SKOLIAD No. 130

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Please send your solutions to problems in this Skoliad by **September 15, 2011**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Niels Henrik Abel Mathematics Contest, 2009–2010, Second Round. Our thanks go to Øyvind Bakke, Norwegian University of Science and Technology, Trondheim, Norway, for providing us with this contest and for permission to publish it. We also thank Rolland Gaudet, University College of Saint Boniface, Winnipeg, MB, for translating this contest from English into French.

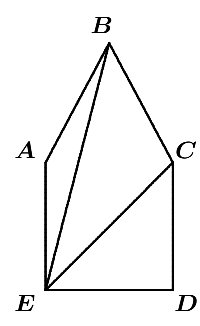
Niels Henrik Abel Mathematics Contest, 2009–2010 2nd Round

100 minutes allowed

1. A four-digit whole number is *interesting* if the number formed by the leftmost two digits is twice as large as the number formed by the rightmost two digits. (For example, **2010** is interesting.) Find the largest whole number, d, such that all interesting numbers are divisible by d.

2. A calculator performs this operation: It multiplies by **2.1**, then erases all digits to the right of the decimal point. For example, if you perform this operation on the number **5**, the result is **10**; if you begin with **11**, the result is **23**. Now, if you begin with the whole number k and perform the operation three times, the result is **201**. Find k.

3. The pentagon $ABCDE$ consists of a square, $ACDE$, with side length **8**, and an isosceles triangle, ABC, such that $AB = BC$. The area of the pentagon is **90**. Find the area of $\triangle BEC$.



4. In how many ways can one choose three different integers between **0.5** and **13.5** such that the sum of the three numbers is divisible by **3**?

5. If a and b are positive integers such that $a^3 - b^3 = 485$, find $a^3 + b^3$.

6. If a and b are positive integers such that $a^3 + b^3 = 2ab(a + b)$, find $a^{-2}b^2 + a^2b^{-2}$.

7. Let D be the midpoint of side AC in $\triangle ABC$. If $\angle CAB = \angle CBD$ and the length of AB is 12, then find the square of the length of BD.

8. If x, y, and z are whole numbers and $xyz+xy+2yz+xz+x+2y+2z = 28$ find $x+y+z$.

9. Henrik's math class needs to choose a committee consisting of two girls and two boys. If the committee can be chosen in 3630 ways, how many students are there in Henrik's math class?

10. Let S be $1!(1^2+1+1) + 2!(2^2+2+1) + 3!(3^2+3+1) + \cdots + 100!(100^2+100+1)$. Find $\dfrac{S+1}{101!}$. (As usual, $k! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (k-1) \cdot k$.)

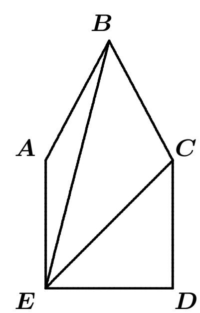
Concours Mathématique Niels Henrik Abel, 2009–2010 $2^{\text{ième}}$ ronde

100 minutes sont accordées

1. Un entier à quatre chiffres est intéressant si l'entier formé par les deux chiffres à l'extrême gauche est deux fois plus grand que l'entier formé par les deux chiffres à l'extrême droite. (Par exemple, 2010 est intéressant.) Déterminer le plus grand entier, d, tel que tous les nombres intéressants sont divisibles par d.

2. Une calculatrice effectue cette opération : elle multiplie par $2{,}1$, puis elle efface tous les chiffres à droite de la décimale. Par exemple, si on effectue cette opération à partir de 5, le résultat est 10 ; à partir de 11, le résultat est 23. Or, si on commence avec un entier k et qu'on effectue cette opération trois fois, le résultat est 201. Déterminer k.

3. Le pentagone $ABCDE$ consiste d'un carré, $ACDE$, de côtés de longueur 8, puis d'un triangle isocèle, ABC, tel que $AB = BC$. La surface du pentagone est 90. Déterminer la surface de $\triangle BEC$.



4. De combien de façons pouvons-nous choisir trois entiers différents entre $0,5$ et $13,5$, tels que la somme des trois entiers soit divisible par 3 ?

5. Si a et b sont des entiers positifs tels que $a^3 - b^3 = 485$, déterminer $a^3 + b^3$.

6. Si a et b sont des entiers positifs tels que $a^3 + b^3 = 2ab(a+b)$, déterminer $a^{-2}b^2 + a^2b^{-2}$.

7. Soit D le mipoint du côté AC dans $\triangle ABC$. Si $\angle CAB = \angle CBD$ et si la longueur de AB est 12, déterminer le carré de la longueur de BD.

