\ &= |3-4\sin^2 x|\cdot|\sin x|\cdot|\cos x\cos 2x\cos 4x\cdots\cos 2^k x|\\ &\le 3\cdot\left|\frac{\sin 2^{k+1}x}{2^{k+1}}\right| \le 3\cdot 2^{-1-k}.\end{aligned}$$

2. *(A. Polyansky)* The incircle of a triangle ABC touches sides BC, AC, AB at points A_1, B_1, C_1, respectively. Segment AA_1 intersects the incircle again at point Q. Line ℓ is parallel to BC and passes through A. Lines A_1C_1 and A_1B_1 intersect ℓ at points P and R, respectively. Prove that $\angle PQR = \angle B_1QC_1$.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give Kandall's presentation.

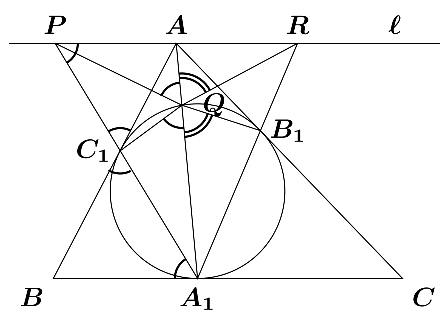
Observe that

$$\begin{aligned}&\angle A_1QC_1 = \frac{1}{2}\widehat{A_1C_1} = \angle BA_1C_1 = \angle BC_1A_1,\\ &\angle BA_1C_1 = \angle APC_1, \quad \angle BC_1A_1 = \angle PC_1A\end{aligned}$$

so $\angle APC_1 = \angle PC_1A$. Therefore, $\angle APC_1$ is supplementary to $\angle AQC_1$, so quadrilateral $PAQC_1$ is cyclic. Consequently,

$$\angle PQA = \angle PC_1A = \angle A_1QC_1$$

and similarly, $\angle RQA = \angle A_1QB_1$. Thus, by angle addition, $\angle PQR = \angle B_1QC_1$.



Next we look at solutions to problems of the XV Olympíada Matemática Rioplatense, Nivel 2, given at [2010; 214] that we started last issue.

4. Let $a_1, a_2, \ldots, a_n$ be positive numbers. The sum of all the products a_ia_j with $i < j$ is equal to 1. Show that there is a number among them such that the sum of the remaining numbers is less than $\sqrt{2}$.

Solved by Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comăneşti, Romania. We give Zvonaru's write-up.

We prove by induction that

$$2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i a_j = \sum_{i=1}^{n}\left(a_i \sum_{\substack{j=1\\ j\neq i}}^{n} a_j\right). \tag{1}$$

For $n = 2$ we have $2a_1a_2 = a_1a_2 + a_2a_1$, which is true.

For $n = 3$ we have $2(a_1a_2 + a_1a_3 + a_2a_3) = a_1(a_2 + a_3) + a_2(a_1 + a_3) + a_3(a_1 + a_2)$, which is also true.

Suppose that (1) is true for some $n = k$, $k > 1$. We have to prove that

$$2\sum_{i=1}^{k}\sum_{j=i+1}^{k+1} a_i a_j = \sum_{i=1}^{k+1}\left(a_i \sum_{\substack{j=1\\ j\neq i}}^{k+1} a_j\right). \tag{2}$$

We have

$$\begin{aligned}
2\sum_{i=1}^{k}\sum_{j=i+1}^{k+1} a_i a_j &= 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} a_i a_j + 2\sum_{i=1}^{k} a_i a_{k+1} \\
&= \sum_{i=1}^{k}\left(a_i \sum_{\substack{j=1\\ j\neq i}}^{k} a_j\right) + \sum_{i=1}^{k} a_i a_{k+1} + a_{k+1}\sum_{j=1}^{k} a_j \\
&= \sum_{i=1}^{k}\left(a_i \left(\sum_{\substack{j=1\\ j\neq i}}^{k} a_j + a_{k+1}\right)\right) + a_{k+1}\sum_{\substack{j=1\\ j\neq k+1}}^{k+1} a_j \\
&= \sum_{i=1}^{k}\left(a_i \sum_{\substack{j=1\\ j\neq i}}^{k+1} a_j\right) + a_{k+1}\sum_{\substack{j=1\\ j\neq k+1}}^{k+1} a_j \\
&= \sum_{i=1}^{k+1}\left(a_i \sum_{\substack{j=1\\ j\neq i}}^{k+1} a_j\right).
\end{aligned}$$

Denoting $s = a_1 + a_2 + \cdots + a_n$ and using (1) we obtain

$$2 = a_1(s - a_1) + a_2(s - a_2) + \cdots + a_n(s - a_n). \tag{3}$$

If there is i such that $s - a_i < \sqrt{2}$, we are done.

Suppose that for $i = 1, 2, \ldots, n$, $s - a_i \geq \sqrt{2}$. Using (3) we deduce that

$$\begin{aligned} 2 &= a_1(s-a_1) + a_2(s-a_2) + \cdots + a_n(s-a_n) \\ &\geq \sqrt{2}(a_1 + a_2 + \cdots + a_n), \end{aligned}$$

hence $a_1 + a_2 + \cdots + a_n \leq \sqrt{2}$. It follows that, for example, $a_1 + a_2 + \cdots + a_{n-1} < \sqrt{2}$, a contradiction with the inequality $s - a_n \geq \sqrt{2}$.

Next we move to solutions to problems of the XV Olympíada Matemática Rioplatense 2006, Nivel 3, given at [2010; 215–216].

1. (a) For each $k \geq 3$, find a positive integer n that can be represented as the sum of exactly k mutually distinct positive divisors of n.

(b) Suppose that n can be expressed as the sum of exactly k mutualy distinct positive divisors of n for some $k \geq 3$. Let p be the smallest prime divisor of n.

Show that
$$\frac{1}{p} + \frac{1}{p+1} + \cdots + \frac{1}{p+k-1} \geq 1 .$$

Solved by Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give Geupel's write-up.

(a) Consider $n = 2^{k-2} \cdot 3$ with the divisors $d_j = 2^{k-2-j} \cdot 3 \quad (1 \leq j \leq k-2)$, $d_{k-1} = 2$, $d_k = 1$. We have

$$\sum_{j=1}^{k} d_j = 3 \sum_{j=1}^{k-2} 2^{k-2-j} + 3 = 3(2^{k-2} - 1) + 3 = 2^{k-2} \cdot 3 = n.$$

(b) Let $d_1 > d_2 > \ldots > d_k$ be divisors of the integer n such that $\sum_{j=1}^{k} d_j = n$.

By $d_1 < n$ we have
$$\frac{n}{d_1} \geq p.$$

Since $\frac{n}{d_1}, \frac{n}{d_2}, \ldots, \frac{n}{d_k}$ is an increasing sequence of integers, we also have

$$\frac{n}{d_2} \geq p+1,\ \frac{n}{d_3} \geq p+2,\ \ldots,\ \frac{n}{d_k} \geq p+k-1.$$

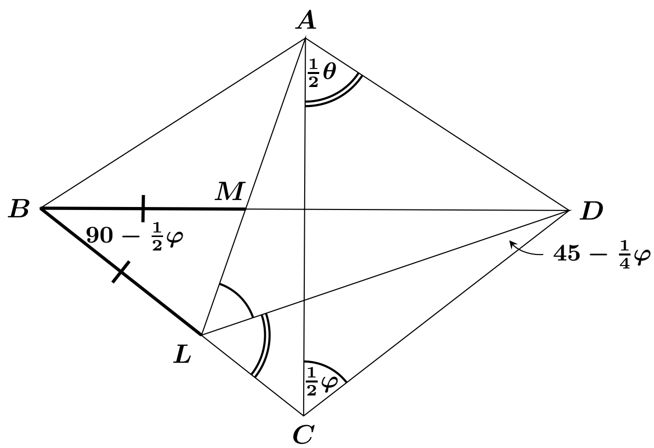
Consequently,

$$\frac{1}{p} + \frac{1}{p+1} + \cdots + \frac{1}{p+k-1} \geq \frac{1}{n} \sum_{j=1}^{k} d_j = 1,$$

which completes the proof.

2. Let $ABCD$ be a convex quadrilateral such that $AB = AD$ and $CB = CD$. The bisector of $\angle BDC$ cuts BC at L, and AL cuts BD at M, and it is known that $BL = BM$. Determine the value of $2\angle BAD + 3\angle BCD$.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Kandall's solution.



Let $\angle BAD = \theta$ and $\angle BCD = \varphi$ (in degrees). We want to determine $2\theta + 3\varphi$.

The line AC bisects $\angle BAD$ and $\angle BCD$; $\angle CBD = 90 - \frac{1}{2}\varphi$, $\angle CDL = \frac{1}{2}(90 - \frac{1}{2}\varphi) = 45 - \frac{1}{4}\varphi$. From $\triangle BLM$, $\angle BLM = \frac{1}{2}(90 + \frac{1}{2}\varphi) = 45 + \frac{1}{4}\varphi$; from $\triangle DLC$, $\angle DLC = 180 - \varphi - (45 - \frac{1}{4}\varphi) = 135 - \frac{3}{4}\varphi$. Consequently, $\angle ALD = 180 - (45 + \frac{1}{4}\varphi) - (135 - \frac{3}{4}\varphi) = \frac{1}{2}\varphi = \angle ACD$.

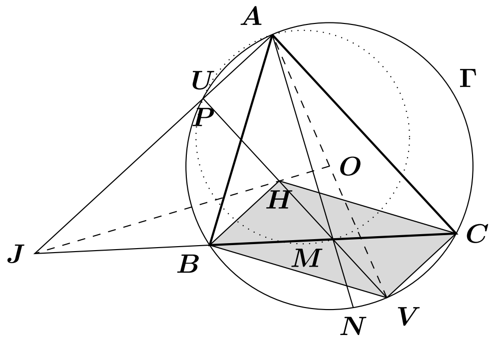
Therefore, quadrilateral $ALCD$ is cyclic, so $\angle DLC = \angle DAC$, that is, $135 - \frac{3}{4}\varphi = \frac{1}{2}\theta$. It follows easily that $2\theta + 3\varphi = 540$.

4. The acute triangle ABC with $(AB \neq AC)$ has circumcircle Γ, circumcentre O and orthocentre H. The midpoint of BC is M and the extension of the median AM intersects Γ at N. The circle of diameter AM intersects again Γ at A and P.

Show that the lines AP, GC, and OH are concurrent if and only if $AH = HN$.

Solved by Michel Bataille, Rouen, France; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Bataille's solution.

Let V denote the point diametrically opposite to A on Γ. Then, $VC \perp CA$, hence $BH \parallel VC$. Similarly $CH \parallel VB$ and it follows that $BHCV$ is a parallelogram with centre M. As a result, the line $HM = HV$ meets Γ again at U such that $AU \perp UM$. Therefore U is also on the circle with diameter AM and $U = P$.

