M475. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON*

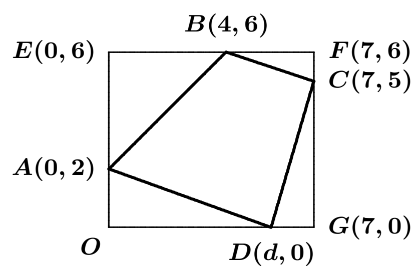
Notons $\lfloor x \rfloor$ le plus grand entier n'excédant pas x. Par exemple, $\lfloor 3{,}1 \rfloor = 3$ et $\lfloor -1{,}4 \rfloor = -2$. Notons $\{x\}$ la partie fractionnaire du nombre réel x, c.-à-d, $\{x\} = x - \lfloor x \rfloor$. Par exemple, $\{3{,}1\} = 0{,}1$ et $\{-1{,}4\} = 0{,}6$. Montrer qu'il existe une infinité de nombres irrationnels x tels que $x \cdot \{x\} = \lfloor x \rfloor$.

Mayhem Solutions

M432. *Proposed by the Mayhem Staff.*

Determine the value of d with $d > 0$ so that the area of the quadrilateral with vertices $A(0, 2)$, $B(4, 6)$, $C(7, 5)$, and $D(d, 0)$ is 24.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.



Let $E = (0, 6), F = (7, 6), G = (7, 0)$ and Ω denote the area function. Then $\Omega(AOD) = \frac{1}{2}(d \times 2) = d$; $\Omega(BEA) = \frac{1}{2}(4 \times 4) = 8$; $\Omega(BFC) = \frac{1}{2}(3 \times 1) = \frac{3}{2}$; and $\Omega(CDG) = \frac{1}{2}(7-d)\times 5 = \frac{5}{2}(7-d)$. Since $\Omega(OEFG) = 7 \times 6 = 42$, we have

$$24 = \Omega(ABCD) = 42 - \left[d + 8 + \frac{3}{2} + \frac{5}{2}(7-d)\right] = 15 + \frac{3}{2}d.$$

Solving we find $d = 6$.

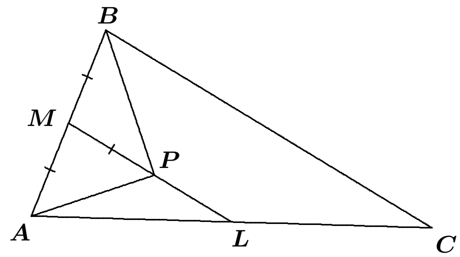
Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; GEOFFREY A. KANDALL, Hamden, CT, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; JOSHUA LONG, Southeast Missouri State University, Cape Girardeau, MO, USA; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania(2 solutions); and JOHN WYNN, student, Auburn University, Montgomery, AL, USA;

Two incorrect solutions were received.

M433. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

In triangle ABC, $AB < BC$, L is the midpoint of AC, and M is the midpoint of AB. Also, P is the point on LM such that $MP = MA$. Prove that $\angle PBA = \angle PBC$.

Solution by Souparna Purohit, student, George Washington Middle School, Ridgewood, NJ, USA.



It is well known that since M and L are the midpoints of AB and AC then $BC \parallel ML$ so $\angle PBC = \angle BPM$. Also, since $PM = AM = BM$, ΔBMP is isosceles. Therefore $\angle ABP = \angle BPM$ which, when combined with $\angle PBC = \angle BPM$, we conclude that $\angle ABP = \angle PBC$, as desired.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain(two solutions); GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

One incorrect solution was received. Several readers pointed out that ΔAPB is right angled with the right angle at P.

M434. *Proposed by Heisu Nicolae, Pîrjol Secondary School, Bacău, Romania.*

Determine all eight-digit positive integers $abcdefgh$ which satisfy the relations $a^3 - b^2 = 2$, $c^3 - d^2 = 4$, $2^e - f^2 = 7$, and $g^3 - h^2 = -1$.

Solution by Arkady Alt, San Jose, CA, USA.

Since $2 \le a^3 \le 9^2 + 2 = 83 \Leftrightarrow 2 \le a \le 4$ and $a^3 - 2$ for such a can only be square for $a = 3$, then $a = 3, b = 5$.

Since $4 \le c^3 \le 9^2 + 4 = 85 \Leftrightarrow 2 \le c \le 4$ and $c^3 - 4$ for such c can only be square for $c = 2$ then $c = 2, d = 2$.

Since $7 \le 2^e \le 9^2 + 7 = 88 \Leftrightarrow 3 \le e \le 6$ and $2^e - 7$ for such e can only be square for $e = 3$, $e = 4$ and $e = 5$ then $(e, f) = (3, 1), (4, 3), (5, 5)$.

