New Editor-in-Chief for *CRUX with MAYHEM*

We happily herald Shawn Godin of Cairine Wilson Secondary School, Orleans, ON, Canada as the next Editor-in-Chief of *Crux Mathematicorum with Mathematical Mayhem* effective 21 March 2011.

With this issue Shawn takes the reins, and though some of you know Shawn as a former *Mayhem* Editor let me provide some further background.

> Shawn grew up outside the small town of Massey in Northern Ontario. He obtained his B. Math from the University of Waterloo in 1987. After a few years trying to decide what to do with his life Shawn returned to school obtaining his B.Ed. in 1991.
>
> Shawn taught at St. Joseph Scollard Hall S.S. in North Bay from 1991 to 1998. During this time he married his wife Julie, and his two sons Samuel and Simon were born. In the summer of 1998 he moved with his family to Orleans where he has taught at Cairine Wilson S.S. ever since (with a three year term at the board office as a consultant). He also managed to go back to school part time and earn an M.Sc. in mathematics from Carleton University in 2002.
>
> Shawn has been involved in many mathematical activities. In 1998 he co-chaired the provincial mathematics education conference. He has been involved with textbook companies developing material for teacher resources and the web as well as writing a few chapters for a grade 11 textbook. He has developed materials for teachers and students for his school board, as well as the provincial ministry of education. Shawn is involved with the problem committees at the Centre for Education in Mathematics and Computing at the University of Waterloo where he currently works on the committee for the Fryer, Galois and Hypatia contests. He is a frequent presenter at local, regional and provincial math education conferences.
>
> In 1997 Shawn met Graham Wright, the former executive director of the CMS, at a conference. When Shawn moved to Ottawa Graham put him to work as Mayhem editor from 2001 to 2006 as well as helping with math camps in Ottawa.
>
> In his free time, Shawn enjoys hanging out with his wife and kids, reading (mainly science fiction and mathematics), and "playing" guitar.

Shawn has already been receiving your submissions, and please send any future submissions to the addresses and e-mail addresses listed inside the back cover, with your full name and affiliation on each page.

Václav (Vazz) Linek, Herald and Guest Editor.

EDITORIAL

Shawn Godin

Hello ***CRUX with MAYHEM***, it is great to be back! I always thought at some point I would come back to the editorial board, but I thought it would be when I retired and not as Editor-in-Chief. Sometimes life throws you a surprise and you have to roll with it.

This is an interesting time for the journal. The readership is down a bit and Robert Woodrow is stepping down as the editor of the Olympiad corner after serving Crux for many, many years, so we are looking to make some changes. Ultimately, we want a publication that meets the needs and wants of its readers, so we will be looking to you for some feedback. In a future issue we will look to get some ideas from you. What do you like about ***CRUX with MAYHEM***? What would you like changed? Are there things that we are not currently doing that we should be doing? Are there things that we are doing that we need to discontinue? Are there alternate formats that we should explore for ***CRUX with MAYHEM***? We will need your input, so please start thinking so that you can give us some feedback later.

It took quite a while for me to be officially appointed the Editor-in-Chief, and as a result we are a few months behind schedule. We will be doing our best to make up some of that time and get us closer to our real time line. To help facilitate this, all deadlines for solutions will appear as if we are on time. Having said that you will have some extra time before the material will be processed. We will post the status of the problems processing on the CMS web site, **cms.math.ca/crux**.

I want to say a quick thank you to Vazz Linek, the rest of the members of the board and the staff at the CMS for all their help getting me on my feet. The learning curve is steep but I am really looking forward to working on ***CRUX with MAYHEM***. I know we can work together and continue the great work that Vazz, Jim, Bruce, Robert, Bill, Leo, Fred and all the other members of the ***CRUX with MAYHEM*** board past and present, have done. Let's get started!

Shawn Godin

SKOLIAD No. 130

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **September 15, 2011**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

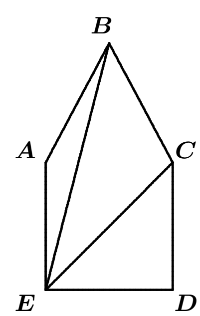
Our contest this month is the Niels Henrik Abel Mathematics Contest, 2009–2010, Second Round. Our thanks go to Øyvind Bakke, Norwegian University of Science and Technology, Trondheim, Norway, for providing us with this contest and for permission to publish it. We also thank Rolland Gaudet, University College of Saint Boniface, Winnipeg, MB, for translating this contest from English into French.