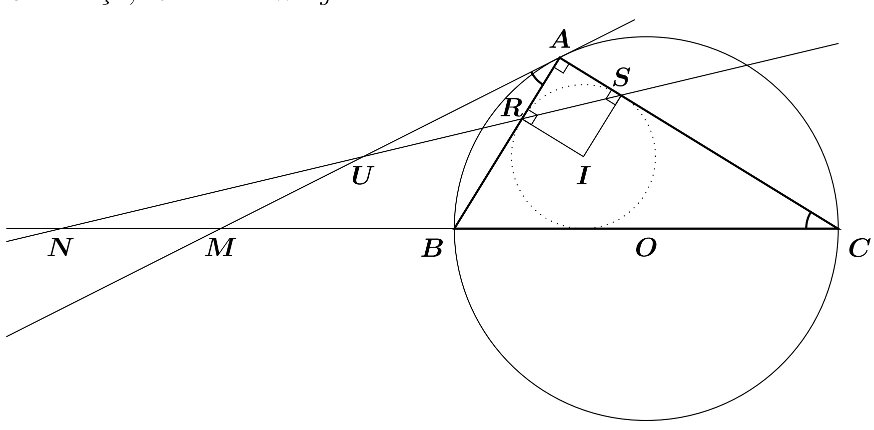


Now, let AP intersect the line BC at J. The point H is also the orthocentre of ΔAJM (since lines AH and MP are altitudes). Thus, the line OH passes through J if and only if $OH \perp AN$. But $OA = ON$, hence OH is perpendicular to AN if and only if OH is the perpendicular bisector of the line segment AN that is, if and only if $HA = HN$. The result follows.

Next we turn to solutions from our readers to problems of the 21 Olimpiada Iberoamericana de Matematico, Guayoquil, given at [2010; 216–217].

1. In the scalene triangle ABC, with $\angle BAC = 90°$, the tangent line to the circumcircle at A intersects the line BC at M. Let S and R be the points where the incircle of ABC touches AC and AB respectively. The line RS intersects the line BC at N. The lines AM and SR meet at U. Show that triangle UMN is isosceles.

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Zelator; and Titu Zvonaru, Comǎneşti, Romania. We give Bataille's solution.



Let I denote the incentre of ΔABC. The quadrilateral $ARIS$ is a rectangle (because $\angle ARI = \angle ASI = \angle RAI = 90°$), even a square since in addition $IR = IS$. It follows that $\angle ARS = 45°$.

Observing that $\gamma = \angle ACB$ and $\angle UAB$ subtend the same arc of the circumcircle, we have

$$\angle NUM = \angle AUR = 180^\circ - \gamma - 135^\circ = 45^\circ - \gamma. \qquad (1)$$

From

$$\begin{aligned}\angle UMN &= 180^\circ - \angle AMB = \angle MAB + \angle MBA \\ &= \gamma + (180^\circ - (90^\circ - \gamma)) = 90^\circ + 2\gamma\end{aligned}$$

we deduce

$$\angle UNM = 180^\circ - (45^\circ - \gamma) - (90^\circ + 2\gamma) = 45^\circ - \gamma. \qquad (2)$$

From (1) and (2), $\angle NUM = \angle UMN$ and so ΔUMN is isosceles with $MN = MU$.

2. Let $a_1, a_2, \ldots, a_n$ be real numbers. Let d be the difference between the smallest and the largest of them, and let $s = \sum_{i<j} |a_i - a_j|$. Show that

$$(n-1)d \ \le\ s\ \le\ \frac{n^2 d}{4}$$

and determine the conditions under which equality holds in each equality.

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománești, Romania. We give Zvonaru's version.

Without loss of generality we may assume that $a_1 \ge a_2 \ge \cdots \ge a_n$, and we denote $d(a_1, \ldots, a_n) = a_1 - a_n$ and $s(a_1, \ldots, a_n) = \sum_{i<j} |a_i - a_j|$.

For $n = 2$, we have $s = d$ and both inequalities are equalities.

For $n = 3$, we have $s = 2d$; the left inequality is equality and the right inequality is true.

Assume that $n \ge 4$. We have

$$\begin{aligned} s(a_1, \ldots, a_n) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_i - a_j) \\ &= a_1 - a_2 + a_2 - a_n + \cdots + a_1 - a_{n-1} + a_{n-1} - a_n \\ &\quad + a_1 - a_n + \sum_{i=2}^{n-2} \sum_{j=i+1}^{n-1} (a_i - a_j) \\ &= (n-1)d(a_1, \ldots a_n) + s(a_2, \ldots, a_{n-1}). \end{aligned}$$

Since $s(a_2, \ldots a_{k-1}) \ge 0$, it results that the left inequality is true. The equality holds if and only if $n = 2$ or $n = 3$ or $n \ge 4$ and $a_2 = \cdots = a_{n-1}$ (that is, $s(a_2, \ldots, a_{n-1}) = 0$).

For the right inequality we proceed by induction. If $n = 2, 3$, the inequality is true. Suppose that the inequality is true for any $K \le n$ and we have to prove that

$$s(a_1, \ldots, a_{n+1})\ \le\ \frac{(n+1)^2}{4} d(a_1, \ldots, a_{n+1}).$$

Since it is obvious that $d(a_2,\ldots,a_n) \le d(a_1,\ldots,a_{n+1})$, we obtain

$$\begin{aligned} s(a_1,\ldots,a_{n+1}) &= nd(a_1,\ldots,a_{n+1}) + s(a_2,\ldots,a_n) \\ &\le nd(a_1,\ldots,a_{n+1}) + \frac{(n-1)^2}{4}d(a_2,\ldots,a_n) \\ &\le nd(a_1,\ldots,a_{n+1}) + \frac{(n-1)^2}{4}d(a_1,\ldots,a_{n+1}) \\ &= \frac{4n+(n-1)^2}{4}d(a_1,\ldots,a_{k+1}) \\ &= \frac{(n+1)^2}{n}d(a_1,\ldots,a_{n+1}) \end{aligned}$$

and the induction is complete.

The equality holds if and only if $n = 2$ or $a_1 = a_2 = \cdots = a_n$.

3. The numbers $1, 2, 3, \ldots, n^2$ are placed in the cells of an $n \times n$ board, one number per cell. A coin is initially placed in the cell containing the number n^2. The coin can move to any of the cells which share a side with the cell it currently occupies.

First, the coin travels from the cell containing the number 1 to the cell containing the number n^2, using the smallest possible number of moves. Then the coin travels from the cell containing the number 1 to the cell containing the number 2 using the smallest number of moves, and then from the cell containing the number 3, and continuing until the coin returns to the initial cell, taking a shortest route each time it travels. The complete trip takes N steps. Determine the smallest and largest possible values of N.

Solved by Oliver Geupel, Brühl, NRW, Germany.

According to `http://www.imomath.com/othercomp/Ib/IbMO06.pdf` the trip should be $n^2 \to 1 \to 2 \to \ldots \to n^2-1 \to n^2$. We will show that $\min N$ is n^2 if n is even and n^2+1 if n is odd. Moreover, we prove that $\max N$ is n^3-2 for even n and n^3-n for odd n.

Since we must enter each of the n^2 cells, we have $\min N \ge n^2$. By colouring the cells alternately black and white like a chess board, we see that we can return to a cell of the colour of the initial cell only after an even number of steps; hence $\min N \ge n^2+1$ if n is odd. Examples of trips that reach these bounds are given in Figures 1 and 2, for even and odd n, respectively.

In each step one edge is passed. Thus, the total number of steps is equal to the number of passed edges, counted with the multiplicity of crossings. Consider a $k \times n$ rectangle consisting of the k leftmost or rightmost columns or of the k uppermost or lowermost rows of the board, where $1 \le k \le n/2$. The interior edge of this rectangle can be passed not more than $2kn$ times, specifically, twice for each cell, that is on entering and on leaving the rectangle.

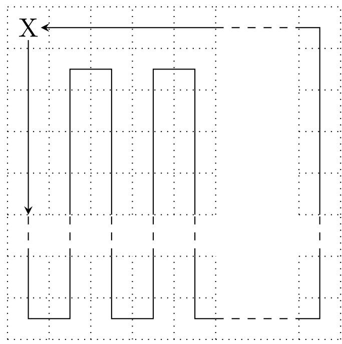


Figure 1

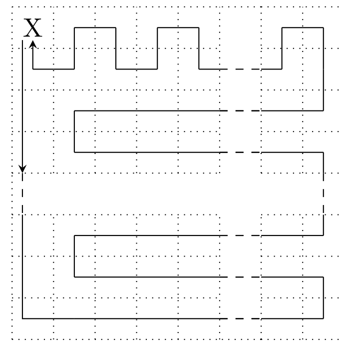


Figure 2

If $\boldsymbol{n}$ is odd, then

$$\boldsymbol{N \leq 4 \sum_{k=1}^{(n-1)/2} 2kn = (n-1)n(n+1) = n^3 - n.}$$

If $\boldsymbol{n}$ is even, the board consists of four quadrants, where we can alternate between the upper-left and the lower-right quadrant as well as between the upper-right and the lower-left quadrant. We must at least twice change the pair of quadrants in order to return to the inital cell thus losing two crossings. Therefore,

$$\boldsymbol{N \leq 4 \sum_{k=1}^{(n-2)/2} 2kn + 2n^2 - 2 = n^3 - 2.}$$

Examples of arrangements that reach these bounds are given in Figures 3 and 4, for $\boldsymbol{n = 2m}$ and $\boldsymbol{n = 2m+1}$, respectively.

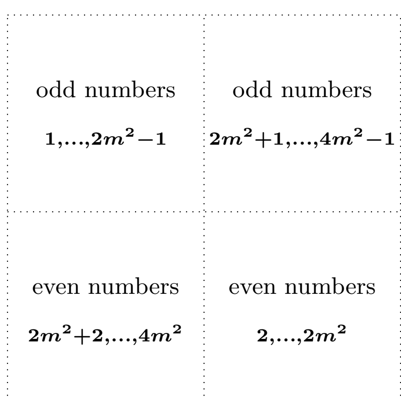


Figure 3

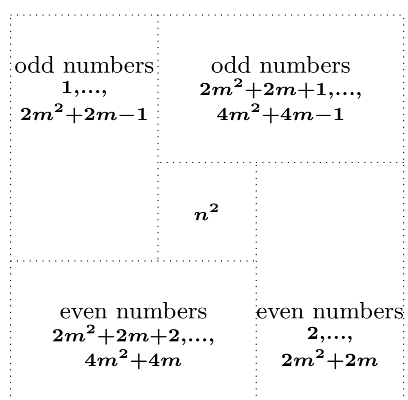


Figure 4

4. Determine all pairs (a, b) of positive integers such that $2a + 1$ and $2b - 1$ are relatively prime and $a + b$ divides $4ab + 1$.

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give Zelator's write-up.

Suppose that a and b are positive integers such that $a + b \mid 4ab + 1$ and $\gcd(2a + 1, 2b - 1) = 1$.

