THE OLYMPIAD CORNER

No. 291

R.E. Woodrow

This number we begin by looking at the files of solutions by readers to problems given in the February 2010 number of the *Corner*, and "A" problems proposed but not used at the 2007 IMO in Vietnam, given at [2010 : 18–19].

A2. Let n be a positive integer, and let x and y be positive real numbers such that $x^n + y^n = 1$. Prove that

$$\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)(1-y)}\,.$$

Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Geupel.

From the identity

$$\frac{1+x^{2k}}{1+x^{4k}} + \frac{(1-x^{3k})(1-x^k)}{x^k(1+x^{4k})} = \frac{1+x^{2k}}{1+x^{4k}} + \frac{(1+x^{4k})-x^k(1+x^{2k})}{x^k(1+x^{4k})} = \frac{1}{x^k}$$

and the premise $0 < x < 1$ we deduce that

$$\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}} < \sum_{k=1}^{n} \frac{1}{x^k} = \frac{1-x^n}{x^n(1-x)}. \tag{1}$$

Similarly we have

$$\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}} < \frac{1-y^n}{y^n(1-y)}. \tag{2}$$

The hypothesis $x^n + y^n = 1$ yields

$$\frac{(1-x^n)(1-y^n)}{x^n y^n} = \frac{1-x^n-y^n+x^n y^n}{x^n y^n} = 1. \tag{3}$$

The desired inequality follows immediately from the relations (1), (2), and (3).

A3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x+f(y)) \;=\; f(x+y)+f(y)$$

for all $x, y \in \mathbb{R}^+$. (Here $\mathbb{R}^+$ denotes the set of all positive real numbers.)

Solved by Mohammed Aassila, Strasbourg, France; and Michel Bataille, Rouen, France. We give Bataille's version.

We show that the unique solution is the function $\phi : x \mapsto 2x$.

Since $2(x+2y) = 2(x+y)+2y$ for all $x, y \in \mathbb{R}^+$, this function ϕ is a solution. Conversely, let f be any solution and x, y be any positive real numbers. On the one hand, adding x on both sides of the given equation and taking the images under f, we successively have

$$\begin{aligned} f(x+f(x+f(y))) &= f((x+f(y))+f(x+y)) \\ &= f(2x+y+f(y))+f(x+y) \\ &= f(2x+2y)+f(y)+f(x+y). \end{aligned}$$

On the other hand, using the given equation, we obtain

$$\begin{aligned} f(x+f(x+f(y))) &= f(2x+f(y))+f(x+f(y)) \\ &= f(2x+y)+f(y)+f(x+y)+f(y). \end{aligned}$$

It follows that

$$f(2x+2y) = f(2x+y)+f(y). \qquad (1)$$

Now, suppose that $0 < a < b$. We prove that $f(a) < f(b)$.

- If $b < 2a$, we take $x = \frac{2a-b}{2}$, $y = b-a$ in (1) and obtain $f(b) = f(a)+f(b-a)$, hence $f(b) > f(a)$.
- If $b > 2a$, with $x = \frac{b-2a}{2}$, $y = a$, (1) gives $f(b) = f(a)+f(b-a)$ and $f(b) > f(a)$ again.
- If $b = 2a$, $f(b) = f(2a) = f\left(2(\frac{a}{2}+\frac{a}{2})\right) = f\left(a+\frac{a}{2}\right)+f\left(\frac{a}{2}\right) > f(a)+f\left(\frac{a}{2}\right) > f(a)$. Thus, f is strictly increasing on $(0,\infty)$ and, as such, is injective.

In addition, if a, b are positive and distinct, say $a < b$, then $f(a+b) = f(a)+f(b)$ (for $f\left(a+2\cdot\frac{b-a}{2}\right)+f(a) = f\left(2a+2\cdot\frac{b-a}{2}\right)$).

Lastly, let $y > 0$. Since $f(y) \neq y$ (otherwise $f(x+f(y)) = f(x+y)$ in contradiction with the given functional equation), we may write $f(y+f(y)) = f(y)+f(f(y))$ as well as $f(y+f(y)) = f(2y)+f(y)$. It follows that $f(f(y)) = f(2y)$ and since f is injective, $f(y) = 2y$, as desired.

Next we look at the "C" problems proposed but not used at the 2007 IMO in Vietnam given at [2010: 19–20].

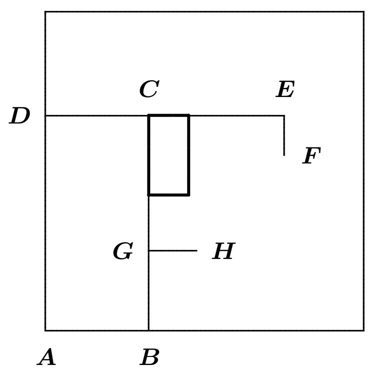
C2. A unit square is dissected into $n > 1$ rectangles such that their sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that one of the rectangles has no point on the boundary of the square.

Solved by Oliver Geupel, Brühl, NRW, Germany.

The proof is by contradiction.

Assume the contrary and consider a counterexample where n is minimal. Let A be one of the vertices of the square, let $\mathcal{E}$ be the locus of the sides of the rectangles of the dissection, and let $ABCD$ be the rectangle that contains the vertex A. Since the square is covered by the rectangles, at least one of the segments BC and DC has an extension in $\mathcal{E}$ beyond the point C. Without loss of generality assume that DC can be extended beyond C where the longest possible extension in $\mathcal{E}$ is up to a point E. Since the line DE intersects the interior of some rectangle, the point E is an interior point of the square. Since the square is covered by the rectangles, a segment EF orthogonal to CE where the points A and F are in the same half-plane relative to the line DE, also belongs to $\mathcal{E}$.

If there were a second rectangle beside $ABCD$ which contains the side BC, then it could be glued together with the rectangle $ABCD$ obtaining a counterexample with $n-1$ rectangles, which contradicts our minimum hypothesis. Hence, there is a segment GH in $\mathcal{E}$ where G is an inner point of BC, and GH is parallel to CE. The rectangle with vertex C and sides on the lines CE and CG is now separated from the boundary of the square by the four segments HG, GC, CE and EF. This is a contradiction which completes the proof.



C5. In the Cartesian coordinate plane let $S_n = \{(x, y) \mid n \leq x < n+1\}$ for each integer n, and paint each region S_n either red or blue. Prove that any rectangle whose side lengths are distinct positive integers may be placed in the plane so that its vertices lie in regions of the same colour.

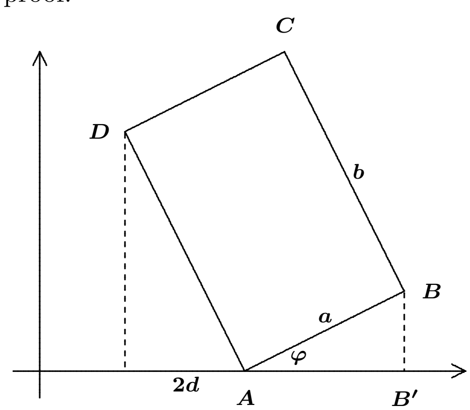
Solved by Oliver Geupel, Brühl, NRW, Germany.

We generalize that any rectangle $ABCD$ with distinct *real* sides $AB = a$ and $BC = b$ may be placed so that its vertices lie in regions of the same colour.

Firstly consider the case where $a \notin \mathbb{Z}$. Without loss of generality, we may assume that S_0 and S_1 have distinct colours. We place $ABCD$ so that A and D have the common x-coordinate $1 - \{a\}$ while B and C have the common x-coordinate $\lceil a\rceil$. We then translate the rectangle with offset $\{a\}$ in the positive x-direction. With this translation, A and D move from S_0 to S_1, while B and D remain in $S_{\lceil a\rceil}$. Consequently, in one of these two positions the vertices A, B, C, D lie in regions of the same colour.

It remains to consider $a, b \in \mathbb{Z}$. Let $d = \gcd(a, b)$, and let a_0, a_1, b_0, b_1 be integers such that $a = a_0 d$, $b = b_0 d$, and $a_0 a_1 + b_0 b_1 = 1$. The proof is by contradiction. Suppose $ABCD$ cannot be placed properly. Then S_0 and S_a have distinct colours, and S_0 and S_b have also distinct colours. By induction, S_0 and S_{ua+vb}, where u and v are integers, have distinct colours if and only if $u + v$ is odd. By $b_0 a - a_0 b = 0$, we see that $b_0 - a_0$ is even. Since a_0 and b_0 are coprime, both are odd. Hence, $a_1 + b_1$ is odd, too. Thus, S_0 and $S_{a_1 a + b_1 b} = S_d$ have distinct colours. We conclude that S_0 and S_{2d} have the same colour.

Assume $a < b$, which implies $b_0 \geq 3$. We pitch the rectangle so that the x-coordinates of A and D as well as the x-coordinates of B and C have distance $2d$. It suffices to prove that the we can translate it so that A and B lie in regions of the same colour. If the angle between the x-axis and the line AB is φ and A' and B' are the projections of A and B, respectively, onto the x-axis, then we obtain $\sin\varphi = \dfrac{2d}{b} = \dfrac{2}{b_0}$ and $A'B' = a\cos\varphi = \dfrac{a}{b_0}\sqrt{b_0^2 - 4} \notin \mathbb{Z}$. By the first part of the proof, we can place A' and B' so that they lie in regions of the same colour. This completes the proof.