Niels Henrik Abel Mathematics Contest, 2009–2010 2nd Round
100 minutes allowed

1. A four-digit whole number is *interesting* if the number formed by the leftmost two digits is twice as large as the number formed by the rightmost two digits. (For example, 2010 is interesting.) Find the largest whole number, d, such that all interesting numbers are divisible by d.

2. A calculator performs this operation: It multiplies by 2.1, then erases all digits to the right of the decimal point. For example, if you perform this operation on the number 5, the result is 10; if you begin with 11, the result is 23. Now, if you begin with the whole number k and perform the operation three times, the result is 201. Find k.

3. The pentagon $ABCDE$ consists of a square, $ACDE$, with side length 8, and an isosceles triangle, ABC, such that $AB = BC$. The area of the pentagon is 90. Find the area of $\triangle BEC$.



4. In how many ways can one choose three different integers between 0.5 and 13.5 such that the sum of the three numbers is divisible by 3?

5. If a and b are positive integers such that $a^3 - b^3 = 485$, find $a^3 + b^3$.

6. If a and b are positive integers such that $a^3 + b^3 = 2ab(a + b)$, find $a^{-2}b^2 + a^2b^{-2}$.

7. Let D be the midpoint of side AC in $\triangle ABC$. If $\angle CAB = \angle CBD$ and the length of AB is 12, then find the square of the length of BD.

8. If x, y, and z are whole numbers and $xyz+xy+2yz+xz+x+2y+2z = 28$ find $x + y + z$.

9. Henrik's math class needs to choose a committee consisting of two girls and two boys. If the committee can be chosen in **3630** ways, how many students are there in Henrik's math class?

10. Let S be $1!(1^2 + 1 + 1) + 2!(2^2 + 2 + 1) + 3!(3^2 + 3 + 1) + \cdots + 100!(100^2 + 100 + 1)$. Find $\dfrac{S+1}{101!}$. (As usual, $k! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (k-1) \cdot k$.)

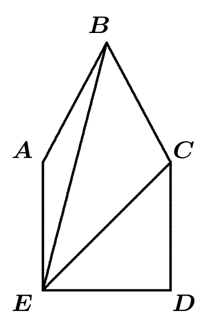
Concours Mathématique Niels Henrik Abel, 2009–2010 $2^{\text{ième}}$ ronde

100 minutes sont accordées

1. Un entier à quatre chiffres est intéressant si l'entier formé par les deux chiffres à l'extrême gauche est deux fois plus grand que l'entier formé par les deux chiffres à l'extrême droite. (Par exemple, **2010** est intéressant.) Déterminer le plus grand entier, d, tel que tous les nombres intéressants sont divisibles par d.

2. Une calculatrice effectue cette opération : elle multiplie par **2,1**, puis elle efface tous les chiffres à droite de la décimale. Par exemple, si on effectue cette opération à partir de **5**, le résultat est **10** ; à partir de **11**, le résultat est **23**. Or, si on commence avec un entier k et qu'on effectue cette opération trois fois, le résultat est **201**. Déterminer k.

3. Le pentagone $ABCDE$ consiste d'un carré, $ACDE$, de côtés de longueur **8**, puis d'un triangle isocèle, ABC, tel que $AB = BC$. La surface du pentagone est **90**. Déterminer la surface de $\triangle BEC$.



4. De combien de façons pouvons-nous choisir trois entiers différents entre $0,5$ et $13,5$, tels que la somme des trois entiers soit divisible par 3 ?

5. Si a et b sont des entiers positifs tels que $a^3 - b^3 = 485$, déterminer $a^3 + b^3$.

6. Si a et b sont des entiers positifs tels que $a^3 + b^3 = 2ab(a + b)$, déterminer $a^{-2}b^2 + a^2b^{-2}$.

7. Soit D le mipoint du côté AC dans $\triangle ABC$. Si $\angle CAB = \angle CBD$ et si la longueur de AB est 12, déterminer le carré de la longueur de BD.