Then

$$4ab + 1 \;=\; k \cdot (a + b)\,, \tag{1}$$

for some positive integer k. From $(2a + 1)(2b + 1) = 4ab + 2(a + b) + 1$ and (1), we obtain

$$(2a + 1)(2b + 1) \;=\; (k + 2) \cdot (a + b)\,. \tag{2}$$

Observe that $a + b$ is relatively prime to $2a + 1$. Indeed, if d is the greatest common divisor of $a + b$ and $2a + 1$, then d must be a divisor of any linear combination (with integer coefficients) of $a + b$ and $2a + 1$. In particular, d must divide $2(a + b) - (2a + 1) = 2b - 1$. Thus $d \mid 2a + 1$ and $d \mid 2b - 1$ and so, by the coprimeness condition in (1), it follows that $d = 1$.

We have shown that,

$$\gcd(a + b, 2a + 1) \;=\; 1. \tag{3}$$

Euclid's lemma in number theory postulates that if an integer divides the product of two other integers, and it is relatively prime to one of those (two) integers, then it must divide the other one.

Clearly then, by Euclid's lemma, (3), and (2), it follows that $a + b$ must divide $2b + 1$. Thus there exists a positive integer, m, such that

$$2b + 1 \;=\; m \cdot (a + b)\,. \tag{4}$$

We rewrite (4) in the form,

$$m \cdot a + (m - 2) \cdot b \;=\; 1\,. \tag{5}$$

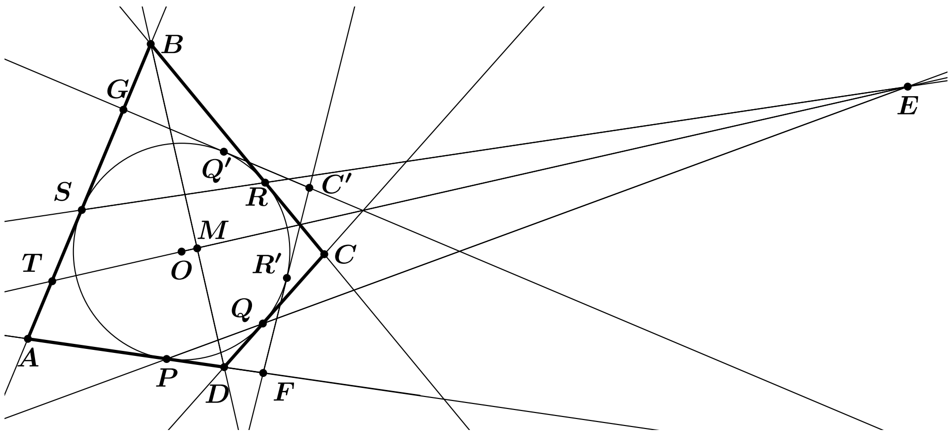
Clearly, if $m \geq 2$, the lefthand side of (5) is greater than 1, since a and b are positive integers. Therefore $m = 1$, which yields, $a - b \;=\; 1$ and $a \;=\; b + 1$.

We have proven that if two positive integers a and b satisfy conditions (1), then $a \;=\; b + 1$. The converse is also true. Indeed if $a \;=\; b + 1$, then $4ab + 1 = 4(b + 1)b + 1 = (2b + 1)^2 = (a + b)^2$, which shows that $a + b$ divides $4ab + 1$. Moreover, we have $2a + 1 \;=\; 2(b + 1) + 1 \;=\; 2b + 3$, which is relatively prime to $2b - 1$, since if $D = \gcd(2b - 1, 2b + 3)$. Then $D \mid [(2b + 3) - (2b - 1)] = 4$. But D is odd, since $2b - 1$ and $2b + 3$ are. Thus $D = 1$.

Conclusion: The pairs $(a, b) = (b + 1, b)$, where b can be any positive integer, are the solution to this problem.

5. The circle $\boldsymbol{\Gamma}$ is inscribed in quadrilateral $\boldsymbol{ABCD}$ with $\boldsymbol{AD}$ and $\boldsymbol{CD}$ tangent to $\boldsymbol{\Gamma}$, at $\boldsymbol{P}$ and $\boldsymbol{Q}$, respectively. If $\boldsymbol{BD}$ intersects $\boldsymbol{\Gamma}$ at $\boldsymbol{X}$ and $\boldsymbol{Y}$ and $\boldsymbol{M}$ is the midpoint of $\boldsymbol{XY}$, prove that $\angle \boldsymbol{AMP} = \angle \boldsymbol{CMQ}$.

Solved by Oliver Geupel, Brühl, NRW, Germany.



We argue in terms of projective geometry, assuming that parallel lines meet at a point of infinity. Let $\boldsymbol{O}$ be the midpoint of $\boldsymbol{\Gamma}$, let $\boldsymbol{E}$ be the pole of $\boldsymbol{BD}$ with respect to $\boldsymbol{\Gamma}$, and let $\boldsymbol{BC}$ and $\boldsymbol{AB}$ be tangent to $\boldsymbol{\Gamma}$ at points $\boldsymbol{R}$ and $\boldsymbol{S}$, respectively. We assume without loss of generality that the points $\boldsymbol{C}$ and $\boldsymbol{E}$ are on the same side of the line $\boldsymbol{BD}$.

Lemma 1. The lines $\boldsymbol{AC}$, $\boldsymbol{PQ}$, and $\boldsymbol{RS}$ meet at $\boldsymbol{E}$. Moreover, $\boldsymbol{BD} \perp \boldsymbol{EM}$, and the points $\boldsymbol{E}$, $\boldsymbol{M}$, and $\boldsymbol{O}$ are collinear.

Proof. Let $\boldsymbol{PQ}$ meet $\boldsymbol{AC}$ at $\boldsymbol{E_1}$, and let $\boldsymbol{RS}$ and $\boldsymbol{AC}$ meet at $\boldsymbol{E_2}$. By $\boldsymbol{PD} = \boldsymbol{PQ}$ and Menelaus' Theorem, for $\triangle \boldsymbol{ACD}$ and the line $\boldsymbol{PQ}$ it holds

$$\frac{\boldsymbol{E_1A}}{\boldsymbol{E_1C}} \cdot \frac{\boldsymbol{QC}}{\boldsymbol{PA}} = \frac{\boldsymbol{E_1A}}{\boldsymbol{E_1C}} \cdot \frac{\boldsymbol{QC}}{\boldsymbol{QD}} \cdot \frac{\boldsymbol{PD}}{\boldsymbol{PA}} = \mathbf{1}$$

Analogously,

$$\frac{\boldsymbol{E_2A}}{\boldsymbol{E_2C}} \cdot \frac{\boldsymbol{RC}}{\boldsymbol{SA}} = \mathbf{1}.$$

Since $\boldsymbol{QC} = \boldsymbol{RC}$ and $\boldsymbol{PA} = \boldsymbol{SA}$, we obtain

$$\frac{\boldsymbol{E_1A}}{\boldsymbol{E_1C}} = \frac{\boldsymbol{E_2A}}{\boldsymbol{E_2C}}$$

and consequently $\boldsymbol{E_1} = \boldsymbol{E_2}$. The point $\boldsymbol{E_1}$ lies on the polar $\boldsymbol{PQ}$ of point $\boldsymbol{D}$. Hence, $\boldsymbol{D}$ is on the polar of $\boldsymbol{E_1}$. Similarly, $\boldsymbol{B}$ is also on the polar of $\boldsymbol{E_1}$. Thus, $\boldsymbol{BD}$ is the polar of $\boldsymbol{E_1}$, that is, $\boldsymbol{E_1}$ coincides with the pole $\boldsymbol{E}$ of $\boldsymbol{BD}$, and $\boldsymbol{BD} \perp \boldsymbol{EM}$. Thus, $\boldsymbol{E}$, $\boldsymbol{M}$, and $\boldsymbol{O}$ are collinear. □

Let $\boldsymbol{P'}$, $\boldsymbol{Q'}$, $\boldsymbol{R'}$, and $\boldsymbol{S'}$ be the reflections of $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{S}$ in the line $\boldsymbol{EM}$.

Lemma 2. The lines $\boldsymbol{PQ'}$, $\boldsymbol{P'Q}$, $\boldsymbol{RS'}$, and $\boldsymbol{R'S}$ meet at $\boldsymbol{M}$.

Proof. Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be the intersection of $\boldsymbol{BD}$ and $\boldsymbol{PQ}$ and the intersection of $\boldsymbol{BD}$ and $\boldsymbol{P'Q'}$, respectively. Let the line $\boldsymbol{UV}$ meet $\boldsymbol{PQ'}$ at point $\boldsymbol{W}$. By Menelaus' Theorem, for $\triangle \boldsymbol{EUV}$ and the line $\boldsymbol{PQ'}$ it holds

$$\frac{PE}{PU} \cdot \frac{WU}{WV} \cdot \frac{Q'V}{Q'E} = 1;$$

hence, by $\boldsymbol{Q'V = QU}$ and $\boldsymbol{Q'E = QE}$,

$$\frac{PE}{PU} : \frac{QE}{QU} = \frac{WV}{WU}.$$

On the other hand, since $\boldsymbol{E}$ and $\boldsymbol{U}$ are harmonic conjugates with respect to the points $\boldsymbol{P}$ and $\boldsymbol{Q}$, we have

$$\frac{PE}{PU} : \frac{QE}{QU} = 1.$$

Consequently, $\boldsymbol{W = M}$, that is, $\boldsymbol{M}$ lies on $\boldsymbol{PQ'}$. Analogously, $\boldsymbol{M}$ lies on $\boldsymbol{P'Q}$, $\boldsymbol{RS'}$, and $\boldsymbol{R'S}$. □

We are now prepared to prove that $\angle \boldsymbol{AMP} = \angle \boldsymbol{CMQ}$.

Since $\boldsymbol{EM}$ bisects $\angle \boldsymbol{PMP'}$, the orthogonal line $\boldsymbol{BD}$ bisects the complementary angle $\boldsymbol{PMQ}$, that is,

$$\angle DMP = \angle DMQ. \tag{1}$$

Let $\boldsymbol{C'}$ be the reflection of $\boldsymbol{C}$ in the line $\boldsymbol{EM}$. Since the tangents of $\boldsymbol{\Gamma}$ in $\boldsymbol{Q}$ and $\boldsymbol{R}$ meet at $\boldsymbol{C}$, we see that the tangents in $\boldsymbol{Q'}$ and $\boldsymbol{R'}$ meet at $\boldsymbol{C'}$. Let the tangents to $\boldsymbol{\Gamma}$ in $\boldsymbol{P}$ and $\boldsymbol{R'}$ meet at point $\boldsymbol{F}$, and let the tangents in $\boldsymbol{S}$ and $\boldsymbol{Q'}$ meet at point $\boldsymbol{G}$.

