Problem of the Month

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Problems involving averages and their properties appear frequently on contests (*Look at the solution to question 1 of Skoliad on page 5 – Ed.*). This month and next, we will look at a few of these problems, at least one of which uses averages in a very subtle way.

Problem 1 (2008 Small c Contest) The average of three numbers is 13. Two numbers are added to this list so that the average of all five numbers is 17. What is the average of the two new numbers?

(A) 21 (B) 25 (C) 23 (D) 30 (E) 15

One of the things about average problems that I like is that there are really only about $1\frac{1}{2}$ things that you need to know about averages in order to be able to do almost all of such problems. (That's not to say that there isn't a plethora of tricks of the trade that can be useful...)

The first of these $1\frac{1}{2}$ important things is how to calculate an average: add up the given numbers, count the given numbers, and divide the sum by the count to get the average. The extra $\frac{1}{2}$ thing to remember is that the sum of the numbers equals the count times the average. Expressing these facts algebraically, we see that if there are n numbers whose sum is S, then the average, a, satisfies the equation $a = \dfrac{S}{n}$. Rearranging this gives $S = na$. (I concede that occasionally we might use the fact that $n = \dfrac{S}{a}$ as well.)

Let's solve Problem 1 using these properties and then look at our answer to see what we can observe.

Solution to Problem 1. Since the average of the original three numbers is 13, then their sum is $3 \times 13 = 39$. Since the average of all five numbers is 17, then the sum of the five numbers is $5 \times 17 = 85$.

The sum of the additional two numbers equals the sum of all five numbers minus the sum of the original three numbers, or $85 - 39 = 46$. Therefore, the average of these two numbers is $\dfrac{46}{2} = 23$. ∎

This problem is particularly nice, in my opinion, because it doesn't require us to use any algebra. Let's look at the data that we have:

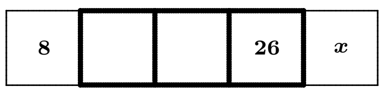
- the average of the first 3 numbers is 13
- the average of all 5 numbers is 17
- the average of the last 2 numbers is 23

Do you notice anything about the position of the overall average relative to the averages of the first and last numbers? You might have noticed that the overall average splits these averages in the ratio $\mathbf{4 : 6}$ which equals $\mathbf{2 : 3}$, which happens to be the ratio of the count of numbers in each partial average (arranged in reverse from what you might quickly guess).

If this rule works in general, then if we had $\mathbf{5}$ numbers with average $\mathbf{22}$ and $\mathbf{3}$ numbers with average $\mathbf{46}$, the average of all $\mathbf{8}$ numbers should split $\mathbf{22}$ and $\mathbf{46}$ in the ratio $\mathbf{3 : 5}$. In other words, the average is $\dfrac{3}{3+5} = \dfrac{3}{8}$ of the way from $\mathbf{22}$ to $\mathbf{46}$, and so equals $22 + \dfrac{3}{8} \times (46 - 22) = 31$. Try solving this problem using the method that we used above to confirm the answer.

Putting this in a more general way, if m numbers have an average of a and n numbers have an average of b with $a < b$, then the average of the $m + n$ numbers splits a and b in the ratio $n : m$ (not $m : n$). Can you prove this? We'll look at another problem next month where this approach is really useful.

Problem 2 (2010 Pascal Contest) In the diagram, each of the five boxes is to contain a number. Each number in a bold outlined box must be the average of the number in the box to the left of it and the number in the box to the right of it. What is the value of x?



(A) 28 (B) 30 (C) 31 (D) 32 (E) 34

Special cases often produce interesting facts. For example, if two numbers x and y have an average of a, then $\dfrac{x+y}{2} = a$ or $x + y = 2a$. Try to use this to solve the following problem algebraically.

Solution to Problem 2. We label the numbers in the empty boxes as y and z, so the numbers in the boxes are thus $8, y, z, 26, x$.

Since the average of z and x is 26, then $x + z = 2(26) = 52$ or $z = 52 - x$. We rewrite the list as $8, y, 52 - x, 26, x$.

Since the average of $\mathbf{26}$ and y is $52 - x$, then $26 + y = 2(52 - x)$ or $y = 104 - 26 - 2x = 78 - 2x$. We rewrite the list as $8, 78 - 2x, 52 - x, 26, x$.

Since the average of $\mathbf{8}$ and $52 - x$ is $78 - 2x$, then $8 + (52 - x) = 2(78 - 2x)$ or $60 - x = 156 - 4x$ and so $3x = 96$ or $x = 32$. ■

Especially while writing a contest, it's very tempting to take the answer that we get and not think about it at all. But let's actually take this a moment to use this answer and go back to the list in terms of x (written as $8, 78 - 2x, 52 - x, 26, x$) and substitute to get the list $8, 14, 20, 26, 32$.

Do you recognize what kind of sequence this list forms? This is an arithmetic sequence. (Look up this term if you've never seen it before.) Do you think that this is a coincidence? (Hint: The answer to this question is almost always no.)

Let's think about this by going back to the list $\boldsymbol{8, y, z, 26, x}$. Let's avoid using algebra, but we'll keep these labels to make things a little clearer. We are told that $\boldsymbol{y}$ is the average of $\mathbf{8}$ and $\boldsymbol{z}$. The important fact to recognize here is that $\boldsymbol{y}$ is halfway between $\mathbf{8}$ and $\boldsymbol{z}$. In other words, the difference $\boldsymbol{y - 8}$ equals $\boldsymbol{z - y}$. Similarly, $\boldsymbol{z}$ is the average of $\mathbf{26}$ and $\boldsymbol{y}$, so $\boldsymbol{26 - z}$ equals $\boldsymbol{z - y}$. But there is a common difference in these two sentences! (And it's no coincidence that I used the phrase common difference...)

Since there is this common difference, then all three differences must be equal. Since $\mathbf{26 - 8 = 18}$, then each of these differences equals $\mathbf{18 \div 3 = 6}$, and so the numbers in the sequence are $\boldsymbol{8, 14, 20, 26, x}$. Can you extend this argument another step to explain why $\boldsymbol{x = 32}$?

So what is the connection between averages and arithmetic sequences? An arithmetic sequence is a sequence with the property that each term after the first is the average of the term before and the term after. This is pretty neat, if you've never seen it before. One last thing to think about – the average is sometimes called the *arithmetic* mean. Coincidence?

Adams, Douglas (1952 - 2001) The first nonabsolute number is the number of people for whom the table is reserved. This will vary during the course of the first three telephone calls to the restaurant, and then bear no apparent relation to the number of people who actually turn up, or to the number of people who subsequently join them after the show/match/party/gig, or to the number of people who leave when they see who else has turned up. The second nonabsolute number is the given time of arrival, which is now known to be one of the most bizarre of mathematical concepts, a recipriversexcluson, a number whose existence can only be defined as being anything other than itself. In other words, the given time of arrival is the one moment of time at which it is impossible that any member of the party will arrive. Recipriversexclusons now play a vital part in many branches of math, including statistics and accountancy and also form the basic equations used to engineer the Somebody Else's Problem field. The third and most mysterious piece of nonabsoluteness of all lies in the relationship between the number of items on the bill, the cost of each item, the number of people at the table and what they are each prepared to pay for. (The number of people who have actually brought any money is only a subphenomenon of this field.) "Life, the Universe and Everything." New York: Harmony Books, 1982.