8. Si x, y et z sont des entiers et si $xyz + xy + 2yz + xz + x + 2y + 2z = 28$, déterminer $x + y + z$.

9. La classe de mathématiques d'Henri a besoin de choisir un comité formé de deux filles et de deux garçons. Si ce comité peut être formé de 3630 façons, combien d'étudiants y a-t-il dans la classe de mathématiques d'Henri ?

10. Soit S égal à $1!(1^2+1+1)+2!(2^2+2+1)+3!(3^2+3+1)+\cdots+100!(100^2+100+1)$. Déterminer $\frac{S+1}{101!}$. (Comme d'habitude, $k! = 1\cdot 2\cdot 3\cdot\cdots\cdot(k-1)\cdot k$.)

Next follow solutions to the selected problems from the 10th Annual Christopher Newport University Regional Mathematics Contest, 2009, given in Skoliad 124 at [2010:129–131]. (*Note: Problems 1, 2, 3 and 4 first appeared on the 2009 Calgary Junior Math Contest. – Ed.*)

1. Elves and ogres live in the land of Pixie. The average height of the elves is 80 cm, the average height of the ogres is 200 cm, and the average height of the elves and the ogres together is 140 cm. If 36 elves live in Pixie, how many ogres live there?

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

Let x be the number of ogres in Pixie. Then the total height of all the ogres is $200x$, and the total height of all the elves is $36\cdot 80 = 2880$. Therefore the total height of all the creatures in Pixie is $2880+200x$. On the other hand, the average height of the $36+x$ creatures in Pixie is 140, so their total height is $140(36+x) = 5040+140x$. Thus $2880+200x = 5040+140x$, so $x = 36$.

Also solved by WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.

2. You are given a two-digit positive integer. If you reverse the digits of your number, the result is a number which is 20% larger than your original number. What is your original number?

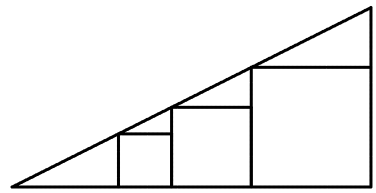
Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Let x be the given two-digit number. Increasing x by 20%, that is, by $\frac{1}{5}x$, yields an integer, so x must be divisible by 5. Thus x ends in 0 or in 5. If the ones digit of x is 0, reversing the digits would decrease the number, so x must end in 5. If the tens digit is larger than 5, reversing the digits would again decrease the number. Thus only the numbers 15, 25, 35, 45, and 55 remain to be checked. Only 45 works out.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.

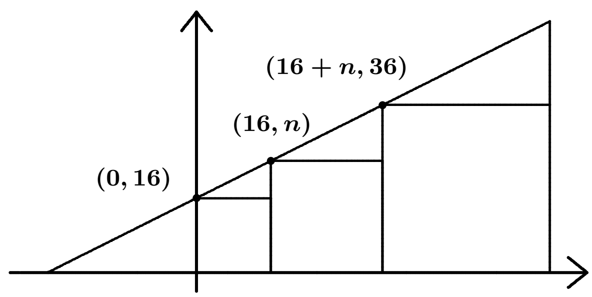
Alternatively, let $10a+b$ *be the two-digit number, where* a *and* b *are digits. Increasing by* 20% *is the same as multiplying by* $\frac{6}{5}$*, so* $10b+a = \frac{6}{5}(10a+b)$*. Thus* $5(10b+a) = 6(10a+b)$, *so $44b = 55a$, so $4b = 5a$. Clearly b is divisible by 5, and since $b = 0 = a$ does not yield a two-digit number, b must be 5. Hence $a = 4$ and the number is 45.*

3. Three squares are placed side-by-side inside a right-angled triangle as shown in the diagram. The side length of the smallest of the three squares is 16. The side length of the largest of the three squares is 36. What is the side length of the middle square?



Solution by Wen-Ting Fan, student, Burnaby North Secondary School, Burnaby, BC.

Impose a coordinate system as in the figure. If the middle square has side length n, then the coordinates are as indicated. Since the slanted line passes through $(0, 16)$, the equation of the line is $y = mx + 16$ for some slope, m.



Using the two other points yields

$$n = 16m + 16 \quad \text{and} \quad 36 = m(16 + n) + 16\,.$$

Therefore $36 = m(16 + 16m + 16) + 16 = 16m^2 + 32m + 16$, so $0 = 16m^2 + 32m - 20 = 4(2m + 5)(2m - 1)$, so $x = -\frac{5}{2}$ or $x = \frac{1}{2}$. In the figure, the slope is clearly positive, so $m = \frac{1}{2}$, and $n = 16m + 16 = 24$.