Consider the circumscribed hexagon $\boldsymbol{APFC'Q'G}$. By Brianchon's Thereom, the lines $\boldsymbol{AC'}$, $\boldsymbol{PQ'}$, and $\boldsymbol{FG}$ are concurrent. Next, consider the circumscribed hexagon $\boldsymbol{FR'C'GSA}$. By Brianchon's Theorem, the lines $\boldsymbol{AC'}$, $\boldsymbol{R'S}$, and $\boldsymbol{FG}$ are concurrent. Therefore, the four lines $\boldsymbol{AC'}$, $\boldsymbol{FG}$, $\boldsymbol{PQ'}$, and $\boldsymbol{R'S}$ are concurrent. By Lemma 2, their intersection is the point $\boldsymbol{M}$. We deduce that the points $\boldsymbol{A}$, $\boldsymbol{C'}$, and $\boldsymbol{M}$ are collinear. Let $\boldsymbol{T}$ be the intersection of $\boldsymbol{AB}$ and $\boldsymbol{EM}$. Then,

$$\angle AMT = \angle C'ME = \angle CME\,. \tag{2}$$

By (1) and (2), we obtain

$$\angle AMP = 90^\circ - \angle DMP - \angle AMT = 90^\circ - \angle DMQ - \angle CME = \angle CMQ,$$

which completes the proof.

That completes the material for this number of the *Corner*.

BOOK REVIEWS

Amar Sodhi

Icons of Mathematics: An Exploration of Twenty Key Images
by Claudi Alsina & Roger B. Nelsen
Dolciani Mathematical Exposition #45
The Mathematical Association of America, 2011
ISBN: 978-0-88385-352-8 (print), 978-0-88385-986-5 (electronic)
Hard cover, 327+xvii pages, US$69.95
Reviewed by ***Edward J. Barbeau****, University of Toronto, Toronto, ON*

The author of a geometry book directed at a general audience has a daunting task. For about 2500 years, amateur and professional mathematicians in Europe and Asia have uncovered an abunadance of fascinating results about simple geometric figures, particularly circles and triangles, and the end is apparently not in sight. Many different techniques have been developed to establish them, each shining its own particular light and allowing for different insights and connections to be made. How can one choose among such a wealth of facts and arguments and arrive at a book that is readable, representative, focussed and compact?

The authors of the book under review use twenty diagrams, or "icons", as an organizing principle. Many of them have a particular historical or cultural significance, such as the Yin-Yang symbol (a circle bisected by two semi-circles of half the diameter), the "windmill" diagram used by Euclid in his proof of Pythagoras' theorem (I:47) (a right triangle with squares constructed outwards on the sides), the three circles of a standard Venn diagram, and a trapezoid figure used by President Garfield in his proof of Pythagoras' theorem. Others are just representative figures to introduce a theme.

The prerequisites for this book are modest. The reader should have a secondary background in Euclidean geometry and trigonometry, along with an active imagination. Most of the results are proved in an informal way; the chief tools are standard Euclidean arguments, algebraic manipulation and *proofs without words* that involve dissections and shifting figures around. A small number of propositions are mentioned without proof, and some are listed as challenges at the end of each chapter, with solutions provided in an appendix. Each chapter is lightened by digressions on mathematical personalities and artefacts, history and background, and occurrences of geometrical figures in ordinary life. For example, on page 203, we find photographs of star-shaped badges with 5, 6, 7 and 8 points worn by law enforcement officers along with speculation about their origin. Two pages later, we learn about the role of star polygons in the design of columns for a modern church in Barcelona by Antoni Gaudi (1852-1926).

This book need not be read straight through. Since the twenty chapters are generally self-contained, readers can forage at random and follow something that catches their fancy. ***CRUX with MAYHEM*** readers will find a lot of familiar material along with results and arguments that will be new to them. The scope of the book can be suggested by a list of some topics that make an appearance. Apart from standard material on triangles, circles and inequalities, the authors touch on Dido's isoperimetric problem, Euclid's construction of the regular solids, reptiles, cevians, the butterfly theorem, conic sections, Reuleux polygons, star polygons, self-similarity, spirals, the Monge sponge and Sierpinski carpet and tilings. The treatment is light but satisfying, and readers wanting more are directed to other resources.

There are some themes that recur throughout the book, such as tilings and dissections, inscribed and escribed figures and cevians. The most prominent of these is the Pythagorean theorem, which is treated in many places. In the opening chapter, Euclid's diagram is generalized to the Vecten configuration (squares escribed on an arbitrary triangle) which leads to an insightful proof of the Cosine Law and the solution to two problems by the American puzzler, Sam Loyd (1841-1911). The second chapter uses the icon of one square inscribed inside another for an ancient Chinese proof of the Pythagorean theorem and continues to get a diagrammatic proof of standard inequalities. The trapezoidal diagram used by U.S. President Garfield (1831-1881) to prove the theorem is pressed into service for some trigonometric equations. The chapter on similar figures provides the occasion to prove the Pythagorean as well as the Reciprocal Pythagorean theorem (that, if $a^2 + b^2 = c^2$, then $(1/a)^2 + (1/b)^2 = (1/h)^2$ where h is the altitude to the hypotenuse). The right triangle has a chapter of its own, in which Pythagorean triples are characterized and the Pythagorean relation is used to derive some inequalities. Another characterization of Pythagorean triples is found using two overlapping squares of areas a^2 and b^2 inside a square of area $c^2 = a^2 + b^2$. An arrangement of tatami rectangular mats in a square is behind a proof of the Pythagorean theorem credited to Bhaskara (1114-1185). In the final chapter, infinitely many demonstrations of the theorem can be had by laying a hypotenuse grid atop a plane tiling of two different squares.

This is an enjoyable and useful book that provides a lot of material that could be presented to secondary students. The challenges at the end of the chapter are generally accessible and often require some insight to obtain an elegant solution. I found very little to quibble with. The approach to the proof of the result on page 133 (the locus of the centres of circles touching an ellipse and passing through a point inside the ellipse) seems backwards when a direct assault is just as easy. In Challenge 3.8, the diagram needs to be justified by the result that if $ABCD$ is a trapezoid with AB and DC both perpendicular to BC, E is the midpoint of BC and $\angle AED = 90°$, then ED bisects angle ADC.

This book suggests how geometry might be rehabilitated in the school curriculum. Now that geometry is no longer seen as the centrepiece of the syllabus and a prerequisite for later mathematics, it can be taught as an important milestone in human intellectual history, a celebration of human ingenuity, a more natural approach to proof and a source of personal pleasure. This book is a fine vehicle to carry out such a course.

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **15 mars 2012**. *Une étoile* (⋆) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

3638. *Proposé par Michel Bataille, Rouen, France.*

Etant donné un triangle ABC, on arrange respectivement les points D, E, F sur les droites BC,CA,AB, de sorte que

$$BD : DC = \lambda : 1 - \lambda, \quad CE : EA = \mu : 1 - \mu, \quad AF : FB = \nu : 1 - \nu.$$

Montrer que DEF est le triangle pédal du triangle ABC si et seulement si

$$(2\lambda - 1)BC^2 + (2\mu - 1)CA^2 + (2\nu - 1)AB^2 = 0.$$

3639. *Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit a, b et c trois nombres réels non négatifs tels que $a + b + c = 3$. Montrer que

$$\frac{a^2b}{a+b+1} + \frac{b^2c}{b+c+1} + \frac{c^2a}{c+a+1} \leq 1\,.$$

3640. *Proposé par Roy Barbara, Université Libanaise, Fanar, Liban.*

On considère la fonction $f(x) = -\sqrt[3]{4x^6 + 6x^3 + 3}$.

(a) Trouver les points fixes de $f(x)$, s'il y en a.

(b) Trouver les points périodiques de période 2 de $f(x)$, s'il y en a.

(c) Montrer que $x = -1$ est l'unique nombre réel tel que x et $f(x)$ sont tous deux des entiers.

3641. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit $0 \leq x_1, x_2, \ldots, x_n < \pi/2$ n nombres réels. Montrer que

$$\left(\frac{1}{n}\sum_{k=1}^{n} \sec(x_k)\right)\left(1 - \left(\frac{1}{n}\sum_{k=1}^{n} \sin(x_k)\right)^2\right)^{1/2} \geq 1\,.$$

3642. *Proposé par Michel Bataille, Rouen, France.*

Evaluer

$$\lim_{n\to\infty} \frac{\int_0^1 (2x^2-5x-1)^n dx}{\int_0^1 (x^2-4x-1)^n dx}\,.$$

3643. *Proposé par Pham Van Thuan, Université de Science des Hanoï, Hanoï, Vietnam.*

Soit u et v deux nombres réels positifs. Montrer que

$$\frac{1}{8}\left(17-\frac{2uv}{u^2+v^2}\right) \;\le\; \sqrt[3]{\frac{u}{v}}+\sqrt[3]{\frac{v}{u}} \;\le\; \sqrt{(u+v)\left(\frac{1}{u}+\frac{1}{v}\right)}\,.$$

Pour chaque inégalité, déterminer quand il y a égalité.

3644. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

On opère une trisection des côtés AB et AC du triangle ABC avec les points D, E et F, G respectivement, de telle sorte que $AE = ED = DB$ et $AF = FG = GC$. La droite BF coupe respectivement CD, CE aux points K, L, tandis que BG coupe CD, CE en N, M respectivement.

Montrer que :

(a) KM est parallèle à BC ;

(b) Aire$(KLM) = \frac{5}{7}$Aire$(KLMN)$.

3645. *Proposé par José Luis Díaz-Barrero et Juan José Egozcue, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit a, b et c trois nombres positifs tels que $a^2+b^2+c^2+2abc=1$. Montrer que

$$\sum_{\text{cyclique}} \sqrt{a\left(\frac{1}{b}-b\right)\left(\frac{1}{c}-c\right)} \;>\; 2\,.$$

3646. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $\alpha \ge 0$ et soit β un nombre positif. Trouver la limite

$$L(\alpha,\beta) \;=\; \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1+\frac{k^\alpha}{n^\beta}\right)^k - n\right)\,.$$

3647. *Proposé par Panagiote Ligouras, École Secondaire Léonard de Vinci, Noci, Italie.*

Montrer que dans un triangle ABC dont les rayons des cercles exinscrits sont r_a, r_b et r_c, on a

$$\sum_{\text{cyclique}} \frac{(r_a + r_b)(r_b + r_c)}{ac} \geq 9\,,$$

où $AB = c$, $BC = a$ et $CA = b$.

3648. *Proposé par Michel Bataille, Rouen, France.*

Trouver tous les nombres réels x, y, z tels que $xyz = 1$ et $x^3 + y^3 + z^3 = \frac{S(S-4)}{4}$ où $S = \frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z}$.

3649. *Proposé par Pham Van Thuan, Université de Science des Hanoï, Hanoï, Vietnam.*

Soit a, b et c trois nombres réels positifs et soit

$$k = (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\,.$$

Montrer que

$$(a^3 + b^3 + c^3)\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) \geq \frac{k^3 - 15k^2 + 63k - 45}{4}\,,$$

et que l'égalité a lieu si et seulement si $(a, b, c) = \left(\frac{k - 5 \pm \sqrt{k^2 - 10k + 9}}{4}, 1, 1\right)$ ou une quelconque de ses permutations.