C7. A convex n-gon P in the plane is given. For every three vertices of P, the triangle determined by them is *good* if all its sides are of unit length. Prove that P has at most $\frac{2}{3}n$ good triangles.

Comment by Mohammed Aassila, Strasbourg, France.

This is not an original problem. It first appeared in J. Pach and R. Pinchasi, *How many unit triangles can be generated by n points in convex position?*, American Math. Monthly 110 (5) 2003 : 40–406.

Next we move to the “G” problems proposed but not used at the 2007 IMO in Vietnam, given at [1020 : 20–21].

G3. Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle of ABC. Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C' respectively. Assume that the points A, B, C, A', B', C' are distinct, and that the lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

Solved by Michel Bataille, Rouen, France.

We denote by Γ the circumcircle of $\triangle ABC$ and by A_0, B_0, C_0 the points of intersection of the lines BB' and CC', CC' and AA', AA' and BB', respectively. We will use areal coordinates relatively to (A, B, C). The equation of the circle Γ is known to be $a^2yz+b^2zx+c^2xy = 0$ where $A = BC$, $b = CA$, $c = AB$. Let $P(x_0, y_0, z_0)$ and $A'(x', y', z')$; note that the line AA' has equation $yz' - zy' = 0$ and that (x_0, y_0, z_0), (x', y', z') are solutions to the system $a^2yz + b^2zx + c^2xy = 0$, $x(y_0 - z_0) = x_0(y - z)$. It readily follows that $\frac{y_0}{z_0}$, $\frac{y'}{z'}$ are solutions of an equation (with unknown U)

$$(c^2x_0)U^2 + \lambda U + (-b^2x_0) = 0$$

for some real λ. From $\frac{y_0}{z_0} \cdot \frac{y'}{z'} = -\frac{b^2}{c^2}$, we deduce $\frac{y'}{-b^2z_0} = \frac{z'}{c^2y_0}$ so that the equation of AA' is $(c^2y_0)y + (b^2z_0)z = 0$. Similarly, the equations of BB' and CC' are $(c^2x_0)x + (a^2z_0)z = 0$ and $(b^2x_0)x + (a^2y_0)y = 0$. Then, it is easily obtained that

$$A_0(-a^2y_0z_0, b^2x_0z_0, c^2x_0y_0), \quad B_0(a^2y_0z_0, -b^2x_0z_0, c^2x_0y_0),$$
$$C_0(a^2y_0z_0, b^2x_0z_0, -c^2x_0y_0).$$

Now, recall that if $M_i(x_i, y_i, z_i)$ with $x_i + y_i + z_i = 1$ for $i = 1, 2, 3$, the ratio $\frac{\text{area}(M_1M_2M_3)}{\text{area}(ABC)}$ is the absolute value of the determinant $\begin{vmatrix} x_1 & y_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$. Here, we have $\frac{\text{area }(A_0B_0C_0)}{\text{area }(ABC)} = |\triangle|$ where

$$\triangle = \frac{a^2y_0z_0}{-a^2y_0z_0 + b^2x_0z_0 + c^2x_0y_0} \cdot \frac{b^2x_0z_0}{a^2y_0z_0 - b^2x_0z_0 + c^2x_0y_0}$$
$$\cdot \frac{c^2x_0y_0}{a^2y_0z_0 + b^2x_0z_0 - c^2x_0y_0} \cdot \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}.$$

Since P is on Γ, $b^2x_0z_0 + c^2x_0y_0 = -a^2y_0z_0$ (for example) so that

$$\triangle = \frac{(a^2y_0z_0)(b^2x_0z_0)(c^2x_0y_0)}{(-2a^2y_0z_0)(-2b^2x_0z_0)(-2c^2x_0y_0)} \cdot 4$$

and

$$\text{area}\ (A_0B_0C_0) \ = \ \frac{1}{2}\text{area}\ (ABC)\,,$$

independent of P.

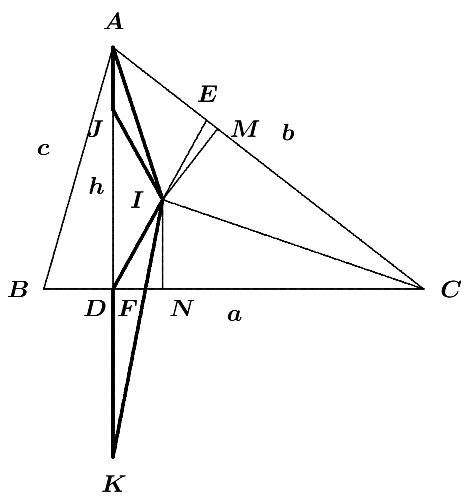
G5. Triangle ABC is acute with $\angle ABC > \angle ACB$, incentre I, and circumradius R. Point D is the foot of the altitude from vertex A, point K lies on line AD such that $AK = 2R$, and D separates A and K. Finally, lines DI and KI meet sides AC and BC at E and F, respectively.

Prove that if $IE = IF$ then $\angle ABC > 3\angle ACB$.

Solved by Oliver Geupel, Brühl, NRW, Germany.

The claim is false. We prove instead that $\angle ABC \leq 3\angle ACB$.

We write $a = BC$, $b = CA$, $c = AB$, $2s = a + b + c$, $h = AD$, $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle ACB$. We denote by r the inradius, by J the point on the segment AD such that $DJ = 2r$, and by M and N the perpendicular projections of I onto AC and BD, respectively.



By standard formulas, we have

$$AI^2 = \frac{r^2}{\sin(A/2)} = \frac{(s-a)(s-b)(s-c)}{s} \cdot \frac{bc}{(s-b)(s-c)} = \frac{(s-a)bc}{s},$$

$$2Rh = bc, \qquad 4Rr = \frac{abc}{[ABC]} \cdot \frac{[ABC]}{s} = \frac{abc}{s}.$$

We obtain $AI^2 = 2R(h - 2r) = AK \cdot AJ$; hence $\dfrac{AI}{AJ} = \dfrac{AK}{AI}$. By $\angle IAJ = \angle KAI$, it follows that $\triangle AIJ \sim \triangle AKI$. Thus, $\angle AIJ = \angle AKI = \angle IKD$. Recognizing the isosceles $\triangle IJD$, we deduce $\angle AJI = 180^\circ - \angle DJI = 180^\circ - \angle IDJ = \angle IDK$.

We obtain $\triangle AIJ \sim \triangle IKD$ and consequently

$$\angle DIK = \angle JAI = \angle BAI - \angle BAD = \frac{\alpha}{2} - (90^\circ - \beta) = \frac{\beta - \gamma}{2}.$$

By $IE = IF$, the triangles EIM and FIN are congruent. The point N is an inner point of the segment CF. Now, if M is between C and E, then $\gamma = 180^\circ - \angle MIN = \angle DIN + \angle EIM > \angle DIK = \dfrac{\beta - \gamma}{2}$; consequently $\beta < 3\gamma$. On the other hand, if the point E is between C and M, or $E = M$, then $\gamma = 180^\circ - \angle MIN = \angle DIK = \dfrac{\beta - \gamma}{2}$ which implies $\beta = 3\gamma$. We are done.

Next we move to the "N" problems proposed but not used at the 2007 IMO in Vietnam, given at [1020 : 21].

N1. Find all pairs (k, n) of positive integers for which $7^k - 3^n$ divides $k^4 + n^2$.

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.

We will demonstrate that there is only one such pair, namely $(k, n) = (2, 4)$. Suppose then that k, n are positive integers such that $(7^k - 3^n) \mid (k^4 + n^2)$; which means that

$$\left\{ \begin{array}{l} k^4 + n^2 \;=\; r \cdot (7^k - 3^n), \\ \text{for some nonzero integer } r \\ k, n \in \mathbb{Z}^+. \end{array} \right\} \tag{1}$$

First we show that both k and n must be even. We do so by ruling out the other three possibilities: both k and n being odd, k odd and n even, or (third possibility) k even and n odd.

Possibility 1. both k and n are odd: $k \equiv n \equiv 1 \bmod 2$.

We then have $k^4 \equiv n^2 \equiv 1 \pmod 8$ (the square of an odd integer is congruent to 1 modulo 8. And so,

$$k^4 + n^2 \;\equiv\; 1 + 1 \;\equiv\; 2 \pmod 8\,. \tag{2}$$

Since k and n are both odd positive integers we also have

$$k \;=\; 2m + 1, \quad n \;=\; 2l + 1; \quad \text{where } m, l \text{ are nonnegative integers.}$$

Thus

$$\begin{aligned} 7^k - 3^n \;&=\; 7^{2m+1} - 3^{2l+1} \;\equiv\; (7^2)^m \cdot 7 - (3^2)^l \cdot 3 \\ &\equiv\; 1 \cdot 7 - 1 \cdot 3 \;\equiv\; 7 - 3 \;\equiv\; 4 \pmod 8\,. \end{aligned} \tag{3}$$

According to (3), 4 divides $7^k - 3^n$. And by (2), the highest power of 2 dividing $k^4 + n^2$; is $2^1 = 2$.

