SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3501. [2010 : 44, 46] *Proposed by Hassan A. ShahAli, Tehran, Iran.*

Let $\mathbb{N}$ be the set of positive integers, E the set of all even positive integers, and O the set of all odd positive integers. A set $S \subseteq \mathbb{N}$ is *closed* if $x + y \in S$ for all distinct x, $y \in S$, and *unclosed* if $x + y \notin S$ for all distinct x, $y \in S$. Prove that if $\mathbb{N}$ is partitioned into A and B, where A is closed and nonempty, and B is unclosed and infinite, then $A = E$ and $B = O$.

Solution by Harry Sedinger, St. Bonaventure University, St. Bonaventure, New York, USA.

We prove first that $1 \in B$.

Assume by contradiction that $1 \in A$. Chose $m, n \in B$ then $m + n \in A$. But since $1 \in A$, it follows that all integers greater than $m + n$ are in A, which contradicts B infinite.

We prove now that $2 \in A$. Again assume by contradiction that $2 \in B$. Chose $n > 2, n \in B$. Then $3 \in A, n + 1 \in A, n + 2 \in A$ and hence $n + 4, n + 5 \in A$. Then $2n + 5, 2n + 6, 2n + 7 \in A$, and since $3 \in A$, it follows that A contains all the integers greater than $2n + 5$, which contradicts B infinite.

Now we show that $3 \in B$. Assume by contradiction $3 \in A$. Then $5 \in A, 7 \in A, 8 \in A$, and then, since $2 \in A$, it follows that A contains all the integers greater than 7. Again this contradicts B infinite.

Next we prove that neither A nor B contains two consecutive integers.

If $n, n + 1$ are in A, then $n \geq 4$, and since $2 \in A$, it follows that A contains all the integers greater than n, a contradiction.

Assume now that B contains two consecutive integers $n, n + 1$. Since B is infinite, there exists $m \in B; m > n + 1$. But then A contains the consecutive integers $m + n, m + n + 1$, a contradiction.

Hence $1 \in B, 2 \in A$ and neither A nor B contains two consecutive integers. Then it follows by induction that $2k - 1 \in B, 2k \in A$ for all $k \geq 1$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW, Germany; JOHAN GUNARDI, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia; MICHAEL JOSEPHY, Universidad de Costa Rica, San Pedro, Costa Rica; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect solution.

3502. [2010 : 44, 46] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find all real solutions of the following system of equations

$$\begin{aligned} x_1^2 + \sqrt{x_2^2 + 21} &= \sqrt{x_2^2 + 77}\,, \\ x_2^2 + \sqrt{x_3^2 + 21} &= \sqrt{x_3^2 + 77}\,, \\ \cdots \quad & \cdots \quad \cdots \\ x_n^2 + \sqrt{x_1^2 + 21} &= \sqrt{x_1^2 + 77}\,. \end{aligned}$$

Similar solutions by Michel Bataille, Rouen, France; and Titu Zvonaru, Cománeşti, Romania. We give Bataille's write up.

Let $u_i = x_i^2$ and $f(x) = \sqrt{x+77} - \sqrt{x+21}$. Then, the equations are:

$$u_1 = f(u_2)\,;\ u_2 = f(u_3)\,;\ \ldots;\ u_{n-1} = f(u_n)\,;\ u_n = f(u_1)\,.$$

We observe that $f([0,\infty)) \subset [0,\infty)$ and $f(4) = 4$. In addition $f'(x) = \frac{1}{2}\left(\frac{1}{\sqrt{x+77}} - \frac{1}{\sqrt{x+21}}\right)$, hence

$$|f'(x)| < \frac{1}{2\sqrt{x+21}} \leq \frac{1}{2\sqrt{21}}\,.$$

Let $k = \frac{1}{2\sqrt{21}}$ and f^m denote the composition $f \circ f \circ f \circ \ldots \circ f$. Then $k < 1$ and it follows by induction that

$$|(f^m)'(x)| < k^m\,,$$

for all $x \in [0,\infty)$ and m positive integer.

Let $1 \leq i \leq n$. Then $u_i \in [0,\infty)$ and $f^n(u_i) = u_i$. We show that $u_i = 4$.

Assume by contradiction that $u_i \neq 4$. Then by the Mean Value Theorem, there exists a c between u_i and 4 so that

$$|u_i - 4| = |(f^n)(u_i) - (f^n)(4)| = |(f^n)'(c)|\,|u_i - 4| \leq k^n\,|u_i - 4|\,.$$

But this contradicts $k < 1$.