Since $0 \le g^3 \le 9^2 - 1 = 80 \Leftrightarrow 0 \le g \le 4$ and $g^3 + 1$ for such g can only be square for $g = 0$ and $g = 2$ then $(g, h) = (0, 1), (2, 3)$.

Thus $abcdefgh = 35\ 223\ 101, 35\ 224\ 301, 35\ 225\ 501, 35\ 223\ 123,$ $35\ 224\ 323, 35\ 225\ 523$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer. Seven incomplete solutions were submitted. Most of the incomplete solutions missed the case where $g = 0$.

M435. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\sum_{k=1}^{n}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}} = \frac{n(n+2)}{n+1}.$$

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

For any $k > 0$ we have

$$\begin{aligned}\left(1+\frac{1}{k}-\frac{1}{k+1}\right)^2 &= 1+\frac{1}{k^2}+\frac{1}{(k+1)^2}+\left(\frac{2}{k}-\frac{2}{k+1}\right)-\frac{2}{k(k+1)}\\ &= 1+\frac{1}{k^2}+\frac{1}{(k+1)^2}+\frac{2}{k(k+1)}-\frac{2}{k(k+1)}\\ &= 1+\frac{1}{k^2}+\frac{1}{(k+1)^2}.\end{aligned}$$

Hence $\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}} = 1+\frac{1}{k}-\frac{1}{k+1}$. Therefore if we let $S=\sum_{k=1}^{n}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}$ then

$$\begin{aligned}S &= \sum_{k=1}^{n}\left(1+\frac{1}{k}-\frac{1}{k+1}\right)\\ &= \left(1+\frac{1}{1}-\frac{1}{2}\right)+\left(1+\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(1+\frac{1}{n}-\frac{1}{n+1}\right)\\ &= n+1-\frac{1}{n+1}=\frac{n(n+2)}{n+1},\end{aligned}$$

and we are done!

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; G.C. GREUBEL, Newport News, VA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

M436. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Determine the smallest possible value of $x + y$, if x and y are positive integers with $\frac{2008}{2009} < \frac{x}{y} < \frac{2009}{2010}$.

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Let $y = x + d$ then $1 - \frac{1}{2009} < \frac{y-d}{y} < 1 - \frac{1}{2010}$ so $\frac{1}{2009} > \frac{d}{y} > \frac{1}{2010}$. If $d = 1$ there is no solution. If $d = 2$, $y = 4019$ is a solution so $x = 4017$ and $x + y = 8036$. If $d > 2$ then $x, y > 6000$ thus $x + y > 12000$. Therefore the minimum value of $x + y$ is 8036.

Also solved by ARKADY ALT, San Jose, CA, USA; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer. Five incorrect solutions were submitted.

Alt, Manes and Wang proved in general that if $\frac{n}{n+1} < \frac{x}{y} < \frac{n+1}{n+2}$ *then the solution with the smallest sum corresponds to* $x = 2n+1$, $y = 2n+3$ *and thus* $x+y = 4n+4$. *In general for any two fractions of non-negative integers, in lowest terms,* $\frac{a}{b} < \frac{c}{d}$ *the value* $\frac{a+c}{b+d}$ *is called the mediant and it satisfies* $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. *If we also have* $bc - ad = 1$ *then the mediant is the fraction with the lowest denominator in the interval* $\left(\frac{a}{b}, \frac{c}{d}\right)$.

M437. *Proposed by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.*

Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For example, $\lfloor 3.1 \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$. Let $\{x\}$ denote the fractional part of the real number x, that is, $\{x\} = x - \lfloor x \rfloor$. For example, $\{3.1\} = 0.1$ and $\{-1.4\} = 0.6$. Determine all rational numbers x such that $x \cdot \{x\} = \lfloor x \rfloor$.

Solution by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

The only rational number x such that $x \cdot \{x\} = \lfloor x \rfloor$ is $x = 0$.

If n is an integer, then $n \cdot \{n\} = 0 = \lfloor n \rfloor = n$ has the only solution $n = 0$. Therefore, 0 is the only integer solution to the equation.

Assume x is a rational number different from an integer such that $x \cdot \{x\} = \lfloor x \rfloor = x - \{x\}$, then $\{x\} = \frac{x}{x+1}$. Therefore, $x\left(\frac{x}{x+1}\right) = \lfloor x \rfloor$ implies $x^2 = \lfloor x \rfloor (x+1)$. Assume $x = \frac{m}{n}$ where m and n are relatively prime integers and $n > 1$. Then

$$\frac{m^2}{n^2} = \lfloor x \rfloor \left(\frac{m}{n} + 1\right) = \lfloor x \rfloor \left(\frac{m+n}{n}\right).$$

As a result, $m^2 = \lfloor x \rfloor (m+n) \cdot n$ so that n is a divisor of m^2, a contradiction since m and n are relatively prime and $n > 1$.

Also solved by ARKADY ALT, San Jose, CA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer. Three incorrect solutions were submitted.