3650. *Proposé par Mehmet Sahin, Ankara, Turquie.*

Soit ABC un triangle acutangle avec $A' \in BC$, $B' \in CA$ et $C' \in AB$ arrangés de telle sorte que

$$\angle ACC' = \angle CBB' = \angle BAA' = 90^\circ\,.$$

Montrer que

(a)
$$|BC'||CA'||AB'| = abc;$$

(b)
$$\frac{AA'}{BC'}\frac{BB'}{CA'}\frac{CC'}{AB'} = \tan(A)\tan(B)\tan(C);$$

(c)
$$\frac{A(ABC)}{A(A'B'C')} = 1 + \frac{4R^2}{(2R + r)^2 - s^2}\,.$$

3638. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle and let points D, E, F lie on lines BC, CA, AB, respectively, such that

$$BD : DC = \lambda : 1 - \lambda, \quad CE : EA = \mu : 1 - \mu, \quad AF : FB = \nu : 1 - \nu.$$

Show that DEF is a pedal triangle with regard to ΔABC if and only if

$$(2\lambda - 1)BC^2 + (2\mu - 1)CA^2 + (2\nu - 1)AB^2 = 0.$$

3639. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{a^2b}{a+b+1} + \frac{b^2c}{b+c+1} + \frac{c^2a}{c+a+1} \leq 1\,.$$

3640. *Proposed by Roy Barbara, Lebanese University, Fanar, Lebanon.*

Consider the function $f(x) = -\sqrt[3]{4x^6 + 6x^3 + 3}$.

(a) Find the fixed points of $f(x)$, if any.

(b) Find the periodic points with period 2 of $f(x)$, if any.

(c) Prove that $x = -1$ is the unique real number such that x and $f(x)$ are both integers.

3641. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $0 \leq x_1, x_2, \dots, x_n < \pi/2$ be real numbers. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n} \sec(x_k)\right)\left(1 - \left(\frac{1}{n}\sum_{k=1}^{n}\sin(x_k)\right)^2\right)^{1/2} \geq 1\,.$$

3642. *Proposed by Michel Bataille, Rouen, France.*

Evaluate

$$\lim_{n\to\infty} \frac{\int_0^1 (2x^2 - 5x - 1)^n dx}{\int_0^1 (x^2 - 4x - 1)^n dx}\,.$$

3643. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let u and v be positive real numbers. Prove that

$$\frac{1}{8}\left(17-\frac{2uv}{u^2+v^2}\right) \;\le\; \sqrt[3]{\frac{u}{v}}+\sqrt[3]{\frac{v}{u}} \;\le\; \sqrt{(u+v)\left(\frac{1}{u}+\frac{1}{v}\right)}$$

For each inequality, determine when equality holds.

3644. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

We trisect the sides AB and AC of triangle ABC with the points D, E and F, G respectively such that $AE=ED=DB$ and $AF=FG=GC$. The line BF intersects CD, CE in the points K, L respectively, while BG intersects CD, CE in N, M respectively.

Prove that:

(a) KM is parallel to BC;

(b) Area$(KLM)=\frac{5}{7}$Area$(KLMN)$.

3645. *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, and c be positive numbers such that $a^2+b^2+c^2+2abc=1$. Prove that

$$\sum_{\text{cyclic}}\sqrt{a\left(\frac{1}{b}-b\right)\left(\frac{1}{c}-c\right)} \;>\; 2\,.$$

3646. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $\alpha\ge 0$ and let β be a positive number. Find the limit

$$L(\alpha,\beta) \;=\; \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1+\frac{k^\alpha}{n^\beta}\right)^k-n\right).$$

3647. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Show that in triangle ABC with exradii r_a, r_b and r_c,

$$\sum_{\text{cyclic}}\frac{(r_a+r_b)(r_b+r_c)}{ac} \;\ge\; 9\,,$$

where $AB=c$, $BC=a$, and $CA=b$.

3648. *Proposed by Michel Bataille, Rouen, France.*

Find all real numbers x, y, z such that $xyz = 1$ and $x^3+y^3+z^3 = \frac{S(S-4)}{4}$ where $S = \frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z}$.

3649. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, and c be three positive real numbers and let

$$k = (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) .$$

Prove that

$$(a^3+b^3+c^3)\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) \ \geq \ \frac{k^3 - 15k^2 + 63k - 45}{4} ,$$

and equality holds if and only if $(a, b, c) = \left(\frac{k-5 \pm \sqrt{k^2-10k+9}}{4}, 1, 1\right)$ or any of its permutations.

3650. *Proposed by Mehmet Sahin, Ankara, Turkey.*

Let ABC be an acute-angled triangle with $A' \in BC$, $B' \in CA$, and $C' \in AB$ arranged so that

$$\angle ACC' = \angle CBB' = \angle BAA' = 90^\circ .$$

Prove that

(a)

$$|BC'||CA'||AB'| = abc;$$

(b)

$$\frac{AA'}{BC'}\frac{BB'}{CA'}\frac{CC'}{AB'} = \tan(A)\tan(B)\tan(C);$$

(c)

$$\frac{A(ABC)}{A(A'B'C')} = 1 + \frac{4R^2}{(2R+r)^2 - s^2} .$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3526. [2010 : 109, 111] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{a}{\sqrt{a^2+2(b+c)^2}} \;\geq\; 1\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

By Hölder inequality we have

$$\left(\sum_{cyclic} \frac{a}{\sqrt{a^2+2(b+c)^2}}\right)^2 \left(\sum_{cyclic} a(a^2+2(b+c)^2)\right) \geq (a+b+c)^3\,.$$

Thus we only need to show that

$$(a+b+c)^3 \geq \sum_{cyclic} a(a^2+2(b+c)^2)\,,$$

or equivalently that

$$a^2b+a^2c+ab^2+ac^2+b^2c+bc^2 \geq 6abc\,.$$

But this is immediate from the AM-GM inequality. This completes the proof.

Equality holds if and only if $a=b=c$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece (2 solutions); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3539. [2010 : 239, 241] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.*

Let A and B be 2×2 square matrices with real entries. Prove that the equations $\det(xA \pm B) = 0$ have all of their roots real if and only if

$$[\mathrm{trace}(AB) - \mathrm{trace}(A)\mathrm{trace}(B)]^2 \;\geq\; 4\det(A)\det(B)\,.$$

Solution by the Henry Ricardo, Tappan, NY, USA.

The result is false without further conditions on the matrix A. For example, letting $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, we find that $\det(xA \pm B) = \det \begin{bmatrix} x \pm 1 & \pm 1 \\ \mp 1 & 0 \end{bmatrix} = 1$, so that $\det(xA \pm B) = 0$ has *no* roots despite the fact that $[\mathrm{trace}(AB) - \mathrm{trace}(A)\mathrm{trace}(B)]^2 = 0 = 4\det(A)\det(B)$.

If A is a 2×2 matrix with real entries, we can use the easily proved result (see Fact 4.9.3 in *Matrix Mathematics (Second Edition)* by Dennis S. Bernstein, Princeton University Press, 2009) that

$$\det(A + B) - \det(A) - \det(B) = \mathrm{trace}(A)\mathrm{trace}(B) - \mathrm{trace}(AB)\,.$$

Thus, using basic properties of the determinant and trace,

$$\det(xA \pm B) = \det(A)x^2 \pm (\mathrm{trace}(A)\mathrm{trace}(B) - \mathrm{trace}(AB))x + \det(B)\,.$$

Now if A is *nonsingular*, the equations $\det(xA \pm B) = 0$ have all of their roots real if and only if $[\mathrm{trace}(AB) - \mathrm{trace}(A)\mathrm{trace}(B)]^2 \geq 4\det(A)\det(B)$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MURIEL BAKER, student, Southeast Missouri State University, Cape Girardeau, Missouri, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; HTET NAING LIN, Southeast Missouri State University, Cape Girardeau, Missouri, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JAMES MEYER, student, Southeast Missouri State University, Cape Girardeau, Missouri, USA; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; JOHN POSTL, St. Bonaventure University, St. Bonaventure, NY, USA; HANNAH PREST, student, Southeast Missouri State University, Cape Girardeau, Missouri, USA; JAMES REID, student, Angelo State University, San Angelo, TX, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; CULLAN SPRINGSTEAD, Southeast Missouri State University, Cape Girardeau, Missouri, USA; ALBERT STADLER, Herrliberg, Switzerland; ELIZABETH WAMSER, student, Southeast Missouri State University, Cape Girardeau, Missouri, USA; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; BRENT WESSEL, student, Southeast Missouri State University, Cape Girardeau, Missouri, USA; DANIEL WINGER, student, St. Bonaventure University, Allegany, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

As the matrices were all 2×2 almost all the other solutions were by direct computation. The proposer was the only other solver to use properties of determinants and the trace. The featured solution was the only solution to note the condition that A needed to be nonsingular.

3540. [2010 : 239, 241] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Triangle ABC has semiperimeter s and area F. A square $PQRS$ with side length x is inscribed in ABC with P and Q on BC, R on AC, and S on AB. Similarly y and z are the sides of squares two vertices of which lie on AC and AB, respectively. Prove that

$$x^{-1}+y^{-1}+z^{-1} \;\leq\; \frac{s(2+\sqrt{3})}{2F}\,.$$

Combination of solutions by John G. Heuver, Grande Prairie, AB and the proposer.

As usual, the side lengths of ΔABC will be denoted by a, b, and c, and the altitudes by h_a, h_b, and h_c. By the similarity of triangles RAS and CAB we have $\frac{x}{a}=\frac{h_a-x}{h_a}$, so that

$$x=\frac{ah_a}{a+h_a}=\frac{2F}{a+h_a},\quad \text{or}\quad x^{-1}=\frac{a+h_a}{2F}.$$

Similarly,

$$y^{-1}=\frac{b+h_b}{2F}\quad \text{and}\quad z^{-1}=\frac{c+h_c}{2F}.$$

This allows us to deduce that

$$x^{-1}+y^{-1}+z^{-1}=\frac{2s+h_a+h_b+h_c}{2F}.$$

The desired conclusion follows from the familiar inequality

$$h_a+h_b+h_c\leq s\sqrt{3};$$

see, for example, [1] page 60, formula 6.1 or 6.2. Equality holds if and only if ΔABC is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD EDEN, student, Purdue University, West Lafayette, IN, USA; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comăneşti, Romania.

Heuver used the Cauchy-Schwarz Inequality to deduce that $h_a+h_b+h_c\leq\sqrt{3}\sqrt{h_a^2+h_b^2+h_c^2}$*, and then applied* $\sqrt{h_a^2+h_b^2+h_c^2}\leq s^2$*, which is (9.8) on page 201 of [2]. Zvonaru finished his solution with* $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{\sqrt{3}}{2r}$*, which is item 5.22 on page 54 of [1].*

References

[1] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969

[2] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989

3541. [2010 : 240, 242] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Triangle ABC has circumcentre O, circumradius R, orthocentre H, side lengths a, b, c, and altitudes AD, BE, CF, where points D, E, F lie on the sides BC, AC, AB, respectively. The Euler line of triangle ABC intersects BC in P and AC in Q, and the quadrilateral $ABPQ$ is cyclic.

Show that $a^2+b^2 = 6R^2$, and express the length of PQ in terms of a, b, c.

Solution by the proposer.