This renders equation (1) contradictory or impossible. Hence possibility 1 is ruled out.

Possibility 2. k is odd and n is even; $k \equiv 1 \pmod 2$, $n \equiv 0 \pmod 2$.

Clearly $7^k - 3^n \equiv 1 - 1 \equiv 0 \pmod 2$, while $k^4 + n^2 \equiv 1 + 0 \equiv 1 \pmod 2$, which renders (1) contradictory modulo 2: the left-hand side is congruent to 0 modulo 2. So this possibility is ruled out as well.

Possibility 3. k is even and n odd; $k \equiv 0 \pmod 2$, $n \equiv 1 \pmod 2$.

Same argument as in Possibility 2; the left-hand side is congruent to 1 but the right-hand side is zero modulo 2. So this possibility is eliminated as well.

We conclude that both positive integers k and n must be even:

$$\left\{ \begin{array}{l} k = 2K \quad \text{and} \quad n = 2N, \\ \text{for some positive integers } K \text{ and } N. \end{array} \right\} \tag{4}$$

From (1) and (4) we obtain,

$$16K^4 + 4N^2 \;=\; k^4 + n^2 \;=\; r \cdot (7^K - 3^N)(7^K + 3^N). \tag{5}$$

According to (5), the positive integer $7^K + 3^N$ is a divisor of $k^4 + n^2$. On the other hand,

$$7^5 \;=\; 16807 > 16 \cdot 5^4 \;=\; 10,000$$

(while $7^K < 16K^4$; for $K = 1, 2, 3, 4$). An easy induction shows that $7^K > 16 \cdot K^4$ for $K \geq 5$, we omit the details. Similarly we have $81 = 3^4 > 4 \cdot 4^2 = 64$ (while $3^N < 4N^2$ for $N = 1, 2, 3$). And an easy induction establishes that $3^N > 4N^2$, for $N \geq 4$.

Therefore for $K \geq 5$ and $N \geq 4$ we have $7^K + e^N > 16K^4 + 4N^2$; which implies that

$$(7^K + 3^N) \cdot |7^K - 3^N| \cdot |r| > 16K^4 + 4N^2 \tag{6}$$

since $|r| \cdot |7^K - 3^N|$ is a positive integer.

Clearly (6) condtradicts (5). We have demonstrated that a necessary condition for (5) to hold true is $K \leq 4$ and $N \leq 3$; which means that there is only up to 12 possible pairs (K, N) that may satisfy (5). We form the following table:

	$7^K + 3^N$	$2^4 \cdot K^4 + 2^2 \cdot N^2$
$K = 1, N = 1$	10	$20 = 16 + 4$
$K = 1, N = 2$	16	$32 = 16 + 16$
$K = 1, N = 3$	34	$52 = 16 + 36$
$K = 2, N = 1$	52	$256 + 4 = 260$
$K = 2, N = 2$	58	$256 + 16 = 272$
$K = 2, N = 3$	76	$256 + 36 = 296$
$K = 3, N = 1$	346	$(16)(81) + 4 = 1300$
$K = 3, N = 2$	352	$(16)(81) + 16 = 1312$
$K = 3, N = 3$	370	$1296 + 36 = 1332$
$K = 4, N = 1$	2404	$4096 + 4 = 4100$
$K = 4, N = 2$	2410	$4096 + 16 = 4112$
$K = 4, N = 3$	2428	$4096 + 36 = 4132$

The above table shows that the only pairs (K, N) for which $7^K + 3^N$ divides $2^4 \cdot K^4 + 2^2 \cdot N^2$ are $(K, N) = (1, 1), (1, 2)$. However, the pair $(1, 1)$

does not satisfy (5) since $\mathbf{20} = \boldsymbol{r} \cdot \mathbf{4} \cdot \mathbf{10}$, is impossible with $\boldsymbol{r} \in \mathbb{Z}$. The other pair, $(\boldsymbol{K}, \boldsymbol{N}) = (\mathbf{1}, \mathbf{2})$ does: $\mathbf{32} = \boldsymbol{r} \cdot (-\mathbf{2})(\mathbf{16})$, satisfied with $\boldsymbol{r} = -\mathbf{1}$. Thus $(\boldsymbol{K}, \boldsymbol{N}) = (\mathbf{1}, \mathbf{2})$ is the only pair; and so $(\boldsymbol{k}, \boldsymbol{n}) = (\mathbf{2}\boldsymbol{K}, \mathbf{2}\boldsymbol{N}) = (\mathbf{2}, \mathbf{4})$.

N2. Let $\boldsymbol{b}, \boldsymbol{n} > \mathbf{1}$ be integers. Suppose that for each $\boldsymbol{k} > \mathbf{1}$ there exists an integer $\boldsymbol{a_k}$ such that $\boldsymbol{b} - \boldsymbol{a_k^n}$ is divisible by $\boldsymbol{k}$. Prove that $\boldsymbol{b} = \boldsymbol{A^n}$ for some integer $\boldsymbol{A}$.

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.

To show that $\boldsymbol{b}$ is the $\boldsymbol{n}^{\text{th}}$ power of an integer, it suffices to show that for every prime number $\boldsymbol{p}$ in the prime factorization of $\boldsymbol{b}$, if $\boldsymbol{p^e}$ is the power of $\boldsymbol{p}$ that appears in the prime factorization of $\boldsymbol{b}$, then the exponent $\boldsymbol{e}$ is a multiple of $\boldsymbol{n}$. We set $\boldsymbol{b} = \boldsymbol{p^e} \cdot \boldsymbol{r}$, $\boldsymbol{r}$ a positive integer such that $\boldsymbol{p}$ does not divide $\boldsymbol{r}$; $(\boldsymbol{r}, \boldsymbol{p}) = \mathbf{1}$, $\boldsymbol{e}$ a positive integer.

We apply the hypothesis of the problem with $\boldsymbol{k} = (\boldsymbol{p^e})^{\boldsymbol{n}} = \boldsymbol{p^{en}}$: $\boldsymbol{k} \mid \boldsymbol{b} - \boldsymbol{a_k^n}$ means that there exists a positive integer $\boldsymbol{\lambda}$ such that

$$\boldsymbol{b} - \boldsymbol{a_k^n} = \boldsymbol{k} \cdot \boldsymbol{\lambda};$$

$$\boldsymbol{b} - \boldsymbol{a_k^n} = \boldsymbol{p^{e \cdot n}} \cdot \boldsymbol{\lambda};$$

and since $\boldsymbol{b} = \boldsymbol{p^e} \cdot \boldsymbol{r}$, we obtain

$$\left\{ \begin{array}{c} \boldsymbol{p^e} \cdot \boldsymbol{r} - \boldsymbol{a_k^n} \;=\; \boldsymbol{p^{e \cdot n}} \cdot \boldsymbol{\lambda}, \\ \boldsymbol{e}, \boldsymbol{r}, \boldsymbol{a_k}, \boldsymbol{\lambda} \text{ positive integers such that } (\boldsymbol{r}, \boldsymbol{p}) = \mathbf{1} \end{array} \right\} \tag{1}$$

Since $\boldsymbol{n} > \mathbf{1}$, it follows that

$$\mathbf{1} \;\leq\; \boldsymbol{e} \;<\; \boldsymbol{e} \cdot \boldsymbol{n} \tag{2}$$

Let $\boldsymbol{p^t}$, $\boldsymbol{t}$ a positive integer, be the highest power of $\boldsymbol{p}$ which divides $\boldsymbol{a_k}$:

$$\left\{ \begin{array}{c} \boldsymbol{a_k} \;=\; \boldsymbol{p^t} \cdot \boldsymbol{b_k}, \\ \boldsymbol{t}, \boldsymbol{b_k} \text{ positive integers such that } (\boldsymbol{b_k}, \boldsymbol{p}) \;=\; \mathbf{1} \end{array} \right\} \tag{3}$$

From (1) and (3) we obtain

$$\boldsymbol{p^e} \cdot \boldsymbol{r} - \boldsymbol{p^{n \cdot t}} \cdot \boldsymbol{b_k^n} = \boldsymbol{p^{e \cdot n}} \cdot \boldsymbol{\lambda}. \tag{4}$$

We claim that (4) implies $\boldsymbol{e} = \boldsymbol{n} \cdot \boldsymbol{t}$. Indeed, if $\boldsymbol{e} \neq \boldsymbol{n} \cdot \boldsymbol{t}$, then either,

Possibility 1. $\boldsymbol{n} \cdot \boldsymbol{t} < \boldsymbol{e}$, or,

Possibility 2. $\boldsymbol{e} < \boldsymbol{n} \cdot \boldsymbol{t}$ holds.