8. Si x, y et z sont des entiers et si $xyz + xy + 2yz + xz + x + 2y + 2z = 28$, déterminer $x + y + z$.

9. La classe de mathématiques d'Henri a besoin de choisir un comité formé de deux filles et de deux garçons. Si ce comité peut être formé de 3630 façons, combien d'étudiants y a-t-il dans la classe de mathématiques d'Henri ?

10. Soit S égal à $1!(1^2 + 1 + 1) + 2!(2^2 + 2 + 1) + 3!(3^2 + 3 + 1) + \cdots + 100!(100^2 + 100 + 1)$. Déterminer $\frac{S+1}{101!}$. (Comme d'habitude, $k! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (k-1) \cdot k$.)

Next follow solutions to the selected problems from the 10th Annual Christopher Newport University Regional Mathematics Contest, 2009, given in Skoliad 124 at [2010:129–131]. (*Note: Problems 1, 2, 3 and 4 first appeared on the 2009 Calgary Junior Math Contest. – Ed.*)

1. Elves and ogres live in the land of Pixie. The average height of the elves is 80 cm, the average height of the ogres is 200 cm, and the average height of the elves and the ogres together is 140 cm. If 36 elves live in Pixie, how many ogres live there?

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

Let x be the number of ogres in Pixie. Then the total height of all the ogres is $200x$, and the total height of all the elves is $36 \cdot 80 = 2880$. Therefore the total height of all the creatures in Pixie is $2880 + 200x$. On the other hand, the average height of the $36 + x$ creatures in Pixie is 140, so their total height is $140(36 + x) = 5040 + 140x$. Thus $2880 + 200x = 5040 + 140x$, so $x = 36$.

Also solved by WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.

2. You are given a two-digit positive integer. If you reverse the digits of your number, the result is a number which is 20% larger than your original number. What is your original number?

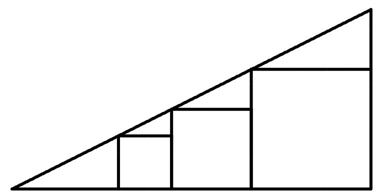
Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Let x be the given two-digit number. Increasing x by 20%, that is, by $\frac{1}{5}x$, yields an integer, so x must be divisible by 5. Thus x ends in 0 or in 5. If the ones digit of x is 0, reversing the digits would decrease the number, so x must end in 5. If the tens digit is larger than 5, reversing the digits would again decrease the number. Thus only the numbers 15, 25, 35, 45, and 55 remain to be checked. Only 45 works out.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.

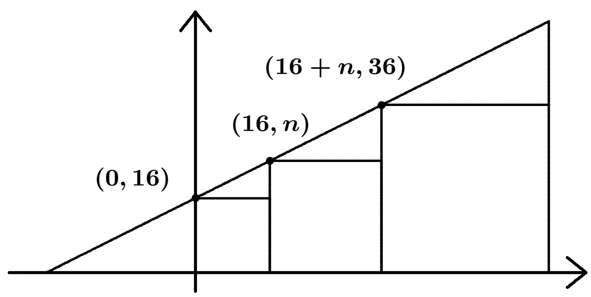
Alternatively, let $10a+b$ be the two-digit number, where a and b are digits. Increasing by 20% is the same as multiplying by $\frac{6}{5}$, so $10b+a = \frac{6}{5}(10a+b)$. Thus $5(10b+a) = 6(10a+b)$, *so $44b = 55a$, so $4b = 5a$. Clearly b is divisible by 5, and since $b = 0 = a$ does not yield a two-digit number, b must be 5. Hence $a = 4$ and the number is 45.*

3. Three squares are placed side-by-side inside a right-angled triangle as shown in the diagram. The side length of the smallest of the three squares is 16. The side length of the largest of the three squares is 36. What is the side length of the middle square?



Solution by Wen-Ting Fan, student, Burnaby North Secondary School, Burnaby, BC.

Impose a coordinate system as in the figure. If the middle square has side length n, then the coordinates are as indicated. Since the slanted line passes through $(0, 16)$, the equation of the line is $y = mx + 16$ for some slope, m.



Using the two other points yields

$$n = 16m + 16 \quad \text{and} \quad 36 = m(16 + n) + 16\,.$$

Therefore $36 = m(16 + 16m + 16) + 16 = 16m^2 + 32m + 16$, so $0 = 16m^2 + 32m - 20 = 4(2m + 5)(2m - 1)$, so $x = -\frac{5}{2}$ or $x = \frac{1}{2}$. In the figure, the slope is clearly positive, so $m = \frac{1}{2}$, and $n = 16m + 16 = 24$.

Also solved by LISA WANG, student, Port Moody Secondary School, Port Moody, BC. You can also solve this problem using similar triangles.

4. Friends Maya and Naya ordered finger food in a restaurant, Maya ordering chicken wings and Naya ordering bite-size ribs. Each wing cost the same amount, and each rib cost the same amount, but one wing was more expensive than one rib. Maya received 20% more pieces than Naya did, and Maya paid 50% more in total than Naya did. The price of one wing was what percentage higher than the price of one rib?