This shows that $u_i = 4$ for all i. Conversely $u_1 = u_2 = .. = u_n = 4$ is obviously a solution for our system.

Thus, all the real solutions of the system are $(\pm 2, \pm 2, \ldots, \pm 2)$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; and the proposer. One incomplete solution was submitted.

3503. [2010 : 44, 47] *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Given a triangle and the midpoints of its sides, with the use of a straight edge and only three uses of a pair of compasses, bisect all three angles of the triangle.

Solution by Mohammed Aassila, Strasbourg, France.

Soit ABC le triangle donné ; I, J et K les milieux respectifs de BC, CA et AB. On va utiliser une **seule fois** le compas.

On trace le cercle de centre K et de rayon $KA = KB$. Soit E le point d'intersection de la droite KJ avec ce cercle. Comme $KE = KB$, alors E est sur la bissectrice de l'angle ABC car KBE triangle isocèle et KJ parallèle à BC.

De même, soit F le point d'intersection du cercle avec la droite KI, alors on a $KF = KA$, et donc F est sur la bissectrice de l'angle BAC car KAF triangle isocèle et KI est parallèle à AC.

Ces deux bissectrices se coupent en H, centre du cercle inscrit et donc sur la troisième bissectrice CH.

Les trois bissectrices sont donc AH, BH et CH.

Also solved by the following readers, with the number of uses of the compass indicated in parentheses: MICHEL BATAILLE, Rouen, France (2); OLIVER GEUPEL, Brühl, NRW, Germany (2); JOHAN GUNARDI, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia (3); GEOFFREY A. KANDALL, Hamden, CT, USA (2); VÁCLAV KONEČNÝ, Big Rapids, MI, USA (2); MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA (2); EDMUND SWYLAN, Riga, Latvia (1); PETER Y. WOO, Biola University, La Mirada, CA, USA (2); TITU ZVONARU, Comăneşti, Romania (2); and the proposer (3).

Swylan was the only other solver who used the compass just once. Bataille, the Missouri State University Problem Solving Group, and Woo all noted that by the Poncelet–Steiner theorem just one use of the compass suffices to carry out the construction.

3504. *Proposed by Mariia Rozhkova, Kiev, Ukraine.*

Given triangle ABC, set $Q = a\cos^2 A + b\cos^2 B + c\cos^2 C$, and let ABC have area S and circumradius R. Prove that

(a) $Q \geq \dfrac{S}{R}$, with equality if and only if ABC is equilateral.

(b) $Q \leq \dfrac{S\sqrt{2}}{R}$ if ABC is not obtuse, with equality if and only if ABC is an isosceles right triangle.

Solution to part (a) by Michel Bataille, Rouen, France; solution to part (b) by the proposer, modified by the editor.

(a) Let $a = BC$, $b = CA$, $c = AB$ and let I and H be the incentre and orthocentre of $\triangle ABC$. Since $2sI = aA + bB + Cc$, we have

$2s\overrightarrow{HI} = a\overrightarrow{HA} + b\overrightarrow{HB} + c\overrightarrow{HC}$, and so

$$\begin{aligned} 4s^2HI^2 &= a^2HA^2 + b^2HB^2 + c^2HC^2 \\ &\quad + 2ab\overrightarrow{HA}\cdot\overrightarrow{HB} + 2bc\overrightarrow{HB}\cdot\overrightarrow{HC} + 2ca\overrightarrow{HC}\cdot\overrightarrow{HA} \\ &= a^2HA^2 + b^2HB^2 + c^2HC^2 + ab(HB^2 + HA^2 - c^2) \\ &\quad + bc(HC^2 + HB^2 - a^2) + ca(HA^2 + HC^2 - b^2) \\ &= 2s(aHA^2 + bHB^2 + cHC^2 - abc)\,. \end{aligned}$$

Now, if A' is the midpoint of BC, then $HA^2 = (2OA')^2 = 4R^2\cos^2 A$ and similar results hold for HB^2 and HC^2. It follows that $2sHI^2 = 4R^2\cdot Q - abc$ and

$$Q = \frac{abc}{4R^2} + \frac{sHI^2}{2R^2}\,.$$

Since $\dfrac{abc}{4R^2} = \dfrac{S}{R}$, we see that $Q \geq \dfrac{S}{R}$, with equality if and only if $H = I$, that is, if and only if ABC is equilateral.