Also solved by LISA WANG, student, Port Moody Secondary School, Port Moody, BC. You can also solve this problem using similar triangles.

4. Friends Maya and Naya ordered finger food in a restaurant, Maya ordering chicken wings and Naya ordering bite-size ribs. Each wing cost the same amount, and each rib cost the same amount, but one wing was more expensive than one rib. Maya received 20% more pieces than Naya did, and Maya paid 50% more in total than Naya did. The price of one wing was what percentage higher than the price of one rib?

Solution by Lisa Wang, student, Port Moody Secondary School, Port Moody, BC.

Say Naya gets n pieces at \$$r$ each. Then Maya gets $1.2n$ pieces at, say, \$$w$ each. Then Naya pays \$$nr$ and Maya pays \$$1.2nw$. Since Maya pays 50% more than Naya, $1.2nw = 1.5nr$, so $w = \frac{1.5}{1.2}r = 1.25r$, so one wing is 25% more expensive than one rib.

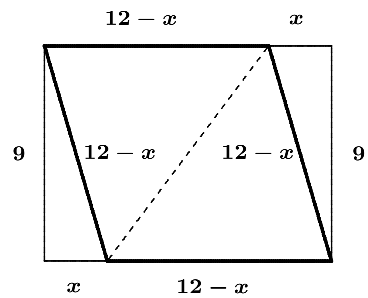
Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC.

5. A 9×12 rectangular piece of paper is folded once so that a pair of diagonally opposite corners coincide. What is the length of the crease?

Solution by Wen-Ting Fan, student, Burnaby North Secondary School, Burnaby, BC.

If you fold the paper as instructed and unfold it again, you obtain the figure below where the section outlined with thick lines used to overlap and the dashed line is the crease. The Pythagorean Theorem now yields that

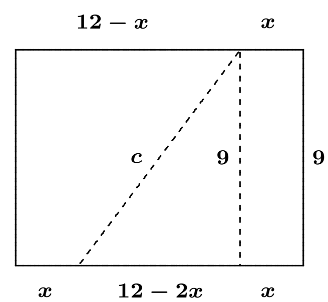
$$\begin{aligned} 9^2 + x^2 &= (12-x)^2 \\ 81 + x^2 &= 144 - 24x + x^2 \\ 24x &= 63 \\ x &= \frac{63}{24} \end{aligned}$$



Then redraw the diagram as in the figure below. Use the Pythagorean Theorem again:

$$c^2 = 9^2 + (12-2x)^2.$$

Since $x = \frac{63}{24}$, $c^2 = \frac{2025}{16}$, so $c = \frac{45}{4}$.



Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.

6. In calm weather, an aircraft can fly from one city to another 200 kilometres north of the first and back in exactly two hours. In a steady north wind, the round trip takes five minutes longer. Find the speed (in kilometres per hour) of the wind.

Solution by the editors.

The airspeed of the plane is $\frac{400}{2} = 200$ kilometres per hour. Let w denote the speed of the wind. Then, if you fly with the wind, the ground speed is $200+w$; and if you fly against the wind, the ground speed is $200-w$. Therefore, the plane takes $\dfrac{200}{200+w}$ hours to fly with the wind and $\dfrac{200}{200-w}$ hours to fly against the wind. If you add these two expressions, you get the total time for the round trip, but this time is given to be $\mathbf{2} + \frac{\mathbf{5}}{\mathbf{60}}$ hours, so

$$\begin{aligned} \frac{200}{200-w} + \frac{200}{200+w} &= \frac{25}{12} \\ \frac{8(200+w)+8(200-w)}{(200-w)(200+w)} &= \frac{1}{12} \\ \frac{3200}{200^2-w^2} &= \frac{1}{12} \\ w = \pm 40\,. \end{aligned}$$

Therefore the speed of the wind is **40** kilometres per hour.

7. A rectangular floor, **24** feet × **40** feet, is covered by squares of sides **1** foot. A chalk line is drawn from one corner to the diagonally opposite corner. How many tiles have a chalk line segment on them?

