PROBLEMS

*Solutions to problems in this issue should arrive no later than **1 August 2010**. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

Note: As ***CRUX with MAYHEM*** is running behind schedule, we will accept solutions past the posted due date. Solutions will be accepted until we process them for publication. Currently we are delayed by about four months. Check the CMS website, **cms.math.ca/crux**, for our status in processing problems.

3601. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Suppose that b is a positive real number such that there are exactly two integers strictly between b and $2b$, and exactly two integers strictly between $2b$ and b^2. Find all possible values of b.

3602. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Prove that if $a_i > 0$ for $i = 1, 2, 3, 4$, then

$$\sum_{\text{cyclic}} \frac{1}{a_i^2 + a_{i+1}^2 + a_{i+2}^2} \geq \frac{12}{(a_1 + a_2 + a_3 + a_4)^2}$$

3603. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let ABC be a given triangle and $0 < \lambda < \frac{1}{2}$. Let D and E be points on AB such that $AD = BE = \lambda \cdot AB$, and F, G points on AC such that $AF = CG = \lambda \cdot AC$. Let $BF \cap CE = H$ and $BG \cap CD = I$. Show that

i) $HI \parallel BC$ and

ii) $HI = \dfrac{1 - 2\lambda}{\lambda^2 - \lambda + 1} BC$.

3604. *Proposed by Michel Bataille, Rouen, France.*

Evaluate

$$\lim_{n \to \infty} \frac{\int_0^1 (x^2 - x - 2)^n dx}{\int_0^1 (4x^2 - 2x - 2)^n dx}.$$

3605. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$ be a real polynomial with positive coefficients and having all its zeros real. Prove that

$$\sqrt[n]{A(1)A(2)\cdots A(n)} \geq (n+1)!$$

3606. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Let ABC be a triangle with $\angle A = 20^\circ$. Let BD be the angle bisector of $\angle ABC$ with D on AC. If $AD = DB + BC$, determine $\angle B$.

3607. *Proposed by George Miliakos, Sparta, Greece.*

Let $c_1 = 9$, $c_2 = 15$, $c_3 = 21$, $c_4 = 25, \ldots$, where c_n is the n^{th} composite odd integer. Evaluate

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}}\,.$$

3608. *Proposed by Michel Bataille, Rouen, France.*

Let

$$f(x) \;=\; \frac{e^{1/x}-1}{e^{1/(x+1)}-1}\,.$$

(a) Show that for all $x \in (0,\infty)$,

$$f(x) \;>\; \sqrt{\frac{x+1}{x}}\,.$$

(b) ★ Prove or disprove:

$$f(x) \;<\; \sqrt{\frac{x+1}{x-1}}$$

for all $x \in (1,\infty)$.

3609. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let r be a real number. and let D, E, and F be points on the sides BC, CA, and AB of a triangle ABC with

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = r.$$

The cevians AD, BE, and CF bound a triangle PQR whose area we denote by $[PQR]$. Find the value of r for which the ratio of the areas, $\frac{[DEF]}{[PQR]}$ equals 4.

3610. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let $S = \{2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, \ldots\}$ be the set of positive integers whose only prime divisors are 2 or 3. Let $a_1 = 2, a_2 = 3, \ldots$, be the elements of S, with $a_1 < a_2 < \ldots$.

(i) Determine $\displaystyle\sum_{i=1}^{\infty}\left(\frac{1}{a_i}\right)$.

(ii) ★ For each positive integer n, let $s(n)$ be the sum of all its divisors including 1 and n itself. Prove $\frac{s(n)}{n} < 3$ for all members of S.

3611. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Given x, y, and z are positive integers such that

$$\frac{x(y+1)}{x-1}, \frac{y(z+1)}{y-1}, \text{ and, } \frac{z(x+1)}{z-1}$$

are positive integers. Find the smallest positive integer N such that $xyz \le N$.

3612. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Find all nonconstant polynomials P such that $P(\{x\}) = \{P(x)\}$, for all $x \in \mathbb{R}$, where $\{a\}$ denotes the fractional part of a.

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3601. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

On suppose que b est un nombre réel positif tel qu'il existe exactement deux entiers strictement compris entre b et $2b$, de même qu'exactement deux entiers strictement compris entre $2b$ et b^2. Trouver toutes les valeurs possibles de b.

3602. *Proposé par Pham Van Thuan, Université de Science de Hanoï, Hanoï, Vietnam.*

Montrer que si $a_i > 0$ pour $i = 1, 2, 3, 4$, alors

$$\sum_{\text{cyclique}} \frac{1}{a_i^2 + a_{i+1}^2 + a_{i+2}^2} \ge \frac{12}{(a_1 + a_2 + a_3 + a_4)^2}$$

3603. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit ABC un triangle et $0 < \lambda < \frac{1}{2}$. Soit D et E deux points sur AB tels que $AD = BE = \lambda \cdot AB$, F et G deux points sur AC tels que $AF = CG = \lambda \cdot AC$. Soit $BF \cap CE = H$ et $BG \cap CD = I$. Montrer que

i) $HI \parallel BC$ et

ii) $HI = \dfrac{1-2\lambda}{\lambda^2-\lambda+1}BC$.