(b) Substituting the well-known relations $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$, $S = 2R^2\sin A\sin B\sin C$ into the inequality and canceling $2R$ from each side yields the equivalent inequality

$$\sin A\cos^2 A + \sin B\cos^2 B + \sin C\cos^2 C \leq \sqrt{2}\sin A\sin B\sin C\,.$$

Assume that $A \geq B \geq C$ and set $\phi = \frac{1}{2}(B + C)$, $\psi = \frac{1}{2}(B - C)$. Then from the hypotheses we have $0 \leq \psi \leq \dfrac{\pi}{4} \leq \phi \leq \dfrac{\pi}{3}$, and

$$A = \pi - 2\phi\,, \qquad B = \phi + \psi\,, \qquad C = \phi - \psi\,.$$

In terms of ϕ and ψ the desired inequality takes the form

$$\begin{aligned} &\sin 2\phi\cos^2 2\phi + \cos^2(\phi+\psi)\sin(\phi+\psi) + \cos^2(\phi-\psi)\sin(\phi-\psi) \\ &\quad \leq \sqrt{2}\sin 2\phi\sin(\phi+\psi)\sin(\phi-\psi)\,. \end{aligned}$$

In this inequality make the replacements $\sin 2\phi = 2\sin\phi\cos\phi$, $\cos 2\phi = \cos^2\phi - \sin^2\phi$, $\cos(\phi\pm\psi) = \cos\phi\cos\psi \mp \sin\phi\sin\psi$, and $\sin(\phi\pm\psi) = \sin\phi\cos\psi \pm \cos\phi\sin\psi$, then expand each side and cancel like terms, then cancel a common term $\sin\phi > 0$ from each side, then apply the identities $\sin^2 x = 1 - \cos^2 x$ for $x = \phi, \psi$. This yields the equivalent inequality

$$\begin{aligned} &\cos^3\phi - (1-\cos^2\phi)\cos\phi + \cos^2\phi\cos^3\psi \\ &+ (1-\cos^2\phi)\cos\psi(1-\cos^2\psi) - 2\cos^2\phi(1-\cos^2\psi)\cos\psi \\ \leq\; &\sqrt{2}(1-\cos^2\phi)\cos\phi\cos^2\psi - \sqrt{2}\cos^3\phi(1-\cos^2\psi) \end{aligned}$$

Set $x = \cos\phi$, $y = \cos\psi$, so $x \in \left[\dfrac{1}{2}, \dfrac{\sqrt{2}}{2}\right]$ and $y \in I = \left[\dfrac{\sqrt{2}}{2}, 1\right]$. Making these substitutions and simplifying yields the equivalent inequality

$$x^3(2+\sqrt{2}) + 4x^2y^3 - 3x^2y - \sqrt{2}xy^2 - x - y^3 + y \leq 0\,.$$

Let $t = x\sqrt{2}$. Then t and y are in the interval I, and we need to prove that

$$f(t,y) = \frac{2+\sqrt{2}}{2\sqrt{2}} + 2t^2y^3 - \frac{3}{2}t^2y - ty^2 - \frac{t}{\sqrt{2}} - y^3 + y \le 0\,.$$

We have

$$f(1,y) = y^3 - y^2 - \frac{1}{2}y + \frac{1}{2} = (y-1)\left(y^2 - \frac{1}{2}\right) \le 0\,, \qquad y \in I\,,$$

with equality only for $y = 1$ and $y = \dfrac{\sqrt{2}}{2}$.

Next we prove that $f(t,y) \le f(1,y)$.

Let $g(y) = 4y^3 - 2y^2 - 3y + 1$ and $h(y) = 4y^3 - 3y$. We then have

$$2[f(t,y) - f(1,y)] = (t-1)\left[t^2(\sqrt{2}+1) + t(\sqrt{2}+1) + th(y) + g(y)\right]\,.$$

Since $t - 1 \le 0$, it suffices to prove that

$$t^2(\sqrt{2}+1) + t(\sqrt{2}+1) + th(y) + g(y) > 0\,. \tag{1}$$

Note that (1) follows from

$$t(\sqrt{2}+1) + g(y) \ge 1\,, \tag{2}$$

$$t(\sqrt{2}+1) + h(y) \ge 1\,, \tag{3}$$

since the left side of (1) is t times the left side of (3) plus the left side of (2).