If Possibility 1 holds then, by (2) we have:

$$\mathbf{1} \;\leq\; \boldsymbol{n} \cdot \boldsymbol{t} \;<\; \boldsymbol{e} \;,\; \boldsymbol{e} \cdot \boldsymbol{n}. \tag{5}$$

And so (4) implies that

$$\boldsymbol{p^{(e - n \cdot t)}} \cdot \boldsymbol{r} - \boldsymbol{b_k^n} \;=\; \boldsymbol{p^{c \cdot n - n \cdot t}} \cdot \boldsymbol{\lambda}, \tag{6}$$

which implies by (5) and (6) that $p \mid b_k^n$; (since p is prime) $p \mid b_k$, contrary to (3).

If possibility 2 holds, then by (2),

$$1 \;\leq\; e < \min\{e\cdot n, n\cdot t\} \;=\; m \tag{7}$$

And so, (4) implies,

$$r - p^{n\cdot t-e}\cdot b_k^n \;=\; p^{e\cdot n-e}\cdot\lambda \tag{8}$$

Thus (7) and (8) imply that $p \mid r$, contrary to (1).

We have proved that $e = n\cdot t$; and so b must be the n^{th} power of an integer.

N4. For every integer $k \geq 2$, prove that 2^{3k} divides the number

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}}$$

but 2^{3k+1} does not.

Comment by Mohammed Aassila, Strasbourg, France.

This is not an original problem. It appeared first in D.B. Fuchs and M.B. Fuchs, *Arithmetic of binomial coefficients*, KVANT 6 (1970).

N5. Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ and every prime p, the number $f(m+n)$ is divisible by p if and only if $f(m)+f(n)$ is divisible by p. ($\mathbb{N}$ is the set of all positive integers.)

Solved by Oliver Geupel, Brühl, NRW, Germany.

The identity $f(n) = n$ is such a function. We prove that it is the unique solution. Suppose that f is any solution.

We show by contradiction that f is injective. Assume the contrary. Consider the equivalence relation on $\mathbb{N}$ defined by $m \sim n$ if and only if $f(m)$ and $f(n)$ have the same prime divisors. By assumption, there are numbers $m < n$ such that $f(m) = f(n)$. For every $k \in \mathbb{N}$ and every prime p, we have

$$p \mid f(m{+}k) \quad\Leftrightarrow\quad p \mid f(m){+}f(k) \quad\Leftrightarrow\quad p \mid f(n){+}f(k) \quad\Leftrightarrow\quad p \mid f(n{+}k).$$

Hence, for every $k \in \mathbb{N}$ it holds $m+k \sim n+k$, i.e. each integer $s > n$ is equivalent to $s-(n-m)$. Let p be a prime such that the least number s with the property $p \mid f(s)$ is greater than n. We have $s \sim s-(n-m)$, but $p \nmid f(s-(n-m))$, a contradiction. This completes the proof that f is injective.

We show by contradiction that

$$f(1) \;=\; 1. \tag{1}$$

Suppose that there were a prime p such that $p \mid f(1)$. Then, for every $n \in \mathbb{N}$, we would have $p \left| \sum_{k=1}^{n} f(1) \right.$ and, by Mathematical Induction, $p \mid f(n)$. Therefore, f is not surjective, a contradiction, which completes the proof of (1).

We prove that for every $m \in \mathbb{N}$ it holds

$$|f(m+1) - f(m)| = 1 \qquad (2)$$

Suppose contrariwise that for any number $m \in \mathbb{N}$ there is a prime p such that $p \mid f(m+1) - f(m)$. Since f is surjective, there is a number $n \in \mathbb{N}$ such that $p \mid f(m) + f(n)$. We obtain $p \mid f(m+n)$ and $p \mid f(m+1) + f(n)$; hence $p \mid f(m+n+1)$. Thus, $p \mid f(1)$, which contradicts (1). This proves that $f(m+1) = f(m) + 1$.

From (1) and (2) and the injectivity of f, it follows by Mathematical Induction that $f(n) = n$ for every $n \in \mathbb{N}$.

Next we turn to solutions to problems of the Bundeswettbewerb Mathematik 2006 given at [2010 : 22].

1. A circle is divided into $2n$ congruent sectors, n of them coloured black and the remaining n sectors coloured white. The white sectors are numbered clockwise from 1 to n, starting anywhere. Afterwards, the black sectors are numbered counter clockwise from 1 to n, again starting anywhere.

Prove that there exist n consecutive sectors having the numbers from 1 to n.

Comment by Mohammed Aassila, Strasbourg, France.

This problem appeared in the 20$^{\text{th}}$ Tournament of the Towns, Spring 1999, A-level, problem 4. The author is V. Proizvolov.

2. Let $\mathbb{Q}^+$ (resp. $\mathbb{R}^+$) denote the set of positive rational (resp. real) numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{R}^+$ that satisfy

$$f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)} \qquad \text{for all } x, y \in \mathbb{Q}^+ .$$

Solved by Michel Bataille, Rouen, France; and by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Bataille's solution.

We show that the only solution is $x \mapsto \dfrac{1}{x^2}$.

Let f satisfy the given functional equation, denoted by (E) in what follows. With $x = y = 1$, (E) readily gives $f(2) = \frac{1}{4}$. Let $a = f(1)$. With $y = 1$ in (E), we obtain

$$f(x+1) = \frac{f(x)}{(2x+1)f(x) + a}, \qquad (1)$$

which successively yields $f(3) = f(2+1) = \dfrac{1}{4a+5}$ and $f(4) = f(3+1) = \dfrac{1}{4a^2+5a+7}$.

However, from (E) with $x = y = 2$, we also deduce $f(4) = \frac{1}{16}$. Comparing with (1), we see that $4a^2 + 5a - 9 = 0$, and, since $a > 0$, it follows that $a = f(1) = 1$.

Now, (1) rewrites as $\dfrac{1}{f(x+1)} = \dfrac{1}{f(x)} + 2x + 1$ and an easy induction shows that

$$\frac{1}{f(x+n)} = \frac{1}{f(x)} + 2nx + n^2 \tag{2}$$

for all positive integers n and rationals x. Thus, $f(n) = \dfrac{1}{n^2}$ for all n in $\mathbb{N}$ ($x = 1$ in (2)) and

$$\frac{1}{f(\frac{1}{n} + n)} = \frac{1}{f(1/n)} + 2 + n^2.$$

Comparing with the result given by (E) with $x = n$ and $y = \frac{1}{n}$, we have

$$\left(f\left(\frac{1}{n}\right)\right)^2 - \left(n^2 - \frac{1}{n^2}\right) f\left(\frac{1}{n}\right) - 1 = 0$$

and $f\left(\frac{1}{n}\right) = n^2$ follows.

Lastly, if $m, n \in \mathbb{N}$, taking $x = m$ and $y = \frac{1}{n}$ in (E) and using (2) lead to

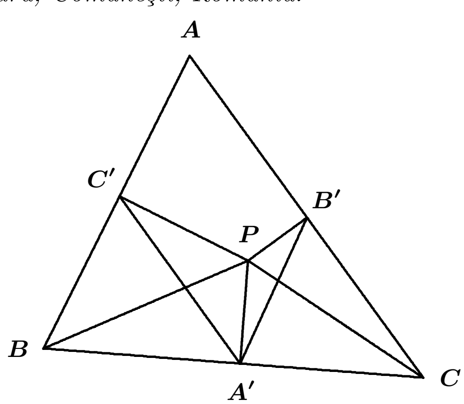
$$\frac{1}{m^2} + n^2 + \frac{2m}{n} f\left(\frac{m}{n}\right) = f\left(\frac{m}{n}\right)\left(\frac{1}{n^2} + \frac{2m}{n} + m^2\right)$$

and a short calculation gives $f\left(\frac{m}{n}\right) = \left(\frac{n}{m}\right)^2$. As a result, $f(x) = \dfrac{1}{x^2}$ for all rational numbers x.

Conversely, it is easily checked that $x \mapsto \dfrac{1}{x^2}$ satisfies (E) for all rational numbers x, y.

3. The point P lies inside the acute-angled triangle ABC and C', A' and B' are the feet of the perpendiculars from P to AB, BC, CA. Find all positions of P such that $\angle BAC = \angle B'A'C'$ and $\angle CBA = \angle C'B'A'$.

Solved by Titu Zvonaru, Comăneşti, Romania.



Since the quadrilaterals $PA'BC'$ and $PA'CB'$ are cyclic, we have

$$\begin{aligned}\angle BAC &= \angle B'A'C' = \angle B'A'P + \angle PA'C' \\ &= \angle PCA + \angle PBA.\end{aligned}$$

It results that

$$\begin{aligned}\angle BPC &= 180^\circ - \angle PBC - \angle PCB \\ &= 180^\circ - (\angle ABC - \angle PBA) - (\angle BCA - \angle PCA) \\ &= 180^\circ - \angle ABC - \angle BCA + \angle PBA + \angle PCA \\ &= \angle BAC + \angle BAC = 2\angle BAC,\end{aligned}$$

hence $\angle BPC = 2\angle BAC$.

We deduce that the point P lies on an arc of a circle which passes through B and C.