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Since $\frac{24}{8} = 3$ and $\frac{40}{8} = 5$, consider instead the 3×5 rectangle on the left in the figure. You can easily count that the diagonal crosses seven squares.

Now tile the 24×40 rectangle with 3×5 rectangles as in the righthand side of the figure. The diagonal of the 24×40 rectangle is also the diagonal of each of eight of the 3×5 rectangles. Therefore the diagonal crosses **56** of the 1×1 squares.

This issue's prize of one copy of ***Crux Mathematicorum*** for the best solutions goes to Wen-Ting Fan, student, Burnaby North Secondary School, Burnaby, BC.

As Skoliad editors we are quite pleased to see envelopes with "exotic" stamps in the mail, but receiving more Canadian solutions would be wonderful. The address is on the inside of the back cover. You do not have to solve the entire featured contest; a well-presented solution to a single problem is enough. The test for "well-presented" is that your classmates at school can understand it. You can send your solution(s) by mail or electronically—even if that means that we miss out on the exotic stamps.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The interim Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School, Orleans, ON). The other staff member is Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON).

Mayhem Problems

*Please send your solutions to the problems in this edition by **15 August 2011**. Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

Note: As ***CRUX with MAYHEM*** is running behind schedule, we will accept solutions past the posted due date. Solutions will be accepted until we process them for publication. Currently we are delayed by about four months. Check the CMS website, **cms.math.ca/crux**, for our status in processing problems.

M470. *Proposed by the Mayhem Staff*

Vazz needs to buy desks and monitors for his new business. A desk costs $\$250$ and a monitor costs $\$260$. Determine all possible ways that he could spend exactly $\$10\,000$ on desks and monitors.

M471. *Proposed by the Mayhem Staff*

Square based pyramid $ABCDE$ has a square base $ABCD$ with side length 10. Its other four edges AE, BE, CE, and DE each have length 20. Determine the volume of the pyramid.

M472. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania*

Suppose that x is a real number. Without using calculus, determine the maximum possible value of $\dfrac{2x^2-8x+17}{x^2-4x+7}$ and the minimum possible value of $\dfrac{x^2+6x+8}{x^2+6x+10}$.

M473. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania*

Determine all pairs (a, b) of positive integers for which $a^2+b^2-2a+b = 5$.

M474. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia*

Let a, b and x be positive integers such that $x^2 - bx + a - 1 = 0$. Prove that $a^2 - b^2$ is not a prime number.

M475. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON*

Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For example, $\lfloor 3.1 \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$. Let $\{x\}$ denote the fractional part of the real number x, that is, $\{x\} = x - \lfloor x \rfloor$. For example, $\{3.1\} = 0.1$ and $\{-1.4\} = 0.6$. Show that there exist infinitely many irrational numbers x such that $x \cdot \{x\} = \lfloor x \rfloor$.

. . . .

M470. *Proposé par l'Équipe de Mayhem*

Vazz doit acheter des pupitres et des écrans pour son nouveau commerce. Un pupitre coûte **250$** et un écran **260$**. Trouver de combien de manières possibles il pourrait dépenser exactement **10 000$** en pupitres et écrans.

M471. *Proposé par l'Équipe de Mayhem*

Une pyramide $ABCDE$ a une base carrée $ABCD$ de côté 10. Les quatre autres arêtes AE, BE, CE et DE sont toutes de longueur 20. Trouver le volume de la pyramide.

M472. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie*

Supposons que x soit un nombre réel. Sans utiliser le calcul différentiel, déterminer la valeur maximale possible de $\dfrac{2x^2 - 8x + 17}{x^2 - 4x + 7}$ et la valeur minimale possible de $\dfrac{x^2 + 6x + 8}{x^2 + 6x + 10}$.

M473. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie*

Déterminer toutes les paires (a, b) d'entiers positifs tels que $a^2 + b^2 - 2a + b = 5$.

M474. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie*

Soit a, b et x trois entiers positis tels que $x^2 - bx + a - 1 = 0$. Montrer que $a^2 - b^2$ n'est pas un nombre premier.