The quadratic equation $g'(y) = 0$ has roots $y = \dfrac{1 \pm \sqrt{10}}{2}$ and $\dfrac{1+\sqrt{10}}{2} < \dfrac{\sqrt{2}}{2}$, so $g(y)$ is strictly increasing on I and $g(y) \ge g\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{-\sqrt{2}}{2}$. Also, $t \ge \dfrac{\sqrt{2}}{2}$, so that $t(\sqrt{2}+1) \ge 1 + \dfrac{\sqrt{2}}{2}$, and (2) follows.

Furthermore, $h(y) \ge g(y)$ for $y \in I$, since $h(y) = g(y) + (2y^2 - 1)$ and $2y^2 - 1 \ge 0$ on I, so (3) follows from (2).

This completes the proof of the inequality in part (b).

Equality holds in $f(t,y) - f(1,y) \le 0$ only for $t = 1$, and equality holds in $f(1,y) \le 0$ only for $y = 1$ and $y = \dfrac{\sqrt{2}}{2}$. These cases correspond to a triangle with $A = \dfrac{\pi}{2}$ and $B = C = \dfrac{\pi}{4}$, or to a degenerate triangle with $A = B = \dfrac{\pi}{2}$ and $C = 0$.

Part (a) *also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; SCOTT BROWN, Auburn University, Montgomery, AL, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; and the proposer. One incomplete solution to part* (a)*, one incorrect solution to part* (a)*, and three incomplete solutions to part* (b) *were received.*

*The proposer said she was influenced by **Crux** problem 3167, which asked to show that* $a\cos^3 A + b\cos^3 B + c\cos^3 C \le abc/4R^2$ *holds for non-obtuse triangles* ABC*. She indicated that the inequality in part* (a) *occurs in a different form on p. 14 of V.P. Soltan and I. Majdan's book* Identities and Inequalities in a Triangle*, Kishinev, 1982 (Russian), although the proof there is of a general nature and different from the one she constructed.*

3505. [2010 : 45, 47, 107, 109] *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

The circles Γ_1 and Γ_2 have a common centre O, and Γ_1 lies inside Γ_2. The point $A \neq O$ lies inside Γ_1; a ray not parallel to AO that starts at A intersects Γ_1 and Γ_2 at the points B and C, respectively. Let tangents to corresponding circles at the points B and C intersect at the point D. Let E be a point on the line BC such that DE is perpendicular to BC. Prove that $AB = EC$ if and only if OA is perpendicular to BC.

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

Let A' be the projection of O on the line $\ell = BC$. A and A' lie on the same side of B on ℓ (because they are interior points of Γ_1), so we have

$$OA \perp \ell \Leftrightarrow A = A' \Leftrightarrow AB = A'B. \qquad (1)$$

Since $\angle OBD = \angle OCD = 90°$, B and C lie on the circle whose diameter is OD, which we will denote by Γ_3; let M be its centre. Since $MB = MC$, the projection of M on ℓ is the midpoint N of BC. The lines OA', MN, and DE are parallel (they are all perpendicular to ℓ). Since $OM = MD$ it follows that $A'N = NE$. Hence, $A'B = A'N - BN = NE - NC = EC$. That is,

$$A'B = EC. \qquad (2)$$

Using (2) and (1) we conclude that $AB = EC$ if and only if $AB = A'B$ if and only if $OA \perp \ell$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; RICHARD EDEN, student, Purdue University, West Lafayette, IN, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHAN GUNARDI, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia; JOHN G. HEUVER, Grande Prairie, AB; JOEL SCHLOSBERG, Bayside, NY, USA; MIHAÏ STOËNESCU, Bischwiller, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.

3506. [2010 : 45, 47] *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Prove that $Q(n) + Q\left(n^2\right) + Q\left(n^3\right)$ is a perfect square for infinitely many positive integers n that are not divisible by 10, where $Q(n)$ is the sum of the digits of n.

Solution by the Missouri State University Problem Solving Group, Springfield, MO, USA.

More generally, we will show that the following result holds: If $Q(n, b)$ denotes the sum of the digits of n in base b, then $Q(n, b) + Q(n^2, b) + Q(n^3, b)$

is a perfect square for infinitely many positive integers n that are not divisible by b.