Because $ABPQ$ is cyclic, $\angle CPQ = \angle BAQ = \angle BAC$. Since DE joins the feet of two altitudes, we also have $\angle BAC = \angle CDE$, whence $PQ||DE$. OH is (by assumption) the same line as PQ. Because P lies on BC, we have $\angle ACH = \angle OCP$ so that ACF and PCO are similar right triangles; that is, $CO \perp PQ$. In the right triangle HOC,

$$CH^2 = OH^2 + OC^2 = OH^2 + R^2. \qquad (1)$$

Formula **5.8(1)** on page 50 of O. Bottema et al., *Geometric Inequalities*, Groningen, 1969 says that

$$OH^2 = 9R^2 - (a^2 + b^2 + c^2). \qquad (2)$$

Moreover, in any triangle $CH = 2R\cos C$ (see, for example Roger A. Johnson *Advanced Euclidean Geometrey*, Paragraph 252(e), page 163), so that equations (1) and (2) gives us

$$9R^2 - (a^2 + b^2 + c^2) + R^2 = 4R^2\cos^2 C.$$

Using the sine law, we replace c by $2R\sin C$ in the last equation to get

$$a^2 + b^2 = 6R^2,$$

as desired.

To determine the length of PQ we first recall that $\angle CPQ = \angle BAC$, which implies that the triangles PQC and ABC are similar. Because CO and CF are corresponding altitudes, we deduce that $\frac{PQ}{CO} = \frac{AB}{CF}$, or (using the definition of sine, the sine law, and the equation from the previous paragraph),

$$PQ = \frac{Rc}{b\sin A} = \frac{2R^2c}{b \cdot 2R\sin A} = \frac{2R^2c}{ba} = \frac{c(a^2+b^2)}{3ab}.$$

Also solved by OLIVER GEUPEL, Brühl, NRW, Germanyand PETER Y. WOO, Biola University, La Mirada, CA, USA.

Although the proposer stated his problem correctly, the published statement of the problem [2010 : 240, 242] contained two errors. Both Geupel and Woo reported the errors, but Woo figured out how to correct them, and he provided a complete solution. One of the editor's errors is worth describing: The proposer correctly called the quadrilateral $ABPQ$ inscriptible *to mean that it can be inscribed in a circle. Unfortunately the word is ambiguous in English, as is its more common version* inscribable*—it can mean "permitting something to be inscribed in it" as in, "This book is inscribable." (that is, it is possible to write an inscription in the book), as well as meaning capable of being inscribed in something, as in, "A rectangle is the only parallelogram that is inscribable in a circle." One should restrict the use of the word to situations where the* *context is clear (that is, when it is clear which object is being inscribed where); otherwise it is preferable to use the word* cyclic *(a cyclic polygon) or* concyclic *(a concyclic set of points). A similar error in **CRUX with MAYHEM** a few years ago brought forth a similar editorial comment [1997 : 530-531].*

3542★. [2010 : 240, 242] *Proposed by Cosmin Pohoaţă, Tudor Vianu National College, Bucharest, Romania.*

The mixtilinear incircles of a triangle ABC are the three circles each tangent to two sides and to the circumcircle internally. Let Γ be the circle tangent to each of these three circles internally. Prove that Γ is orthogonal to the circle passing through the incentre and the isodynamic points of the triangle ABC.

[*Ed.: Let Γ_A be the circle passing through A and the intersection points of the internal and external angle bisectors at A with the line BC. The isodynamic points are the two points that Γ_A, Γ_B, and Γ_C have in common.*]

No solutions to this problem were submitted. Problem 3542 therefore remains open.

3543. [2010 : 240, 242] *Proposed by Mehmet Mehmet Şahin, Ankara, Turkey.*

Triangle ABC has inradius r, circumradius R, and angle bisectors $[AD]$, $[BE]$, $[CF]$, where points D, E, F lie on the sides BC, AC, AB, respectively. Let R' be the circumradius of triangle DEF. Prove that

$$R' \leq \frac{R^4}{16r^3}.$$

A combination of solutions by Arkady Alt, San Jose, CA, USA and Michel Bataille, Rouen, France.

We will prove the inequality $R' \leq \frac{R}{2}$, which implies the required inequality, since $\frac{R}{2} \leq \frac{R^4}{16r^3}$ is equivalent to Euler's inequality, $2r \leq R$.

Let $a = BC$, $b = CA$, $c = AB$, $s = \frac{1}{2}(a+b+c)$, and let $[\cdot]$ denote the area of the enclosed figure.

Since $\frac{BD}{c} = \frac{DC}{b} = \frac{a}{c+b}$, we have $BD = \frac{ca}{b+c}$ and $CD = \frac{ab}{b+c}$. Similarly, $AE = \frac{bc}{a+c}$, $CE = \frac{ab}{a+c}$, $AF = \frac{bc}{a+b}$, $BF = \frac{ca}{a+b}$, and it follows that

$$[BDF] = \frac{1}{2} \cdot \frac{ca}{b+c} \cdot \frac{ca}{a+b} \cdot \sin B = \frac{a^2bc^2}{4R(b+c)(a+b)}\,,$$
$$[CED] = \frac{a^2b^2c}{4R(b+c)(a+c)}\,,$$
$$[AFE] = \frac{ab^2c^2}{4R(a+b)(a+c)}\,.$$

Using $[ABC] = \frac{abc}{4R}$, a straightforward calculation yields

$$[EDF] = [ABC] - [BDF] - [CED] - [AFE] = \frac{(abc)^2}{2R(a+b)(b+c)(c+a)}\,.$$

From the Law of Cosines,

$$EF^2 = \frac{(bc)^2}{(a+c)^2} + \frac{(bc)^2}{(a+b)^2} - \frac{2(bc)^2}{(a+b)(a+c)} \cdot \cos A = \frac{(abc)^2}{(a+b)^2(a+c)^2} \cdot K_a\,,$$

where $K_a = \frac{1}{a^2}[(a+c)^2 + (a+b)^2 - 2(a+b)(a+c)\cos A]$. Now,

$$\begin{aligned} K_a &= \frac{1}{a^2}\Big[2a^2 + b^2 + c^2 + 2ab + 2ac \\ &\qquad - 2a^2\cos A - 2ab\cos A - 2ac\cos A - 2bc\cos A\Big] \\ &= \frac{1}{a^2}\Big[a^2 + 2a(1-\cos A)(a+b+c)\Big] \\ &= 1 + \frac{8s}{a}\sin^2(A/2) \\ &= 1 + \frac{8s(s-b)(s-c)}{abc} \\ &= 1 + \frac{8}{4Rrs} \cdot \frac{r^2s^2}{s-a} = 1 + \frac{2rs}{R(s-a)}\,. \end{aligned}$$

As a result, $EF = \frac{abc}{(a+b)(a+c)}\sqrt{K_a}$, and similarly

$$FD = \frac{abc}{(a+b)(b+c)}\sqrt{K_b}\,, \qquad DE = \frac{abc}{(a+c)(b+c)}\sqrt{K_c}\,,$$

where $K_b = 1 + \frac{2rs}{R(s-b)}$ and $K_c = 1 + \frac{2rs}{R(s-c)}$.
From these results, we obtain

$$R' = \frac{EF \cdot FD \cdot DE}{4[EDF]} = \frac{Rabc}{2(a+b)(b+c)(c+a)}\sqrt{K_aK_bK_c}\,.$$

Using $abc = 4Rrs$ and $(a+b)(b+c)(c+a) = 2s(s^2+r^2+2rR)$, it readily follows that

$$R' = \frac{rR^2}{s^2+r^2+2rR}\sqrt{K_aK_bK_c}\,,$$

and we will have proved $R' \leq \frac{R}{2}$ if we can show that

$$\sqrt{K_aK_bK_c} \leq \frac{R}{2} \cdot \frac{s^2+r^2+2rR}{rR^2} = 1 + \frac{s^2+r^2}{2rR}\,. \qquad (1)$$

With this aim, we compute $K_aK_bK_c$ as follows

$$K_aK_bK_c = 1 + \left(\frac{2rs}{R}\right)S + \left(\frac{4r^2s^2}{R^2}\right)T + \frac{8r^3s^3}{R^3(s-a)(s-b)(s-c)}\,.$$

where

$$S = \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} = \frac{r+4R}{rs}\,,$$

$$T = \frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)} = \frac{1}{r^2}\,,$$

$$r^2s = (s-a)(s-b)(s-c)\,.$$

This yields

$$K_aK_bK_c = 9 + \frac{2r}{R} + \frac{4s^2}{R^2} + \frac{8rs^2}{R^3}\,.$$

Now, from the well-known inequalities $2r \leq R$ and $2s \leq 3R\sqrt{3}$, we obtain $K_aK_bK_c \leq 9+1+27+27 = 64$, and so

$$\sqrt{K_aK_bK_c} \leq 8\,. \qquad (2)$$

But we also have $s \geq 3r\sqrt{3}$, hence

$$1 + \frac{s^2+r^2}{2rR} \geq 1 + \frac{27r^2+r^2}{2r\cdot 2r} = 8\,, \qquad (3)$$

and (1) directly follows from (2) and (3).

Also solved by OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

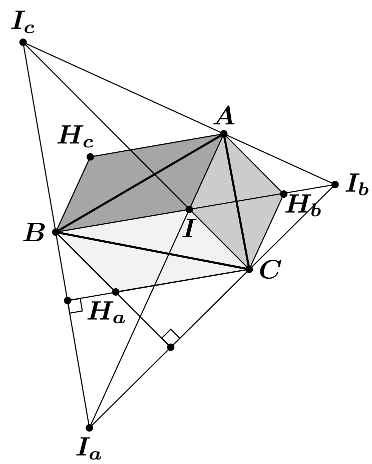
3544. [2010 : 240, 242] *Proposed by Mehmet Şahin, Ankara, Turkey.*

Triangle ABC has excentres I_a, I_b, I_c and H_a, H_b, H_c are the orthocentres of triangles I_aBC, I_bCA, I_cAB, respectively. Prove that

$$\text{Area}(H_aCH_bAH_cB) \;=\; 2\text{Area}(ABC)\,.$$

Similar approaches by the solvers listed below with an asterisk.