Solution by Lisa Wang, student, Port Moody Secondary School, Port Moody, BC.

Say Naya gets n pieces at \$$r$ each. Then Maya gets $1.2n$ pieces at, say, \$$w$ each. Then Naya pays \$$nr$ and Maya pays \$$1.2nw$. Since Maya pays 50% more than Naya, $1.2nw = 1.5nr$, so $w = \frac{1.5}{1.2}r = 1.25r$, so one wing is 25% more expensive than one rib.

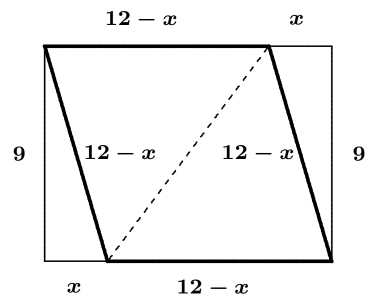
Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC.

5. A 9×12 rectangular piece of paper is folded once so that a pair of diagonally opposite corners coincide. What is the length of the crease?

Solution by Wen-Ting Fan, student, Burnaby North Secondary School, Burnaby, BC.

If you fold the paper as instructed and unfold it again, you obtain the figure below where the section outlined with thick lines used to overlap and the dashed line is the crease. The Pythagorean Theorem now yields that

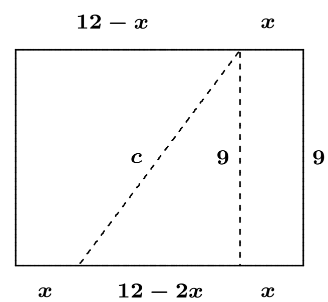
$$\begin{aligned} 9^2 + x^2 &= (12 - x)^2 \\ 81 + x^2 &= 144 - 24x + x^2 \\ 24x &= 63 \\ x &= \frac{63}{24} \end{aligned}$$



Then redraw the diagram as in the figure below. Use the Pythagorean Theorem again:

$$c^2 = 9^2 + (12 - 2x)^2.$$

Since $x = \frac{63}{24}$, $c^2 = \frac{2025}{16}$, so $c = \frac{45}{4}$.



Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.

6. In calm weather, an aircraft can fly from one city to another 200 kilometres north of the first and back in exactly two hours. In a steady north wind, the round trip takes five minutes longer. Find the speed (in kilometres per hour) of the wind.

Solution by the editors.

The airspeed of the plane is $\frac{400}{2} = 200$ kilometres per hour. Let w denote the speed of the wind. Then, if you fly with the wind, the ground speed is $200 + w$; and if you fly against the wind, the ground speed is $200 - w$. Therefore, the plane takes $\dfrac{200}{200 + w}$ hours to fly with the wind and $\dfrac{200}{200 - w}$ hours to fly against the wind. If you add these two expressions, you get the total time for the round trip, but this time is given to be $\mathbf{2 + \frac{5}{60}}$ hours, so

$$\begin{aligned} \frac{200}{200-w} + \frac{200}{200+w} &= \frac{25}{12} \\ \frac{8(200+w) + 8(200-w)}{(200-w)(200+w)} &= \frac{1}{12} \\ \frac{3200}{200^2 - w^2} &= \frac{1}{12} \\ w = \pm 40\,. \end{aligned}$$

Therefore the speed of the wind is **40** kilometres per hour.

7. A rectangular floor, **24** feet $\times$ **40** feet, is covered by squares of sides **1** foot. A chalk line is drawn from one corner to the diagonally opposite corner. How many tiles have a chalk line segment on them?

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Since $\frac{24}{8} = 3$ and $\frac{40}{8} = 5$, consider instead the $\mathbf{3 \times 5}$ rectangle on the left in the figure. You can easily count that the diagonal crosses seven squares.

Now tile the $\mathbf{24 \times 40}$ rectangle with $\mathbf{3 \times 5}$ rectangles as in the righthand side of the figure. The diagonal of the $\mathbf{24 \times 40}$ rectangle is also the diagonal of each of eight of the $\mathbf{3 \times 5}$ rectangles. Therefore the diagonal crosses **56** of the $\mathbf{1 \times 1}$ squares.

This issue's prize of one copy of ***Crux Mathematicorum*** for the best solutions goes to Wen-Ting Fan, student, Burnaby North Secondary School, Burnaby, BC.

As Skoliad editors we are quite pleased to see envelopes with "exotic" stamps in the mail, but receiving more Canadian solutions would be wonderful. The address is on the inside of the back cover. You do not have to solve the entire featured contest; a well-presented solution to a single problem is enough. The test for "well-presented" is that your classmates at school can understand it. You can send your solution(s) by mail or electronically—even if that means that we miss out on the exotic stamps.