Let a be the square-free part of $b-1$, $k = a\ell^2$ with $\ell \in \mathbb{N}$, and $n = b^k - 1$. Now n consists of k $b-1$'s when written in base b and hence $Q(n) = k(b-1)$. Using the binomial theorem, it is easy to see that the base b representation of n^2 consists of $k-1$ $b-1$'s, one $b-2$, $k-1$ 0's, and one 1 (hence $Q(n^2, b) = k(b-1)$) and n^3 consists of $k-1$ $b-1$'s, one $b-3$, $k-1$ 0's, one 2, and k $b-1$'s (hence $Q(n^3, b) = 2k(b-1)$). Therefore $Q(n,b) + Q(n^2,b) + Q(n^3,b) = 4k(b-1) = 4a(b-1)\ell^2$, but since a is the square-free part of $b-1$, $a(b-1)$ is a perfect square and we're done.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHAN GUNARDI, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PETER HURTHIG, Columbia College, Vancouver, BC; MICHAEL JOSEPHY, Universidad de Costa Rica, Costa Rica; R. LAUMEN, Deurne, Belgium; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

The other submitted solutions were similar and they can be summarized by the following sequences $n_k = 2 \times 10^k + 7$ $(k \geq 2)$, $n_k = 7 \times 10^k + 2$ $(k \geq 2)$, $n_k = 10^{k^2} - 1$ $(k \geq 1)$, $n_k = 10^k + 17$ $(k \geq 4)$, *and* $n_k = 18 \times 10^k + 18$ $(k \geq 4)$.

3507. [2010 : 45, 47] *Proposed by Pham Huu Duc, Ballajura, Australia.*

Let a, b, and c be positive real numbers. Prove that

$$\sqrt{\frac{a(b+c)}{a^2+bc}} + \sqrt{\frac{b(c+a)}{b^2+ca}} + \sqrt{\frac{c(a+b)}{c^2+ab}} \leq \sqrt{2(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right)}\,.$$

Solution by Joe Howard, Portales, NM, USA, modified by the editor.

By the Cauchy-Schwarz inequality, we have that

$$\left(\sum_{\text{cyc}} \sqrt{\frac{a(b+c)}{a^2+bc}}\right)^2 \leq 2(a+b+c)\left(\sum_{\text{cyc}} \frac{a}{a^2+bc}\right)$$

Thus, it suffices to show that

$$\sum_{\text{cyc}} \frac{a}{a^2+bc} \leq \sum_{\text{cyc}} \frac{1}{a+b}$$

Without loss of generality, let $a \geq b \geq c$. Then $(a-c)(b-c) \geq 0$, so $c^2 + ab \geq ac + bc$ and $\frac{c}{c^2+ab} \leq \frac{c}{ac+bc} = \frac{1}{a+b}$. Therefore, it now suffices to show that

$$\frac{a}{a^2+bc} + \frac{b}{b^2+ac} \leq \frac{1}{b+c} + \frac{1}{a+c}$$

Since this inequality holds for $a = b$, we can assume that $a > b \geq c$. This simplifies to

$$a^2c^2 + b^2c^2 + a^3b + ab^3 + ab^2c + a^2bc \leq 2abc^2 + a^4 + b^4 + a^3c + b^3c$$

$$c^2(a-b)^2 + b^2c(a-b) + b^3(a-b) \leq a^3(a-b) + a^2c(a-b)$$

Since $a - b > 0$, we obtain

$$c^2(a-b) \leq a^3 - b^3 + c(a^2 - b^2)$$

$$c^2(a-b) \leq (a-b)(a^2+ab+b^2) + c(a-b)(a+b)$$

$$c^2 \leq a^2 + ab + b^2 + c(a+b)$$

The result follows.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.

3508. [2010 : 45, 47] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, c, d be nonnegative real numbers such that $a+b+c+d=4$. Prove that

$$a\sqrt{bc} + b\sqrt{cd} + c\sqrt{da} + d\sqrt{ab} \;\leq\; 2\left(1+\sqrt{abcd}\right).$$

Solution by the proposer.