Similarly, we deduce that the point P lies on an arc of a circle which passes through A and C.

It follows that there exists at most one point P satisfying the given condition.

Since it is easy to see that the circumcentre satisfies the problem (the medial triangle is similar to the given triangle), we conclude that the desired point is the circumcentre.

And next we look at solutions for problems of the Bundeswettebwerb Mathematik 2007 given at [2010 : 22].

1. Show that one can distribute the integers from 1 to 4014 on the vertices and the midpoints of the sides of a regular 2007-gon so that the sum of the three numbers along any side is constant.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Starting at any vertex number the vertices counterclockwise from v_1 to v_{2007}. Setting $v_{2008} = v_1$, label as m_k the midpoint of side $v_k v_{k+1}$. For $k = 1, 2, \ldots, 2007$, assign the integer $2k - 1$ to m_k. For $k = 1, 2, \ldots, 1004$, assign the integer $2k$ to $v_{2009-2k}$, and for $k = 1005, 1006, \ldots, 2007$, assign the integer $2k$ to $v_{4016-2k}$. Then m_j has the value $2j - 1$, v_{2j} has the value $4016 - 2j$, and v_{2j+1} has the value $2008 - 2j$. Hence, the sum on side $v_{2j} m_{2j} v_{2j+1}$ is $(4016 - 2j) + (4j - 1) + (2008 - 2j) = 6023$, and the sum on side $v_{2j-1} m_{2j-1} v_{2j}$ is $(20080(2j - 2)) + (4j - 3) + (4016 - 2j) = 6023$.

2. Each positive integer is coloured either red or green so that

(a) The sum of three (not necessarily different) red numbers is red.

(b) The sum of three (not necessarily different) green numbers is green.

(c) There is at least one green number and one red number.

Find all colourings that satisfy these conditions.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and by Titu Zvonaru, Comǎneşti, Romania. We give Zvonaru's response.

Let R be the set of red numbers and G be the set of green numbers.

Let a, b be two positive integers such that $a \neq b$, $a \in R$, $b \in G$. It is easy to see that

$$a + a + a = 3a, \quad a + a + 3a = 5a, \quad a + a + 5a = 7a, \ldots \in R$$

and

$$b + b + b = 3b, \quad b + b + 3b = 5b, \quad b + b + 5b = 7b, \ldots \in G.$$

It results that

$$R \text{ and } G \text{ are infinite sets.} \tag{1}$$

To make a choice, we assume that $1 \in R$: It follows that R contains all odd positive integers.

If we suppose that the even integer $2k \in R$, then an easy induction shows that every integer t, with $t \geq 2k$ belongs to R.

Since $1 \in R$ and $2k \in R$, then $1 + 1 + 2k = 2(k + 1) \in R$ and so on.

We deduce that $\{t, t \geq 2k\} \subset R$, hence the set G is finite — a contradiction with (1).

It results that we have two possibilities:

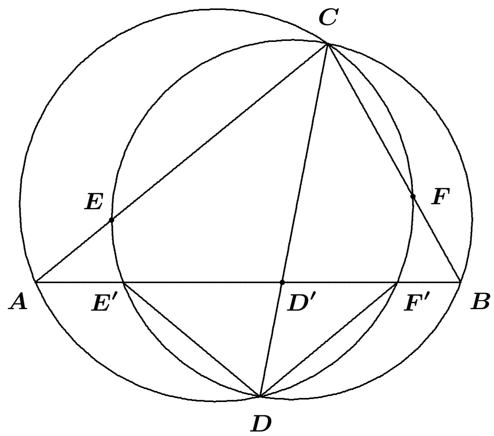
- (i) R is the set of all odd positive integers; G is the set of all even positive integers.

- (ii) R is the set of all even positive integers; G is the set of all odd positive integers.

3. In triangle ABC the points E and F lie in the interiors of sides AC and BC (respectively) so that $|AE| = |BF|$. Furthermore, the circle through A, C and F and the circle through B, C and E intersect in a point $D \neq C$.

Prove that the line CD is the bisector of $\angle ACB$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by Michel Bataille, Rouen, France. We give the solution of Amengual Covas.



Let the circle through B, C and E intersect AB at $E' \neq B$.

Let the circle through A, C, and F intersect AB at $F' \neq A$.

We denote by D' the point where the line CD intersects AB.

Observing that E, E', B and C are concyclic, as are F, F', A and C, we have $AE \cdot AC = AE' \cdot AB$ and $BF \cdot BC = BF' \cdot BA$ implying $\dfrac{AE \cdot AC}{BF \cdot BC} = \dfrac{AE' \cdot AB}{BF' \cdot BA}$ which simplifies to

$$\frac{AC}{BC} = \frac{AE'}{BF'} \tag{1}$$

because $AE = BF$ by hypothesis.

Also, since E, E', B and C are concyclic, as are F, F', A, and C, then $DD' \cdot D'C = BD' \cdot D'E'$ and $DD' \cdot D'C = AD' \cdot D'F'$ implying $AD' \cdot D'F' = BD' \cdot D'E'$.

Hence

$$\begin{aligned} \frac{AD'}{BD'} &= \frac{D'E'}{D'F'} \\ &= \frac{AD' - D'E'}{BD' - D'F'} \\ &= \frac{AE'}{BF'} \\ &= \frac{AC}{BC} \quad \text{by (1).} \end{aligned}$$

By the converse of the internal angle bisector theorem, then, CD' is the bisector of $\angle ACB$. That is, the line CD bisects $\angle ACB$.

4. Let a be a positive integer. How many nonnegative integers x satisfy

$$\left\lfloor \frac{x}{a} \right\rfloor = \left\lfloor \frac{x}{a+1} \right\rfloor ?$$

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Bataille.

We show that the required number is $\dfrac{a(a+1)}{2}$.

By long division, we may write any nonnegative integer x as $qa(a+1)+r$ for some nonnegative integers q and r such that $r < a(a+1)$. Using the fact that $\lfloor m+u \rfloor = m + \lfloor u \rfloor$ for $m \in \mathbb{Z}$ and $u \in \mathbb{R}$, the proposed equation becomes

$$q + \left\lfloor \frac{r}{a} \right\rfloor = \left\lfloor \frac{r}{a+1} \right\rfloor . \tag{1}$$

Now, $\dfrac{r}{a} \geq \dfrac{r}{a+1}$, hence $\left\lfloor \dfrac{r}{a} \right\rfloor \geq \left\lfloor \dfrac{r}{a+1} \right\rfloor$ and (1) cannot be satisfied if $q \geq 1$. Thus, the desired number is also the number of elements $r \in A$ with $A = \{0, 1, 2, \dots, a(a+1) - 1\}$ such that

$$\left\lfloor \frac{r}{a} \right\rfloor = \left\lfloor \frac{r}{a+1} \right\rfloor . \tag{2}$$

Any $r \in A$ satisfies $ka \leq r < (k+1)a$ for some unique $k \in \{0, 1, 2, \dots, a\}$ and then $\left\lfloor \dfrac{r}{a} \right\rfloor = k$. Observing that such an r satisfies $\dfrac{r}{a+1} < \dfrac{(k+1)a}{a+1} < k+1$, we see that this r is a solution if and only if $\dfrac{r}{a+1} \geq k$, that is $r \geq ka + k$. As a result, solutions r with $ka \leq r < (k+1)a$ exist if and only if $k \leq a-1$, in which case the solutions are $ka+k,\ ka+k+1,\ \dots,\ ka+a-1$. Thus, for each $k \in \{0, 1, 2, \dots, a-1\}$, we obtain $(a-1)-(k-1) = a-k$ solutions and no other solution exists. In conclusion the number of solutions is

$$(a-0) + (a-1) + \cdots + (a-(a-2)) + (a-(a-1)) = \frac{a(a+1)}{2}.$$

Next we move to the March 2010 number of the *Corner* and solutions to problems of the Republic of Moldova Selection tests for BMO 2007 and IMO 2007 given at [2010 : 81–83].

1. In triangle ABC the points M, N and P are the midpoints of the sides BC, AC and AB, respectively. The lines AM, BN and CP intersect the circumcircle of ABC at A_1, B_1 and C_1, respectively. Prove that the area of the triangle ABC does not exceed the sum of the areas of the triangles BA_1C, AB_1C and AC_1B.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Michel Bataille, Rouen, France; by Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India; and by Titu Zvonaru, Cománeşti, Romania. We give the solution of De.

Notation: $[XYZ]$ = Area of the triangle XYZ. m_a = median on the side of a triangle with length a.