M475. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON*

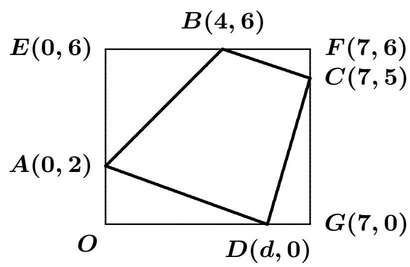
Notons $\lfloor x \rfloor$ le plus grand entier n'excédant pas x. Par exemple, $\lfloor 3{,}1 \rfloor = 3$ et $\lfloor -1{,}4 \rfloor = -2$. Notons $\{x\}$ la partie fractionnaire du nombre réel x, c.-à-d, $\{x\} = x - \lfloor x \rfloor$. Par exemple, $\{3{,}1\} = 0{,}1$ et $\{-1{,}4\} = 0{,}6$. Montrer qu'il existe une infinité de nombres irrationnels x tels que $x \cdot \{x\} = \lfloor x \rfloor$.

Mayhem Solutions

M432. *Proposed by the Mayhem Staff.*

Determine the value of d with $d > 0$ so that the area of the quadrilateral with vertices $A(0,2)$, $B(4,6)$, $C(7,5)$, and $D(d,0)$ is 24.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.



Let $E = (0,6), F = (7,6), G = (7,0)$ and Ω denote the area function. Then $\Omega(AOD) = \frac{1}{2}(d \times 2) = d$; $\Omega(BEA) = \frac{1}{2}(4 \times 4) = 8$; $\Omega(BFC) = \frac{1}{2}(3 \times 1) = \frac{3}{2}$; and $\Omega(CDG) = \frac{1}{2}(7-d)\times 5 = \frac{5}{2}(7-d)$. Since $\Omega(OEFG) = 7 \times 6 = 42$, we have

$$24 = \Omega(ABCD) = 42 - \left[d + 8 + \frac{3}{2} + \frac{5}{2}(7-d)\right] = 15 + \frac{3}{2}d.$$

Solving we find $d = 6$.

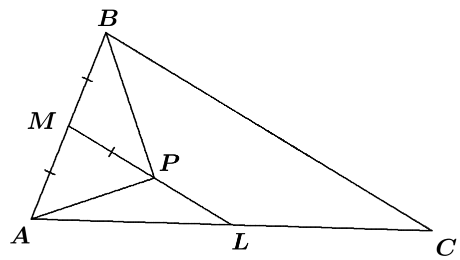
Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; GEOFFREY A. KANDALL, Hamden, CT, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; JOSHUA LONG, Southeast Missouri State University, Cape Girardeau, MO, USA; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania(2 solutions); and JOHN WYNN, student, Auburn University, Montgomery, AL, USA;

Two incorrect solutions were received.

M433. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

In triangle ABC, $AB < BC$, L is the midpoint of AC, and M is the midpoint of AB. Also, P is the point on LM such that $MP = MA$. Prove that $\angle PBA = \angle PBC$.

Solution by Souparna Purohit, student, George Washington Middle School, Ridgewood, NJ, USA.



It is well known that since M and L are the midpoints of AB and AC then $BC \parallel ML$ so $\angle PBC = \angle BPM$. Also, since $PM = AM = BM$, ΔBMP is isosceles. Therefore $\angle ABP = \angle BPM$ which, when combined with $\angle PBC = \angle BPM$, we conclude that $\angle ABP = \angle PBC$, as desired.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain(two solutions); GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

One incorrect solution was received. Several readers pointed out that ΔAPB is right angled with the right angle at P.

M434. *Proposed by Heisu Nicolae, Pîrjol Secondary School, Bacău, Romania.*

Determine all eight-digit positive integers $abcdefgh$ which satisfy the relations $a^3 - b^2 = 2$, $c^3 - d^2 = 4$, $2^e - f^2 = 7$, and $g^3 - h^2 = -1$.

Solution by Arkady Alt, San Jose, CA, USA.

Since $2 \le a^3 \le 9^2 + 2 = 83 \Leftrightarrow 2 \le a \le 4$ and $a^3 - 2$ for such a can only be square for $a = 3$, then $a = 3, b = 5$.

Since $4 \le c^3 \le 9^2 + 4 = 85 \Leftrightarrow 2 \le c \le 4$ and $c^3 - 4$ for such c can only be square for $c = 2$ then $c = 2, d = 2$.

Since $7 \le 2^e \le 9^2 + 7 = 88 \Leftrightarrow 3 \le e \le 6$ and $2^e - 7$ for such e can only be square for $e = 3$, $e = 4$ and $e = 5$ then $(e, f) = (3, 1), (4, 3), (5, 5)$.