Let (x,y,z,t) be a permutation of (a,b,c,d) such that $x \geq y \geq z \geq t$. We clearly have $\sqrt{x} \geq \sqrt{y} \geq \sqrt{z} \geq \sqrt{t}$ and

$$\sqrt{xyz} \;\geq\; \sqrt{xyt} \;\geq\; \sqrt{xzt} \;\geq\; \sqrt{yzt},$$

and therefore, by the Rearrangement Inequality, we have

$$\sqrt{x}\sqrt{xyz} + \sqrt{y}\sqrt{xyt} + \sqrt{z}\sqrt{xzt} + \sqrt{t}\sqrt{yzt}$$
$$\geq\; \sqrt{a}\sqrt{abc} + \sqrt{b}\sqrt{bcd} + \sqrt{c}\sqrt{cda} + \sqrt{d}\sqrt{dab}.$$

It remains to prove that

$$\sqrt{x}\sqrt{xyz} + \sqrt{y}\sqrt{xyt} + \sqrt{z}\sqrt{xzt} + \sqrt{t}\sqrt{yzt} \;\leq\; 2(1+\sqrt{abcd}),$$

or

$$(\sqrt{xy}+\sqrt{zt})(\sqrt{xz}+\sqrt{yt}) \;\leq\; 2(1+\sqrt{xyzt}).$$

Since $uv \leq \frac{1}{2}\left(u^2+v^2\right)$, it is enough to prove that

$$(\sqrt{xy}+\sqrt{zt})^2 + (\sqrt{xz}+\sqrt{yt})^2 \;\leq\; 4(1+\sqrt{xyzt}),$$

or

$$xy + zt + xz + yt \le 4,$$

which is equivalent to $(x + t)(y + z) \le 4$. This is clearly true by the AM–GM Inequality, since $x + y + z + t = 4$, and we are done.

There were 2 incomplete solutions.

3509. [2010 : 45, 48] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a + b + c = 3$. For each positive real number k, find the maximum value of

$$\left(a^2b + k\right)\left(b^2c + k\right)\left(c^2a + k\right).$$

The proposer's submitted solution is distributed among several other solutions to several other problem proposals, while all of the other submitted solutions to this problem were either incomplete or incorrect.

The editor has therefore elected to leave this problem open until a correct and complete "one piece" solution is received.

3510. [2010 : 45, 48] *Proposed by Cosmin Pohoaţă, Tudor Vianu National College, Bucharest, Romania.*

Let d be a line exterior to a given circle Γ with centre O. Let A be the orthogonal projection of O on the line d, M be a point on Γ, and X, Y be the intersections of Γ, d with the circle Γ' of diameter AM. Prove that the line XY passes through a fixed point as M moves about Γ.

I. *Solution by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia, modified by the editor.*

When M lies on OA the lines OA and XY coincide, so that a fixed point would necessarily lie on OA. For any position of M on Γ off OA let Q denote the intersection of XY and OA. We must prove that the position of Q is independent of the choice of M. Let P be the centre of Γ'; we first show that $OPQX$ is cyclic. To that end, note that $\angle OPM$ is both an exterior angle of ΔAOP and half the apex angle of the isosceles triangle PXM. Therefore,

$$\angle OAP + \angle POA = \angle OPM = 90^\circ - \angle AMX. \qquad (1)$$

Also, because $MY \perp YA$ and $YA \perp OA$, it follows that $MY || OA$ and, thus, $\angle OAP = \angle YMA$. From (1) therefore,

$$\angle OAP + \angle POA - \angle OAP = 90^\circ - \angle AMX - \angle YMA$$

that is,

$$\angle POA = 90^\circ - \angle YMX. \qquad (2)$$

But $\angle POQ = \angle POA$ and, because the angle at the center P of Γ' is twice the corresponding inscribed angle at M, $\angle YMX = \frac{1}{2}\angle YPX$. Note also that $90° - \frac{1}{2}\angle YPX = \angle PXY = \angle PXQ$. Consequently, equation (2) becomes $\angle POQ = \angle PXQ$, and we conclude that O, P, Q, X are concyclic, as desired. This now implies that $\angle OQP = \angle OXP$; moreover, because triangles PXM and OMX are isosceles we have $\angle OXP = \angle PMO$. Define R to be the point where the line parallel to PQ through M meets OA. We have $\angle ORM = \angle OQP = \angle PMO = \angle AMO$. Therefore, triangles MRO and AMO are similar and we have $\frac{OR}{OM} = \frac{OM}{OA}$; hence

$$OR = \frac{OM^2}{OA},$$

so that R is a fixed point. But because P is the midpoint of AM and $PQ||MR$, Q must be the midpoint of AR. We conclude that line XY passes through the midpoint of the fixed segment AR as M moves about Γ.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