We see at once that

$$\frac{[BA_1C]}{[ABC]} = \frac{MA_1}{AM}, \quad \frac{[AB_1C]}{[ABC]} = \frac{NB_1}{BN}, \quad \frac{[AC_1B]}{[ABC]} = \frac{PC_1}{CP}.$$

Let $BC = a$, $CA = b$ and $AB = c$. Chords AA_1 and BC of the circumcircle of triangle ABC intersect at M. Therefore $AM \cdot MA_1 = BM \cdot MC$. Also M is the midpoint of BC. Therefore $BM = MC = \frac{1}{2}a$ and hence $MA_1 = \dfrac{a^2}{4AM}$. Thus

$$\frac{MA_1}{AM} = \frac{a^2}{4AM^2} = \left(\frac{a}{2m_a}\right)^2.$$

Similarly we can show that $\dfrac{NB_1}{BN} = \left(\dfrac{b}{2m_b}\right)^2$ and $\dfrac{PC_1}{CP} = \left(\dfrac{c}{2m_c}\right)^2$. Therefore

$$\frac{[BA_1C]}{[ABC]} + \frac{[AB_1C]}{[ABC]} + \frac{[AC_1B]}{[ABC]} = \frac{1}{4}\left(\frac{a^2}{m_a^2} + \frac{b^2}{m_b^2} + \frac{c^2}{m_c^2}\right) \dots \qquad (1)$$

Recall that

$$\begin{aligned} a^2 &= \frac{4}{9}(2(m_b^2 + m_c^2) - m_a^2) \\ b^2 &= \frac{4}{9}(2(m_c^2 + m_a^2) - m_b^2) \\ c^2 &= \frac{4}{9}(2(m_a^2 + m_b^2) - m_c^2). \end{aligned}$$

Using these in (1) we obtain

$$\frac{1}{4}\left(\frac{a^2}{m_a^2} + \frac{b^2}{m_b^2} + \frac{c^2}{m_c^2}\right) = \frac{1}{9}\left(2\left(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - 3\right) \dots, \qquad (2)$$

where

$$x = \frac{m_a^2}{m_b^2}, \quad y = \frac{m_b^2}{m_c^2}, \quad \text{and} \quad z = \frac{m_c^2}{m_a^2}.$$

Now, $2(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}) - 3 \geq 9$ because $x + \frac{1}{x} \geq 2$, $y + \frac{1}{y} \geq 2$ and $z + \frac{1}{z} \geq 2$. Thus from (1) and (2) we can conclude that

$$[BA_1C] + [AB_1C] + [AC_1B] \geq [ABC].$$

2. Let p be a prime number, $p \neq 2$, and $m_1, m_2, \dots, m_p$ positive consecutive integers, and σ a permutation of the set $A = \{1, 2, \dots, p\}$. Prove that the set A contains 2 distinct numbers k and l such that p divides $m_k \cdot m_{\sigma(k)} - m_l \cdot m_{\sigma(l)}$.

Solved by Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India.

The residue classes modulo p of the p consecutive positive integers is $C = \{0, 1, 2, \ldots, p-1\}$. Let $m_k \equiv 0 \pmod p$ for some $k \in A$ and $\sigma(k) \neq k$. Then there exists some $l \in A$, $l \neq k$ such that $\sigma(l) = k$. Thus we obtain two distinct positive integers m and l such that $p \mid (m_k m_{\sigma(k)} - m_l m_{\sigma(l)})$.

Now suppose for some positive integer $j \in A$ we have $m_j \equiv 0 \pmod p$ and $\sigma(j) = j$. Consider the set $A' = A - \{j\}$.

Claim. There exist distinct positive integers k and l in A' such that $p \mid (m_k m_{\sigma(k)} - m_l m_{\sigma(l)})$.

To prove it assume on the contrary that no such pair of positive integers exists. Then $\{m_r m_{\sigma(r)} \pmod p : r \in A'\} = \{1, 2, \ldots, p-1\}$. Now observe that

$$\prod_{r \in A'} m_r m_{\sigma(r)} \equiv (p-1)! \pmod p \ldots. \tag{1}$$

Again observe that

$$\prod_{r \in A'} m_r m_{\sigma(r)} \equiv \left(\prod_{r \in A'} m_r\right)\left(\prod_{r \in A'} m_{\sigma(r)}\right) \equiv ((p-1)!)^2 \pmod p) \ldots. \tag{2}$$

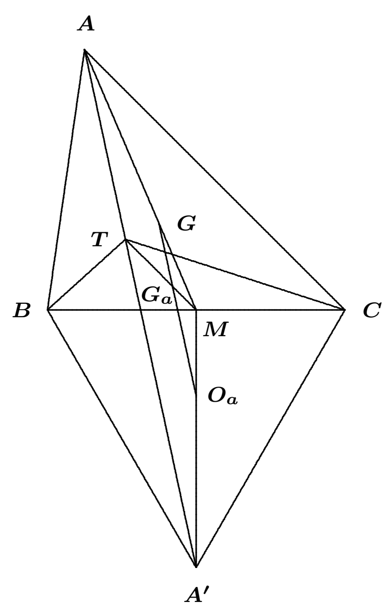
From (1) and (2) we conclude that $(p-1)! \equiv 1 \pmod p$ and this contradicts Wilson's Theorem. Thus our assumption is wrong and the claim is correct.

3. Inside the triangle ABC there exists a point T such that

$$m(\angle ATB) = m(\angle BTC) = m(\angle CTA) = 120^\circ.$$

Prove that the Euler lines of the triangles ATB, BTC and ATC are concurrent.

Solved by Titu Zvonaru, Comăneşti, Romania.



Let M be the midpoint of BC, and let G_a be the centroid of the triangle BTC. Let A' be the point on the other side of BC as A such that $\triangle BA'C$ is equilateral.

It is known that the point T lies on AA'. Since $\angle BTC + \angle BA'C = 180^\circ$, the quadrilateral $BA'CT$ is cyclic; it results that the circumcentre O_a of $\triangle BA'C$ is the circumcentre of $\triangle BTC$. We deduce that O_aG_a is the Euler line of $\triangle BTC$. Since $\dfrac{G_aM}{G_aT} = \dfrac{1}{3} = \dfrac{MO_a}{O_aA'}$, we obtain that $O_aG_a \| A'T$.

Now, denoting $G = O_aG_a \cap AM$, by similarity, it results that $\dfrac{GM}{GA} = \dfrac{O_aM}{O_aA'} = \dfrac{1}{3}$, hence G is the centroid of $\triangle ABC$.

Similarly, we deduce that the Euler lines of $\triangle CTA$ and $\triangle ATB$ pass through the centroid of $\triangle ABC$, hence the three Euler lines are concurrent.

5. Determine the smallest positive integers m and k such that:

(a) there exist $2m+1$ consecutive positive integers whose sum of cubes is a perfect cube;

(b) there exist $2k+1$ consecutive positive integers whose sum of squares is a perfect square.

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We use Manes' solution.

For part (a), $m = 1$ since there exist three consecutive positive integers whose cubes sum to a perfect cube; namely $3^3 + 4^3 + 5^3 = 6^3$. For part (b), we will show that $k = 5$.

Let $s(n, 2k+1) = n^2 + (n+1)^2 + \cdots + (n+2k)^2$ be the sum of the squares of $2k+1$ consecutive integers, the smallest of which is n. If $k = 1$, then

$$s(n-1,3) \;=\; (n-1)^2 + n^2 + (n+1)^2 \;=\; 3n^2 + 2 \;\equiv\; 2 \pmod{3}\,.$$

Therefore, $s(n-1,3)$ is not a perfect square since x^2 is not congruent to 2 modulo 3 for any integer x.

If $k = 2$, then

$$\begin{aligned} s(n-2,5) &= (n-2)^2 + (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 \\ &= 5(n^2+2) \;\equiv\; 2 \text{ or } 3 \pmod{4}\,. \end{aligned}$$

Thus, $s(n-2,5)$ is not a perfect square since $x^2 \equiv 0$ or $1 \pmod{4}$ for every integer x.

If $k = 3$, assume that $s(n-3,7) = r^2$ for some integer r. This equation reduces to $7(n^2+4) = r^2$. Hence, 7 divides r so that $r = 7t$ for some integer t. Therefore, $n^2 + 4 = 7t^2$ and so, $n^2 + 4 \equiv 0 \pmod{7}$ or $n^2 \equiv 3 \pmod{7}$, a contradiction since 3 is not a quadratic residue of 7. Thus, $s(n-3,7)$ is not a perfect square.

If $k = 4$, assume that $s(n-4,9) = r^2$ for some integer r. This equation reduces to $3(3n^2+20) = r^2$. Therefore, 3 divides r so that $r^2 = 9t^2$ for some integer t, whence $3n^2+20 = 3t^2$. Thus, 3 divides 20, a contradiction that shows $s(n-4,9)$ is not a perfect square.

If $k = 5$, assume that $s(n-5,11) = m^2$ for some integer m. Then $s(n-5,11) = 11(n^2+10) = m^2$ implies that 11 divides m so that $m^2 = 11^2t^2$ for some integer t. Therefore $n^2 + 10 = 11t^2$ or $n^2 - 1 \equiv 0 \pmod{11}$. Thus, $n \equiv \pm 1 \pmod{11}$, and so $n = 11j \pm 1$ for some integer j. Then

$$\begin{aligned} s(n-5,11) &= 11(n^2+10) \;=\; 11[(11j \pm 1)^2 + 10] \\ &= 11^2[11j^2 \pm 2j + 1] \;=\; 11^2[10j^2 + (j \pm 1)^2]. \end{aligned}$$

The problem now reduces to finding the smallest value of j so that $10j^2+(j\pm1)^2$ is a perfect square. The value $j=1$ is easily dispensed with. However, for $j=2$, $10j^2+(j+1)^2=7^2$ and $n=11j+1=23$. Accordingly, $s(18,11)=77^2$ or

$$18^2+19^2+20^2+\cdots+28^2 \;=\; 77^2.$$

6. Let I be the incenter of triangle ABC and let R be the circumradius. Prove that $AI+BI+CI \;\leq\; 3R$.

Solved by Arkady Alt, San Jose, CA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Titu Zvonaru, Cománeşti, Romania. We give the version of Amengual Covas.