3604. *Proposé par Michel Bataille, Rouen, France.*

Calculer

$$\lim_{n\to\infty} \frac{\int_0^1 (x^2-x-2)^n dx}{\int_0^1 (4x^2-2x-2)^n dx}.$$

3605. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit $A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + 1$ un polynôme réel à coefficients positifs n'ayant que des racines réelles. Montrer que

$$\sqrt[n]{A(1)A(2)\cdots A(n)} \geq (n+1)!$$

3606. *Proposé par Václav Konečný, Big Rapids, MI, É-U.*

Soit ABC un triangle avec $\angle A = 20°$. Soit BD la bissectrice de l'angle au sommet B avec D sur AC. Si $AD = DB + BC$, trouver $\angle B$.

3607. *Proposé par George Miliakos, Sparte, Grèce.*

Soit $c_1 = 9$, $c_2 = 15$, $c_3 = 21$, $c_4 = 25$, $\ldots$, où c_n désigne le n^{e} entier impair non premier. Calculer

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}}.$$

3608. *Proposé par Michel Bataille, Rouen, France.*

Soit
$$f(x) \ = \ \frac{e^{1/x}-1}{e^{1/(x+1)}-1}\,.$$

(a) Montrer que pour tout $x \in (0, \infty)$,
$$f(x) \ > \ \sqrt{\frac{x+1}{x}}\,.$$

(b) ★ Trouver si oui ou non, on a
$$f(x) \ < \ \sqrt{\frac{x+1}{x-1}}$$
pour tous les $x \in (1, \infty)$

3609. *Proposé par Panagiote Ligouras, École Secondaire Léonard de Vinci, Noci, Italie.*

Soit r un nombre réel et D, E et F des points sur les côtés BC, CA et AB d'un triangle ABC avec
$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = r.$$
Les céviennes AD, BE et CF limitent un triangle PQR dont on désigne l'aire par $[PQR]$. Trouver la valeur de r pour laquelle le rapport $\frac{[DEF]}{[PQR]}$ des aires est égal à 4.

3610. *Proposé par Peter Y. Woo, Université Biola, La Mirada, CA, É-U.*

Soit $S = \{2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, \ldots\}$ l'ensemble des entiers positifs dont les seuls diviseurs premiers sont 2 ou 3. Notons $a_1 = 2, a_2 = 3, \ldots$ les éléments de S, avec $a_1 < a_2 < \ldots$.

(i) Trouver $\displaystyle\sum_{i=1}^{\infty} \left(\frac{1}{a_i}\right)$.

(ii) ★ Pour chaque entier positif n, soit $s(n)$ la somme de tous ses diviseurs, y compris 1 et n lui-même. Montrer que $\frac{s(n)}{n} < 3$ pour tous les éléments de S.

3611. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

On donne trois entiers positifs x, y et z tels que

$$\frac{x(y+1)}{x-1}, \frac{y(z+1)}{y-1} \text{ et } \frac{z(x+1)}{z-1}$$

sont des entiers positifs. Trouver le plus petit entier positif N tel que $xyz \leq N$.

3612. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Trouver tous les polynômes non constants P tels que $P(\{x\}) = \{P(x)\}$ pour tout $x \in \mathbb{R}$, où $\{a\}$ désigne la partie fractionnaire de a.

Just a reminder, it makes it easier for us if problem proposals and solutions are sent to us in electronic format. Material sent in TEX or LATEX is preferred, but we will also accept pdf, Microsoft Word as well as handwritten material (mailed or scanned).

When sending electronic solutions, please name the files in a meaningful way to identify yourself and the problem. For example, if I was to submit a solution to problem 3603 from this issue, I would name it **February_3603_Godin.tex**.

Please place each solution on its own separate sheet(s) with the problem number, your name and affiliation on each page. Multiple solutions on one page means we have to do lots of photocopying and it increases the chances that something will get overlooked or misfiled.

As always no problem is ever closed. We always accept new solutions and generalizations to past problems. Also, in the last issue [2010 : 545, 547], Chris Fisher published a list of unsolved problems from *Crux*. Below is a sample of one of these unsolved problems:

609★. [1981 : 49; 1982 : 27-28] *Proposed by Ian June L. Garces, Ateneo de Manila University, The Philippines.*

$A_1B_1C_1D_1$ is a convex quadrilateral inscribed in a circle and M_1, N_1, P_1, Q_1 are the mid-points of sides B_1C_1, C_1D_1, D_1A_1, A_1B_1, respectively. The chords A_1M_1, B_1N_1, C_1P_1, D_1Q_1 meet the circle again in A_2, B_2, C_2, D_2, respectively. Quadrilateral $A_3B_3C_3D_3$ is formed from $A_2B_2C_2D_2$ as the latter was formed from $A_1B_1C_1D_1$, and the procedure is repeated indefinitely. Prove that quadrilateral $A_nB_nC_nD_n$ "tends to" a square as $n \to \infty$.

What happens if $A_1B_1C_1D_1$ is not convex?

Enjoy!