We do not need the assumption that d is exterior to Γ so long as it is neither tangent to Γ nor passing through O. Let p be the power of A with respect to Γ. Note that p is negative if A is inside Γ. Let i be the transformation defined by $i(U) = V$ if and only if A, U, V are collinear and as signed lengths, $AU \cdot AV = p$. [*Editor's comment.* Zhou called this transformation an inversion. When A is outside Γ, then i is indeed an inversion in the circle with centre A that is orthogonal to Γ; when A is inside Γ, i is the commutative product of a halfturn about A and inversion in the circle with centre A whose diameter is the chord intercepted from d by Γ.] Note that d and Γ are invariant under i. Let $M' = i(M), X' = i(X)$, and $Y' = i(Y)$. Then $i(\Gamma')$ is the line passing through M', X', Y' and $i(XY)$is the circle passing through A, X', Y'. Since $AX \perp MX$, we have $AM' \perp M'X'$, whence MX' is a diameter of Γ. Let B be the point symmetrical with A about O. Then the triangles OBX' and OAM are congruent, which implies that $\angle OBP = \angle OAM$. This last angle equals the directed angle between the lines $Y'A$ and $Y'X'$ (because corresponding sides are perpendicular). We conclude that B is on the circle through A, X', and Y', whence $i(B)$ is the fixed point through which XY passes.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece(2 solutions); MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comăneşti, Romania; and the proposer. There was one incorrect submission.

The final equation of solution I implies that R is the inverse of A with respect to Γ, something Woo proved in his solution using projective geometry. Many of the other solutions easily solved the problem with the help of coordinates or trigonometry.

3511. [2010 : 46, 48] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, and d be nonnegative real numbers. Prove that

$$\prod_{\text{cyclic}} \left(a^2+b^2+c^2\right) \;\leq\; \frac{1}{64}(a+b+c+d)^8 .$$

Solution by George Apostolopoulos, Messolonghi, Greece.

We write $f(a,b,c,d) = \prod_{\text{cyclic}} \left(a^2+b^2+c^2\right)$, and without loss of generality we assume that $a \geq b \geq c \geq d$.

Since

$$\begin{aligned} b^2+c^2+d^2 &\leq \left(b+\frac{c+d}{2}\right)^2 , \\ c^2+d^2+a^2 &\leq \left(a+\frac{c+d}{2}\right)^2 , \\ a^2+b^2+c^2 &\leq \left(a+\frac{c+d}{2}\right)^2 + \left(b+\frac{c+d}{2}\right)^2 , \end{aligned}$$

and

$$a^2+c^2+d^2 \leq \left(a+\frac{c+d}{2}\right)^2 + \left(b+\frac{c+d}{2}\right)^2 ,$$

we obtain $f(a,b,c,d) \;\leq\; f\left(a+\frac{c+d}{2}, b+\frac{c+d}{2}, 0, 0\right)$. Let $x = a+\frac{c+d}{2}$ and $y = b+\frac{c+d}{2}$, so that $x+y = a+b+c+d$.

We now need to prove that $\left(x^2+y^2\right)^2 x^2y^2 \leq \frac{1}{64}(x+y)^8$. However, this inequality follows from an application of the AM–GM inequality:

$$\left(x^2+y^2\right)xy = \frac{1}{2}\left(x^2+y^2\right)(2xy) \leq \frac{1}{2}\left(\frac{x^2+y^2+2xy}{2}\right)^2 = \frac{1}{8}\,(x+y)^4 ,$$

and the proof is complete.

Equality holds precisely when two of a, b, c, d are equal and the remaining two are zero.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece (second solution); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OLIVER GEUPEL, Brühl, NRW, Germany; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3512. [2010 : 46, 48] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let α be a real number and let $p \geq 1$. Find

$$\lim_{n\to\infty} \prod_{k=1}^{n} \frac{n^p + (\alpha - 1)k^{p-1}}{n^p - k^{p-1}}.$$

Solution by Albert Stadler, Herrliberg, Switzerland.