We write a, b, c respectively for the lengths of the sides BC, CA, AB, and $s=\frac{a+b+c}{2}$ for the semiperimeter. Let r be the radius of the incircle.

Note that $\cos\frac{A}{2}$ may be expressed as $\frac{s-a}{AI}$, and also as $\sqrt{\frac{s(s-a)}{bc}}$. Equating these and solving for AI, we get $AI=\sqrt{\frac{bc(s-a)}{s}}$, with symmetric results for BI and CI.

Applying the Cauchy's inequality $\vec{u}\cdot\vec{v} \;\leq\; \|\vec{u}\|\cdot\|\vec{v}\|$ with $\vec{u}=\left(\sqrt{bc},\sqrt{ca},\sqrt{ab}\right)$ and $\vec{v}=\left(\sqrt{\frac{s-a}{s}},\sqrt{\frac{s-b}{s}},\sqrt{\frac{s-c}{s}}\right)$, we now get

$$\begin{aligned} AI+BI+CI &= \sqrt{\frac{bc(s-a)}{s}}+\sqrt{\frac{ca(s-b)}{s}}+\sqrt{\frac{ab(s-c)}{s}} \\ &\leq \sqrt{ab+bc+ca} \end{aligned} \qquad (1)$$

Using the relations $ab+bc+ca=r^2+4Rr+s^2$, $r\leq\frac{R}{2}$ and $s\leq\frac{3\sqrt{3}}{2}R$, we have

$$ab+bc+ca\leq\left(\frac{R}{2}\right)^2+4R\cdot\frac{R}{2}+\left(\frac{3\sqrt{3}}{2}R\right)^2=9R^2 \qquad (2)$$

By (1) and (2), we obtain the desired inequality. Since equality in (1) and (2) holds if and only if $a=b=c$, it holds in the required inequality if and only if $\triangle ABC$ is equilateral.

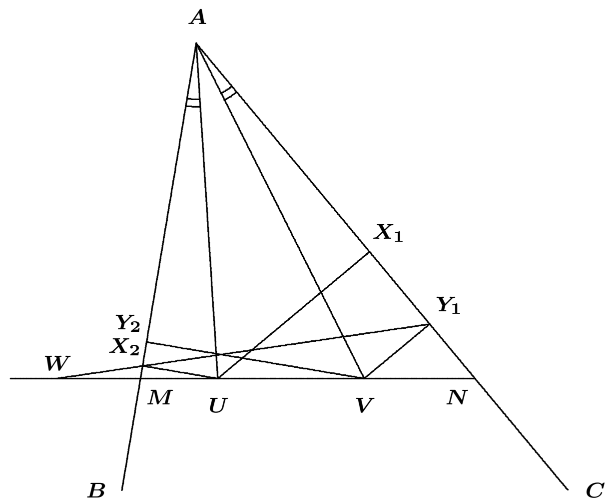
Comment. Also solved, by using the Erdős-Mordell inequality, on page 38 of the book Experiences in Problem Solving: A W. J. Blundon Commemorative, Atlantic Provinces Council on the Sciences, Canada, (1994).

7. Let U and V be two points inside the angle BAC such that

$$m(\angle BAU) \;=\; m(\angle CAV)\,.$$

Denote projections from U and V on the angle sides AC, AB as X_1, X_2 and Y_1, Y_2 respectively. Let W be the intersection of the lines X_2Y_1 and X_1Y_2. Prove that U, V, W are collinear.

Solved by Titu Zvonaru, Comăneşti, Romania.



Let the lines UV and AB intersect at M, and the lines UV and AC intersect at N. We denote

$$\alpha = \angle BAC, \quad \gamma = \angle ANM, \quad \beta = \angle AMN, \quad \varphi = \angle BAV = \angle CAV.$$

Suppose that the lines X_2Y_1 and MN intersect at the point W. We will prove that the points W, Y_2, X_1 are collinear.

By Menelaus' theorem we obtain:

$$\frac{WM}{WN} \cdot \frac{Y_1N}{Y_1A} \cdot \frac{X_2A}{X_2N} = 1 \quad \Leftrightarrow \quad \frac{WM}{WN} \cdot \frac{VN\cos\gamma}{AV\cos\varphi} \cdot \frac{AV\cos\varphi}{MU\cos\beta} = 1$$
$$\Leftrightarrow \quad \frac{WM}{WN} = \frac{AV \cdot MU\cos\beta}{AU \cdot VN\cos\gamma} \qquad (1)$$

By the converse of Menelaus' theorem, we have to prove that

$$\frac{WM}{WN} \cdot \frac{X_1N}{X_1A} \cdot \frac{Y_2A}{Y_2M} = 1 \quad \Leftrightarrow \quad \frac{WM}{WN} \cdot \frac{UN\cos\gamma}{AU\cos(\alpha-\varphi)} \cdot \frac{AV\cos(\alpha-\varphi)}{MV\cos\beta}$$
$$\Leftrightarrow \quad \frac{WM}{WN} = \frac{AU \cdot MV \cdot \cos\beta}{AV \cdot UN \cdot \cos\gamma}.$$

By (1), it suffices to prove that

$$\frac{AV \cdot MU}{AU \cdot VN} = \frac{AU \cdot MV}{AV \cdot UN} \Leftrightarrow \frac{AU^2}{AV^2} = \frac{MU \cdot UN}{NV \cdot VM},$$

which is Steiner's Theorem with respect to isogonal cevians.

Here a proof: We denote by $[XYZ]$ the area of $\triangle XYZ$. We have

$$\begin{aligned}\frac{MU}{NV}\cdot\frac{UN}{VM} &= \frac{[AMU]}{[AVM]}\cdot\frac{[AUN]}{[ANV]}\\ &= \frac{AM\cdot AU\cdot\sin\varphi}{AN\cdot AV\sin(\alpha-\varphi)}\cdot\frac{AU\cdot AN\sin(\alpha-\varphi)}{AV\cdot AM\cdot\sin\gamma}\\ &= \frac{AU^2}{AV^2}\,.\end{aligned}$$

9. Let $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ be real numbers in the interval $[0,1]$. Let $S = a_1^3 + a_2^3 + \cdots + a_n^3$. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{2n+1+S-a_i^3} \;\leq\; \frac{1}{3}\,.$$

Solved by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and by Titu Zvonaru, Cománeşti, Romania. We give the solution of Díaz-Barrero.

Since a_i, $1 \leq i \leq n$, lie in the interval $[0,1]$, then $1 - a_i^3 \geq 0$ for $1 \leq i \leq n$, and

$$\sum_{i=1}^{n} \frac{a_i}{2n+1+S-a_i^3} \leq \sum_{i=1}^{n} \frac{a_i}{2n+S}.$$

So, it will suffice to prove that

$$\sum_{i=1}^{n} \frac{a_i}{2n+S} \leq \frac{1}{3}$$

or equivalently,

$$3(a_1+a_2+\ldots+a_n) \leq 2n+S = \sum_{i=1}^{n}(1+1+a_i^3)$$

which trivially holds on account of AM-GM inequality. Indeed,

$$\sum_{i=1}^{n}(1+1+a_i^3) \geq \sum_{i=1}^{n} 3\sqrt[3]{1\cdot 1\cdot a_i^3} = 3\sum_{i=1}^{n} a_i.$$

Equality holds when $a_1 = a_2 = \ldots = a_n = 1$, and we are done.

10. Find all polynomials f with integer coefficients, such that $f(p)$ is a prime for every prime p.

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

Note that the polynomial $f(x) = x$ and the constant polynomials $f(x) = c$ where c is a prime satisfy the requirements in the problem. They are the only such polynomials that do.

Assume that f is a polynomial with integer coefficients such that if p is a prime, then $f(p)$ is a prime and also assume that $f(x) \neq x$ and $f(x) \neq c$. Then there exists a prime π such that $\gcd(\pi, f(\pi)) = 1$ since otherwise p divides $f(p)$ for all primes p and it would follow that either $f(x) = 0$ or $f(p) = p$ so that $f(x) = x$, both contradictions. By Dirichlet's Theorem, there exist infinitely many integers n_i such that $n_i f(\pi) + \pi$ is a prime, say $n_i f(\pi) + \pi = q_i$, $i = 1, 2, 3, \ldots$. Since $q_i - \pi$ divides $f(q_i) - f(\pi)$ we get that $f(\pi)$ divides $f(q_i)$ for each $i \geq 1$. Hence $f(x) = f(\pi)$, a contradiction. As a result, the only polynomials f with integer coefficients for which $f(p)$ is a prime for every prime p are $f(x) = x$ and $f(x) = c$ where c is a prime.