Let I be the incenter of $\triangle ABC$. Since H_aB and IC are each perpendicular to I_aC, $H_aB \parallel IC$. Similarly, $H_aC \parallel IB$, so H_aBIC is a parallelogram. Similarly, H_bCIA and H_cAIB are parallelograms. Since $\angle BI_aC = 90^\circ - \frac{1}{2}\angle BAC$, $\angle I_aBC = 90^\circ - \frac{1}{2}\angle ABC$, and $\angle I_aCB = 90^\circ - \frac{1}{2}\angle ACB$, $\triangle I_aBC$ is acute, so H_a is inside $\triangle I_aBC$. Similarly, H_b is inside $\triangle I_bCA$ and H_c is inside $\triangle I_cAB$. Since A, B and C lie on segments I_bI_c, I_cI_a and I_aI_b respectively and I is inside $\triangle ABC$, $H_aCH_bAH_cB$ is convex and I is in its interior. Therefore,



$$\begin{aligned}
&\text{Area}(H_aCH_bAH_cB)\\
&\quad = \text{Area}(H_aBIC) + \text{Area}(H_bCIA) + \text{Area}(H_cAIB)\\
&\quad = 2\text{Area}(BIC) + 2\text{Area}(CIA) + 2\text{Area}(AIB) = 2\text{Area}(ABC)\,.
\end{aligned}$$

*Solved by *ARKADY ALT, San Jose, CA, USA; *GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; *EMMANUEL LANCE CHRISTOPHER, Ateneo de Manila University, The Philippines; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; *RICHARD EDEN, student, Purdue University, West Lafayette, IN, USA; OLEH FAYNSHTEYN, Leipzig, Germany; *OLIVER GEUPEL, Brühl, NRW, Germany; *JOEL SCHLOSBERG, Bayside, NY, USA; *PETER Y. WOO, Biola University, La Mirada, CA, USA; *TITU ZVONARU, Cománeşti, Romania; and the proposer.*

3545. [2010 : 240, 242] *Proposed by Michel Bataille, Rouen, France.*

Given a line ℓ and points A and B with $A \notin \ell$ and $B \in \ell$, find the locus of points P in their plane such that $PA + QB = PQ$ for a unique point Q of ℓ.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Proposition 1 *If $\ell \perp AB$, then $\mathcal{L}$ consists of the line parallel to ℓ through A with the exception of the point A, and the mid-perpendicular line of the segment AB.*

Proof: We consider Cartesian (x, y)-coordinates such that, without loss of generality, $A = (2, 0)$, $B = (0, 0)$, and ℓ is the line $x = 0$.

For any points $P = (p, r)$, $Q = (0, y)$, the condition

$$PA + QB = PQ \tag{1}$$

is equivalent to

$$\sqrt{(p-2)^2 + r^2} + |y| = \sqrt{p^2 + (r-y)^2}\,.$$

After squaring both sides and erasing equal terms on both sides, this successively becomes equivalent to

$$(p-2)^2 + r^2 + y^2 + 2|y|\sqrt{(p-2)^2 + r^2} = p^2 + (r-y)^2\,,$$

and

$$[y < 0 \wedge (r-s)y = 2(p-1)] \vee [y \geq 0 \wedge (r+s)y = 2(p-1)]\,, \tag{2}$$

where $s = \sqrt{(p-2)^2 + r^2}$.

We consider the three cases $p = 1$, $p = 2$, and $p \notin \{1, 2\}$ in succession.

If $p = 1$, then, by the definition of s, $r - s$ and $r + s$ are both nonzero. Hence, $y = 0$ is the unique solution of the logical expression (2), that is, the mid-perpendicular of AB belongs to the locus $\mathcal{L}$.

Let $p = 2$. If $r < 0$, then the first alternative of the condition (2) yields the solution $y = 1/r$, while the second alternative is contradictory. Similarly, if $r > 0$, there is the unique solution $y = 1/r$. If $r = 0$ then the condition (2) is contradictory. Therefore, the parallel to ℓ through A with the exception of the point A is included in $\mathcal{L}$.

Finally, let $p \notin \{1, 2\}$. By $(r-s)(r+s) = r^2 - s^2 < 0$, so numbers $2(p-1)/(r-s)$ and $2(p-1)/(r+s)$ have opposite signs. Therefore there cannot be exactly one value of y for which (2) is true. Thus, the number of points Q with the property (1) is either 0 or 2, that is, $P \notin \mathcal{L}$. □

Proposition 2 *Assume that ℓ and AB are not perpendicular. Let g be the perpendicular to ℓ through B. Let π denote the parabola with directrix g and focus A. Let $\mathcal{R}$ be the region that is bounded by π and that contains the point A, where the boundary π belongs to $\mathcal{R}$. Let m be the mid-perpendicular of the segment AB. Then m is tangential to π and the locus $\mathcal{L}$ consists of the region R and the line m with the exception of their tangential point.*

Proof: We consider Cartesian (x, y)-coordinates such that, without loss of generality, $A = (a, 2)$, $a > 0$, $B = (0, 0)$, and ℓ is the line $x = 0$. For any points $P = (p, r)$, $Q = (0, y)$, the condition (1) is equivalent to

$$\sqrt{(p-a)^2 + (r-2)^2} + |y| = \sqrt{p^2 + (y-r)^2}.$$

After squaring both sides, this becomes equivalent to

$$[y < 0 \wedge (r-s)y = t] \vee [y \geq 0 \wedge (r+s)y = t], \tag{3}$$

where $s = \sqrt{(p-a)^2 + (r-2)^2}$ and $t = pa + 2r - 2 - a^2/2$. The parabola π is given by the equation $4(y-1) = (x-a)^2$. We have $t = \vec{P}\cdot\vec{A} - \dfrac{1}{2}\vec{A}^2 = \left(P - \dfrac{1}{2}\vec{A}\right)\cdot\vec{A}$. Hence, the condition $t = 0$ holds if and only if P is on m. It is easy to check that m touches π at the point $(0, a^2/4 + 1)$ on the line ℓ.

We consider in succession the three cases where P is above π, on π, and below π.

If P is above π, then $(r-s)(r+s) = r^2 - s^2 = [Pg]^2 - [PA]^2 > 0$. Hence the numbers $t/(r-s)$ and $t/(r+s)$ have the same sign. Thus, exactly one of the two alternatives (3) is satisfiable, and the solution is unique. Therefore, the interior of $\mathcal{R}$ is included in $\mathcal{L}$.

Let P be on π. Then $r - s = [Pg] - [PA] = 0$, $r + s > 0$, $r = (p-a)^2/4+1$, $t = p^2/2$. If $p \neq 0$ then $t > 0$ holds. Hence $t/(r+s) > 0$. Thus, the first alternative in (3) is contradictory, while the second alternative yields a unique solution. Otherwise, if $p = 0$, then each $y < 0$ satisfies the condition (3). Therefore, the parabola π with the exception of its tangential point with m belongs to $\mathcal{L}$.

Finally, let P be below π. We consider the situations for $t \neq 0$ and $t = 0$ in succession.

Firstly, let $t \neq 0$. By $(r-s)(r+s) = r^2 - s^2 = [Pg]^2 - [PA]^2 < 0$, the numbers $t/(r-s)$ and $t/(r+s)$ have opposite signs. Therefore, either both alternatives in (3) are satisfiable or both are contradictory. Thus, the number of points Q with the property (1) is either 0 or 2, that is, $P \notin \mathcal{L}$.

Finally, let $t = 0$. Then P is on m so $PA = PB$. The condition (1) is therefore equivalent to $PB + BQ = PQ$, which is satisfied if and only if the points B, P, and Q are collinear where B belongs to the line segment PQ. The unique point Q with this property is $Q = B$. We have shown that the line m with the exception of its tangential point with the parabola π is included in the locus $\mathcal{L}$. $\square$

Also incompletely solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and the proposer whose solutions were relatively short and more geometric.

3546. [2010 : 240, 242] *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer. Prove that

$$0 \;<\; \sum_{k=0}^{\binom{n}{2}} \frac{(-1)^k}{n+k}\binom{\binom{n}{2}}{k} \;\leq\; \frac{1}{n^n}\,.$$

Solution by the proposer.

The inequality holds for $n = 1$, so we can assume that $n \geq 2$. Let $N = \binom{n}{2}$

and let S denote the central sum,

$$S = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \int_0^1 x^{n+k-1} dx = \int_0^1 x^{n-1} \sum_{k=0}^{N} \binom{N}{k} (-x)^k dx$$
$$= \int_0^1 x^{n-1} (1-x)^N dx\,.$$

It follows that $S > 0$. For $k = 1, 2, \ldots, n-1$ and by the AM-GM inequality,

$$(1-x)x^k = k^k \left((1-x) \cdot \frac{x}{k} \cdot \frac{x}{k} \cdots \frac{x}{k} \right)$$
$$\leq k^k \left(\frac{(1-x) + k(x/k)}{k+1} \right)^{k+1} = \frac{k^k}{(k+1)^{k+1}}\,.$$

Since $N = 1 + 2 + \cdots + (n-1)$,

$$(1-x)^{n-1} x^N = [(1-x)x] \cdot [(1-x)x^2] \cdots [(1-x)x^{n-1}]$$
$$\leq \frac{1^1}{2^2} \cdot \frac{2^2}{3^3} \cdots \frac{(n-1)^{n-1}}{n^n} = \frac{1}{n^n}\,.$$

Thus,

$$S = \int_0^1 x^N (1-x)^{n-1} dx \leq \frac{1}{n^n}\,,$$

which completes the proof.

Also solved by Oliver Geupel, Brühl, NRW, Germany; and Albert Stadler, Herrliberg, Switzerland.

3547. [2010 : 241, 243] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Triangle ABC has perimeter equal to 1, inradius r, circumradius R, and side lengths a, b, c. Prove that

$$\frac{a}{\sqrt{1-a}} + \frac{b}{\sqrt{1-b}} + \frac{c}{\sqrt{1-c}} \geq \sqrt{\frac{2}{1 + 4r(r+4R)}}\,.$$

Solution by Richard Eden, student, Purdue University, West Lafayette, IN, USA.

The relations $R = \dfrac{abc}{4rs}$ and $r = \sqrt{(s-a)(s-b)(s-c)/s}$, where s is the semiperimeter, are well-known. Since $s = 1/2$, then $16Rr = 8abc$ and $4r^2 = 8(1/2-a)(1/2-b)(1/2-c) = (1-2a)(1-2b)(1-2c)$. Therefore,

$$1 + 4r(r+4R) = 1 + (1-2a)(1-2b)(1-2c) + 8abc$$
$$= 4(ab+bc+ca) - 2(a+b+c) + 2$$
$$= 4(ab+bc+ca)$$

The inequality to prove is then

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{a+c}}+\frac{c}{\sqrt{a+b}} \geq \frac{1}{\sqrt{2(ab+bc+ca)}}$$

The function $f(x) = 1/\sqrt{x}$ is convex for $x > 0$. Since $a + b + c = 1$, we can think of a, b, c as weights and apply Jensen's inequality,

$$\begin{aligned}\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{a+c}}+\frac{c}{\sqrt{a+b}} &= af(b+c)+bf(a+c)+cf(a+b)\\ &\geq f(a(b+c)+b(a+c)+c(b+c))\\ &= f(2(ab+bc+ca)\\ &= \frac{1}{\sqrt{2(ab+bc+ca)}}\,,\end{aligned}$$

which completes the proof.

Also solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Oliver Geupel, Brühl, NRW, Germany; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland; Peter Y. Woo, Biola University, La Mirada, CA, USA; and the proposer.