Let $p_n = \prod\limits_{k=1}^{n} \dfrac{n^p + (\alpha-1)k^{p-1}}{n^p - k^{p-1}}$. Then

$$\begin{aligned}\ln p_n &= \sum_{k=1}^{n} \ln\left(\frac{1 + (\alpha-1)\left(\frac{k}{n}\right)^{p-1}\cdot\frac{1}{n}}{1 - \left(\frac{k}{n}\right)^{p-1}\cdot\frac{1}{n}}\right) \\ &= \sum_{k=1}^{n} \ln\left(1 + (\alpha-1)\left(\frac{k}{n}\right)^{p-1}\cdot\frac{1}{n}\right) \\ &\quad - \sum_{k=1}^{n} \ln\left(1 - \left(\frac{k}{n}\right)^{p-1}\cdot\frac{1}{n}\right). \end{aligned} \tag{1}$$

For each $k = 1, 2, \dots, n$, we have, as $n \to \infty$

$$\ln\left(1 + (\alpha-1)\left(\frac{k}{n}\right)^{p-1}\cdot\frac{1}{n}\right) = (\alpha-1)\left(\frac{k}{n}\right)^{p-1}\cdot\frac{1}{n} + O\left(\frac{1}{n^2}\right), \tag{2}$$

$$\ln\left(1 - \left(\frac{k}{n}\right)^{p-1}\cdot\frac{1}{n}\right) = -\left(\frac{k}{n}\right)^{p-1}\cdot\frac{1}{n} + O\left(\frac{1}{n^2}\right). \tag{3}$$

From (1), (2), and (3) we have

$$\ln p_n = \alpha \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{p-1}\cdot\frac{1}{n} + O\left(\frac{1}{n^2}\right). \tag{4}$$

Note that the sum $\sum\limits_{n=1}^{n} S_n = \left(\dfrac{k}{n}\right)^{p-1}\cdot\dfrac{1}{n}$ is a Riemann sum for the function $f(x) = x^{p-1}$ over the interval $[0,1]$. Hence,

$$\lim_{n\to\infty} S_n = \int_0^1 x^{p-1}\,dx = \frac{1}{p}. \tag{5}$$

From (4) and (5) we have $\lim\limits_{n\to\infty} \ln p_n = \dfrac{\alpha}{p}$, hence $\lim\limits_{n\to\infty} p_n = e^{\alpha/p}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; ANASTASIOS KOTRONIS, Heraklion, Greece; and the proposer. Two incorrect solutions were submitted.

3513. [2010 : 46, 48] *Proposed by Hassan A. ShahAli, Tehran, Iran.*

Let α and β be positive real numbers, and r be a positive rational number. Prove that there exist infinitely many integers m and n such that

$$\frac{\lfloor m\alpha \rfloor}{\lfloor n\beta \rfloor} = r,$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

Solution by Oliver Geupel, Brühl, NRW, Germany, modified by the editor.

We show that the statement is false. It is well known (see D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom, *The USSR Olympiad Problem Book: selected problems and theorems of elementary mathematics*, Dover, New York, 1993, Problem 108) that if α and β are positive irrational numbers such that $\frac{1}{\alpha}+\frac{1}{\beta} = 1$, then all positive integers appear with no duplications in the two sequences $\lfloor \alpha \rfloor$, $\lfloor 2\alpha \rfloor$, $\lfloor 3\alpha \rfloor$, $\dots$ and $\lfloor \beta \rfloor$, $\lfloor 2\beta \rfloor$, $\lfloor 3\beta \rfloor$, $\dots$. [*Ed.: These are known in the literature as complementary Beatty sequences.*]

If we take $r = 1$, $\alpha = \sqrt{2}$, and $\beta = 2 + \sqrt{2}$, then clearly $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, so there are no positive integers m and n such that $\frac{\lfloor m\alpha \rfloor}{\lfloor n\beta \rfloor} = 1$. It is also clear that $m \neq 0$, $n \neq 0$, and if m and n are of opposite signs, then $\lfloor m\alpha \rfloor \neq \lfloor n\beta \rfloor$.

Finally, suppose $m < 0$ and $n < 0$. Using the trivial fact that $\lfloor x \rfloor + \lfloor -x \rfloor = -1$ for all reals x which are not integers, we see that the sequences $\lfloor -\alpha \rfloor$, $\lfloor -2\alpha \rfloor$, $\lfloor -3\alpha \rfloor$, $\dots$ and $\lfloor -\beta \rfloor$, $\lfloor -2\beta \rfloor$, $\lfloor -3\beta \rfloor$, $\dots$ contain all nonnegative integers with no duplications. Hence it is again impossible for $\lfloor m\alpha \rfloor = \lfloor n\beta \rfloor$ to hold, and our proof is complete.

Two incorrect solutions were submitted.