11. Let ABC be a triangle with $a = BC$, $b = AC$, $c = AB$, inradius r and circumradius R. Let r_A, r_B and r_C be the radii of the excircles of the triangle ABC. Prove that

$$a^2\left(\frac{2}{r_A} - \frac{r}{r_B r_C}\right) + b^2\left(\frac{2}{r_B} - \frac{r}{r_C r_A}\right) + c^2\left(\frac{2}{r_C} - \frac{r}{r_B r_A}\right) = 4(R+3r)\,.$$

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give Kandall's write-up.

Let $K = [ABC]$, $s = \frac{a+b+c}{2}$. It is well known (Heron) that

$$\begin{aligned} 16K^2 &= (a+b+c)(-a+b+c)(a-b+c)(a+b-c) \\ &= -(a^4+b^4+c^4) + 2(a^2b^2+b^2c^2+a^2c^2) \end{aligned}$$

and $K = rs = r_A(s-a) = r_B(s-b) = r_C(s-c) = \dfrac{abc}{4R}$.

Let

$$X = a^2\left(\frac{2}{r_A} - \frac{r}{r_B r_c}\right) = b^2\left(\frac{2}{r_B} - \frac{r}{r_C r_A}\right) + c^2\left(\frac{2}{r_C} - \frac{r}{r_B r_A}\right).$$

We have

$$\begin{aligned} a^2\left(\frac{2}{r_A} - \frac{r}{r_B r_C}\right) &= a^2\left(\frac{2(s-b)}{K} - \frac{K}{s}\cdot\frac{s-b}{K}\cdot\frac{s-c}{K}\right) \\ &= \frac{a^2}{sK}\left(2\cdot\frac{a+b+c}{2}\cdot\frac{a-b+c}{2}\right. \\ &\qquad \left. - \frac{a-b+c}{2}\cdot\frac{a+b-c}{2}\right) \\ &= \frac{1}{4sK}(-3a^4 + 3a^2b^2 + 3a^2c^2 + 2a^2bc). \end{aligned}$$

Similarly,

$$b^2\left(\frac{2}{r_B}-\frac{r}{r_Cr_A}\right) = \frac{1}{4sK}(-3b^4+3a^2b^2+3b^2c^2+2ab^2c),$$

$$c^2\left(\frac{2}{r_C}-\frac{r}{r_Br_A}\right) = \frac{1}{4sK}(-3c^4+3a^2c^2+3b^2c^2+2abc^2).$$

Consequently,

$$\begin{aligned} X &= \frac{1}{4Sk}(-3(A^4+B^4+C^4) \\ &\qquad +6(A^2B^2+B^2C^2+A^2C^2)+2ABC(A+B+C)) \\ &= \frac{1}{4(\frac{K}{r})K}\left(3\cdot 16K^2+2\cdot 4RK\cdot 2\frac{K}{r}\right) \\ &= 4(R+3r), \end{aligned}$$

as required.

12. Consider n distinct points in the plane $n \geq 3$, arranged such that the number $r(n)$ of segments of length l is maximized. Prove that $r(n) \leq \frac{n^2}{3}$.

Solved by Oliver Geupel, Brühl, NRW, Germany.

We will apply Turán's Theorem: Let G be a simple graph with n vertices which does not contain a complete subgraph containing p vertices. Let r be the remainder of n modulo p. Then the number of edges of G is not greater than

$$f(n,p) = \frac{(p-2)n^2-r(p-1-r)}{2(p-1)}.$$

Consider the graph G whose vertices are the n given points and where two vertices P and Q are connected by an edge if and only if $PQ = l$. Clearly, G does not contain a complete subgraph with 4 vertices. By Turán's Theorem, the number of edges of G is not greater than

$$f(n,4) = \frac{2n^2-r(3-r)}{6} \leq \frac{n^2}{3}.$$

Remark. Problem A-6 of the 49th William Lowell Putnam Competition (1978) asked for proving the inequality $r(n) < 2n^{3/2}$. This is a sharper upper bound for each $n \geq 36$. See: *The William Lowell Putnam Mathematical Competition problems and solutions: 1965-1984*, ed. by G.L. Alexanderson, L.F. Klosinski, and L.C. Larson, MAA, 1986, p. 104f.

14. Let $b_1, b_2, \ldots, b_n$ $(n \geq 1)$ be nonnegative real numbers at least one of which is positive. Prove that $P(X) = X^n - b_1X^{n-1} - \cdots - b_{n-1}X - b_n$, has a single positive root p, which is simple, and that the absolute value of each root of $P(X)$ is not greater than p.

Solved by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Oliver Geupel, Brühl, NRW, Germany. We give the solution by Díaz-Barrero.

We consider the continuous function $A : (0, +\infty) \to \mathbb{R}$ defined by

$$A(x) = \frac{b_1}{x} + \frac{b_2}{x^2} + \ldots + \frac{b_n}{x^n} - 1$$

Since $A'(x) = -\frac{b_1}{x^2} - \frac{2b_2}{x^3} - \ldots - \frac{nb_n}{x^{n+1}} < 0$ for all $x > 0$, then A is a decreasing continuous function. Furthermore, $\lim_{x\to+\infty} A(x) = -1$ and $\lim_{x\to 0^+} A(x) = +\infty.$ Therefore, on account of Bolzano's theorem the equation $A(x) = -\frac{P(x)}{x^n} = 0$ has only one positive root, say $p,$ which is a zero of polynomial P. On the other hand, from $A'(p) < 0$ follows $P'(p) > 0$ and p is a simple zero.

To see that all the zeros of P have modulus less than or equal to p we argue by contradiction. Assume that x_0 is a zero of P and let $|x_0| = \alpha$ with $\alpha > p$. Then $A(\alpha) < A(p) = 0$ and $P(\alpha) > 0.$ On the other hand from $x_0^n = b_n + b_{n-1}x_0 + \ldots + b_1 x_0^{n-1}$ we have

$$|x_0^n| = |b_n + b_{n-1}x_0 + \ldots + b_1 x_0^{n-1}| \le b_n + b_{n-1}|x_0| + \ldots + b_1|x_0^{n-1}|$$

from which follows

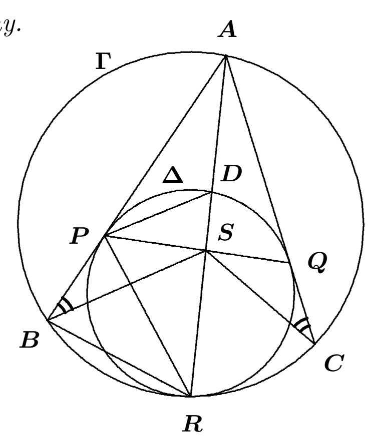
$$P(\alpha) = \alpha^n - b_1\alpha^{n-1} - \ldots - b_{n-1}\alpha - b_n < 0$$

Contradiction, and we are done.

Comment. The first part of the statement also follows immediately applying the well-known Descartes Rule of Signs.

15. A circle is tangent to the sides AB and AC of the triangle ABC and to its circumcircle at P, Q and R respectively. If $PQ \cap AR = \{S\}$ prove that $m(\angle SBA) = m(\angle SCA)$.

Solved by Oliver Geupel, Brühl, NRW, Germany.



Let Γ be the circle through A, B, and C, and let Δ be the circle through P, Q, and R. Let D be the second intersection of Δ and the line AR. Since AP is tangent to Δ, we have $\angle PDR = \angle BPR$. Since Δ and Γ have a common tangent t at R, it holds $\angle DPR = \angle(AR, t) = \angle ABR = \angle PBR$. Hence, the triangles DPR and PBR are similar. It follows that $\angle ARP = \angle PRD = \angle BRP$, that is, the line RP is the internal bisector of $\angle R$ in $\triangle ABR$.

Therefore,

$$\frac{AP}{BP} = \frac{AR}{BR}.$$

Similarly,

$$\frac{AQ}{CQ} = \frac{AR}{CR}.$$

By $AP = AQ$, we obtain

$$\frac{BP}{BR} = \frac{CQ}{CR}.$$

By

$$\frac{PS}{QS} = \frac{\sin\angle PAS}{\sin\angle QAS} = \frac{\sin\angle BAR}{\sin\angle CAR} = \frac{BR}{CR},$$

we deduce that

$$\frac{BP}{CQ} = \frac{PS}{QS}.$$

By $\angle BPS = \angle CQS$, it follows that $\triangle BPS \sim \triangle CQS$. Consequently,

$$\angle SBA = \angle SBP = \angle SCQ = \angle SCA,$$

which completes the proof.

16. Prove that there are infinitely many primes p for which there exists a positive integer n such that p divides $n! + 1$ and n does not divide $p - 1$.

Comment by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

A proof is by Paul Erdős, (c.f. p. 558 of G.E. Hardy and M.V. Subbarao, “A modified Problem of Pillai and Some Related Questions”, The American Math. Monthly, Vol 109, pp. 554–559).