3548. [2010 : 241, 243] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let x, y, and z be nonnegative real numbers. Prove that

$$\sum_{\text{cyclic}} \sqrt{x^2-xy+y^2} \;\leq\; x+y+z+\sqrt{x^2+y^2+z^2-xy-yz-zx}\,.$$

I. *Solution by the proposer, modified slightly by the editor.*

Due to complete symmetry we may assume without loss of generality that $x = \min\{x, y, z\}$. Then

$$\sqrt{x^2-xy+y^2} = \sqrt{x(x-y)+y^2} \leq y \qquad \text{and}$$
$$\sqrt{x^2-xz+z^2} = \sqrt{x(x-z)+z^2} \leq z\,.$$

Hence it suffices to show that

$$\sqrt{y^2-yz+z^2}-x \leq \sqrt{x^2+y^2+z^2-xy-yz-zx}\,. \qquad (1)$$

Since $\sqrt{y^2-yz+z^2}-x = \sqrt{(y-z)^2+yz}-x \geq \sqrt{x^2}-x = 0$, we may square both sides of (1) to obtain

$$x^2+y^2+z^2-yz-2x\sqrt{y^2-yz+z^2} \leq x^2+y^2+z^2-xy-yz-zx$$

or

$$\boldsymbol{x(y+z) \le 2x\sqrt{y^2 - yz + z^2}}\,. \tag{2}$$

Squaring both sides, (2) is equivalent, in succession, to

$$\begin{aligned} \boldsymbol{x^2(y^2+2yz+z^2)} &\le \boldsymbol{4x^2(y^2-yz+z^2)} \\ \boldsymbol{3x^2y^2 - 6x^2yz + 3x^2z^2} &\ge \boldsymbol{0} \\ \boldsymbol{3x^2(y-z)^2} &\ge \boldsymbol{0}\,. \end{aligned}$$

The proof is complete.

II. *Solution by Albert Stadler, Herrliberg, Switzerland, expanded by the editor.*

We show that the proposed inequality follows from the result below, known as Hlawka Inequality [see D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993; p.521]:

If $\boldsymbol{V}$ *is an inner product space, then for all* $\boldsymbol{a, b, c} \in \boldsymbol{V}$,

$$||\boldsymbol{a}+\boldsymbol{b}|| + ||\boldsymbol{b}+\boldsymbol{c}|| + ||\boldsymbol{c}+\boldsymbol{a}|| \le ||\boldsymbol{a}|| + ||\boldsymbol{b}|| + ||\boldsymbol{c}|| + ||\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}||\,,$$

where $||\cdot||$ *denotes the norm induced by the inner product.*

If we let $\boldsymbol{V} = \mathbb{R}^2$ and set $\boldsymbol{a} = \boldsymbol{x(1,0)}$, $\boldsymbol{b} = \boldsymbol{y}\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $\boldsymbol{c} = \boldsymbol{z}\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, then $||\boldsymbol{a}|| = \sqrt{x^2} = x$, $||\boldsymbol{b}|| = \sqrt{y^2} = y$, $||\boldsymbol{c}|| = \sqrt{z^2} = z$. Furthermore,

$$\begin{aligned} ||\boldsymbol{a}+\boldsymbol{b}|| &= \left|\left|\left(x - \frac{1}{2}y, \frac{\sqrt{3}}{2}y\right)\right|\right| = \sqrt{\left(x - \frac{1}{2}y\right)^2 + \frac{3}{4}y^2} = \sqrt{x^2 - xy + y^2}\,, \\ ||\boldsymbol{b}+\boldsymbol{c}|| &= \left|\left|\left(-\frac{1}{2}(y+z), \frac{\sqrt{3}}{2}(y-z)\right)\right|\right| \\ &= \sqrt{\frac{1}{4}(y+z)^2 + \frac{3}{4}(y-z)^2} = \sqrt{y^2 - yz + z^2}\,, \\ ||\boldsymbol{c}+\boldsymbol{a}|| &= \left|\left|\left(x - \frac{1}{2}z, -\frac{\sqrt{3}}{2}z\right)\right|\right| = \sqrt{\left(x - \frac{1}{2}z\right)^2 + \frac{3}{4}z^2} = \sqrt{x^2 - zx + z^2}\,, \end{aligned}$$

and

$$\begin{aligned} ||\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}|| &= \left|\left|\left(x - \frac{1}{2}(y+z), \frac{\sqrt{3}}{2}(y-z)\right)\right|\right| \\ &= \sqrt{\left(x - \frac{1}{2}(y+z)\right)^2 + \frac{3}{4}(y-z)^2} \\ &= \sqrt{x^2 + y^2 + z^2 - xy - yz - zx}\,. \end{aligned}$$

The result follows.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; RICHARD EDEN, student, Purdue University, West Lafayette, IN, USA; OLEH FAYNSHTEYN, Leipzig, Germany; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and PETER Y. WOO, Biola University, La Mirada, CA, USA. As usual, Stan Wagon verified the validity of the inequality by using "FindInstance" (in 13 seconds).

3549. [2010 : 241, 243] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that $\left(1+a^2b\right)\left(1+b^2c\right)\left(1+c^2a\right)\leq 5+3abc$.

Solution by Arkady Alt, San Jose, CA, USA.

Note first that the given inequality is equivalent to

$$a^2b+b^2c+c^2a+abc(ab^2+bc^2+ca^2)+a^3b^3c^3\leq 4+3abc\,. \qquad (1)$$

We first establish the following lemma:

Lemma *If a, b, and c are nonnegative real numbers, then*

$$a^2b+b^2c+c^2a+abc\leq\frac{4}{27}(a+b+c)^3\,. \qquad (2)$$

Proof: Due to the cyclic symmetry of a, b, and c in (2) we may assume, without loss of generality, that $c=\min\{a,b,c\}$. We consider two cases separately:

Case (i). Suppose $b\leq a$. By the AM-GM Inequality, we have

$$\frac{4}{27}(a+b+c)^3=\frac{1}{2}\left(\frac{2b+2(a+c)}{3}\right)^3\geq\frac{1}{2}\left(2b(a+c)^2\right)=b(a+c)^2\,. \quad (3)$$

Since

$$\begin{aligned}b(a+c)^2-(a^2b+b^2c+c^2a+abc)&=abc+bc^2-b^2c-c^2a\\&=c(ab+bc-b^2-ca)\\&=c(a-b)(b-c)\geq 0\end{aligned}$$

we have

$$a^2b+b^2c+c^2a+abc\leq b(a+c)^2 \qquad (4)$$

and (2) follows from (3) and (4).

Case (ii). Suppose $b>a$. We have

$$\begin{aligned}2(a^2b+b^2c+c^2a+abc)&=\sum_{cyclic}(a^2b+ab^2)+2abc+\sum_{cyclic}(a^2b-ab^2)\\&=(a+b)(b+c)(c+a)-(a-b)(b-c)(c-a)\,. \qquad (5)\end{aligned}$$

By the AM-GM Inequality we have

$$(a+b)(b+c)(c+a) \le \left(\frac{(a+b)+(b+c)+(c+a)}{3}\right)^3 = \frac{8}{27}(a+b+c)^3$$

so

$$\begin{aligned}\frac{4}{27}(a+b+c)^3 &\ge \frac{1}{2}(a+b)(b+c)(c+a)\\ &= a^2b+b^2c+c^2a+abc+\frac{1}{2}(a-b)(b-c)(c-a)\\ &\ge a^2b+b^2c+c^2a+abc\end{aligned}$$

since $(a-b)(b-c)(c-a) \ge 0$.

This completes the proof of the lemma.

Since $a+b+c=3$, (2) becomes $a^2b+b^2c+c^2a \le 4-abc$ and since $4-abc$ is invariant under the interchanging of a and b, we have $\max\{a^2b+b^2c+c^2a, ab^2+bc^2+ca^2\} \le 4-abc$. Therefore,

$$\sum_{cyclic} a^2b + abc\sum_{cyclic} ab^2 + a^3b^3c^3 \le (1+abc)(4-abc)+a^3b^3c^3\,.$$

Finally, since $abc \le \left(\frac{a+b+c}{3}\right)^3 = 1$ we have

$$\begin{aligned}4+3abc-\left(\sum_{cyclic} a^2b + abc\sum_{cyclic} ab^2 + a^3b^3c^3\right)\\ \ge 4+3abc-(1+abc)(4-abc)-a^3b^3c^3\\ = a^2b^2c^2-a^3b^3c^3 = a^2b^2c^2(1-abc) \ge 0\end{aligned}$$

which establishes (1) and completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer. Stan Wagon gave his usual verification using "FindInstances".

3550. [2010 : 241, 243] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Find the sum

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\left(\ln 2-\sum_{i=1}^{n+m}\frac{1}{n+m+i}\right).$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let

$$a_m=\ln 2-\sum_{i=1}^{m}\frac{1}{m+i},$$

and

$$b_n=\sum_{m=1}^{\infty}(-1)^{m+n}a_{m+n}.$$

We have

$$\begin{aligned} a_m &= \ln 2-\sum_{i=1}^{2m}\frac{(-1)^{i-1}}{i} \\ &= \int_0^1\frac{1}{1+x}dx-\sum_{i=1}^{2m}\int_0^1(-1)^{i-1}x^{i-1}dx \\ &= \int_0^1\frac{1}{1+x}dx-\int_0^1\frac{1-x^{2m}}{1+x}dx=\int_0^1\frac{x^{2m}}{1+x}dx\,; \end{aligned}$$

hence

$$\begin{aligned} \sum_{i=1}^{m}(-1)^{n+1}a_{n+1} &= \int_0^1\frac{1}{1+x}\sum_{i=1}^{m}(-1)^{n+i}x^{2(n+i)}dx \\ &= \int_0^1\frac{(-x^2)^{n+1}}{1+x}\cdot\frac{1-(-x^2)^m}{1+x^2}dx \end{aligned}$$

Let $f_m(x)=\dfrac{(-x^2)^{n+1}}{1+x}\cdot\dfrac{1-(-x^2)^m}{1+x^2}$ and $f(x)=\dfrac{(-x^2)^{n+1}}{(1+x)(1+x^2)}$.

Then for all $0\le x\le$ we have $0\le|f_m(x)|\le 2f(x)$. Also, on $[0,1)$ $f_m(x)\to f(x)$ pointwise. Then, by the Lebesgue Dominated Convergence Theorem we have

$$b_n=\lim_{m\to\infty}\int_0^1\frac{(-x^2)^{n+1}}{1+x}\cdot\frac{1-(-x^2)^m}{1+x^2}dx=\int_0^1\frac{(-x^2)^{n+1}}{(1+x)(1+x^2)}dx.$$

It follows that

$$\begin{aligned} \sum_{i=1}^{N}b_n &= \int_0^1\frac{1}{(1+x)(1+x^2)}\sum_{n=1}^{N}(-x^2)^{n+1}dx \\ &= \int_0^1\frac{x^4}{(1+x)(1+x^2)}\cdot\frac{1-(-x^2)^N}{1+x^2}dx \end{aligned}$$

Applying again the Lebesgue Dominated Convergence Theorem we get:

$$\sum_{i=1}^{\infty} b_n = \int_0^1 \frac{x^4}{(1+x)(1+x^2)^2}$$
$$= \frac{1}{8}\left[\frac{2(1+x)}{1+x^2} + 3\ln(1+x^2) + 2\ln(1+x) - 4\arctan(x)\right]_0^1$$
$$= \frac{5\ln 2 - \pi}{8}$$

Thus, the sum is $\frac{5\ln 2-\pi}{8}$.

Also solved by PAUL BRACKEN, University of Texas, Edinburg, TX, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; and the proposer .