Since $0 \le g^3 \le 9^2 - 1 = 80 \Leftrightarrow 0 \le g \le 4$ and $g^3 + 1$ for such g can only be square for $g = 0$ and $g = 2$ then $(g, h) = (0, 1), (2, 3)$.

Thus $abcdefgh = 35\ 223\ 101, 35\ 224\ 301, 35\ 225\ 501, 35\ 223\ 123$, $35\ 224\ 323, 35\ 225\ 523$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer. Seven incomplete solutions were submitted. Most of the incomplete solutions missed the case where $g = 0$.

M435. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\sum_{k=1}^{n}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}} = \frac{n(n+2)}{n+1}.$$

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

For any $k>0$ we have

$$\begin{aligned}\left(1+\frac{1}{k}-\frac{1}{k+1}\right)^2 &= 1+\frac{1}{k^2}+\frac{1}{(k+1)^2}+\left(\frac{2}{k}-\frac{2}{k+1}\right)-\frac{2}{k(k+1)}\\ &= 1+\frac{1}{k^2}+\frac{1}{(k+1)^2}+\frac{2}{k(k+1)}-\frac{2}{k(k+1)}\\ &= 1+\frac{1}{k^2}+\frac{1}{(k+1)^2}.\end{aligned}$$

Hence $\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}} = 1+\frac{1}{k}-\frac{1}{k+1}$. Therefore if we let $S=\sum_{k=1}^{n}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}$ then

$$\begin{aligned}S &= \sum_{k=1}^{n}\left(1+\frac{1}{k}-\frac{1}{k+1}\right)\\ &= \left(1+\frac{1}{1}-\frac{1}{2}\right)+\left(1+\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(1+\frac{1}{n}-\frac{1}{n+1}\right)\\ &= n+1-\frac{1}{n+1}=\frac{n(n+2)}{n+1},\end{aligned}$$

and we are done!

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; G.C. GREUBEL, Newport News, VA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

M436. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Determine the smallest possible value of $x+y$, if x and y are positive integers with $\frac{2008}{2009} < \frac{x}{y} < \frac{2009}{2010}$.

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Let $y = x + d$ then $1 - \frac{1}{2009} < \frac{y-d}{y} < 1 - \frac{1}{2010}$ so $\frac{1}{2009} > \frac{d}{y} > \frac{1}{2010}$. If $d = 1$ there is no solution. If $d = 2$, $y = 4019$ is a solution so $x = 4017$ and $x + y = 8036$. If $d > 2$ then $x, y > 6000$ thus $x + y > 12000$. Therefore the minimum value of $x + y$ is 8036.

Also solved by ARKADY ALT, San Jose, CA, USA; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer. Five incorrect solutions were submitted.

Alt, Manes and Wang proved in general that if $\frac{n}{n+1} < \frac{x}{y} < \frac{n+1}{n+2}$ then the solution with the smallest sum corresponds to $x = 2n+1$, $y = 2n+3$ and thus $x+y = 4n+4$. In general for any two fractions of non-negative integers, in lowest terms, $\frac{a}{b} < \frac{c}{d}$ the value $\frac{a+c}{b+d}$ is called the mediant and it satisfies $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. If we also have $bc - ad = 1$ then the mediant is the fraction with the lowest denominator in the interval $\left(\frac{a}{b}, \frac{c}{d}\right)$.

M437. *Proposed by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.*

Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For example, $\lfloor 3.1 \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$. Let $\{x\}$ denote the fractional part of the real number x, that is, $\{x\} = x - \lfloor x \rfloor$. For example, $\{3.1\} = 0.1$ and $\{-1.4\} = 0.6$. Determine all rational numbers x such that $x \cdot \{x\} = \lfloor x \rfloor$.

Solution by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

The only rational number x such that $x \cdot \{x\} = \lfloor x \rfloor$ is $x = 0$.

If n is an integer, then $n \cdot \{n\} = 0 = \lfloor n \rfloor = n$ has the only solution $n = 0$. Therefore, 0 is the only integer solution to the equation.

Assume x is a rational number different from an integer such that $x \cdot \{x\} = \lfloor x \rfloor = x - \{x\}$, then $\{x\} = \frac{x}{x+1}$. Therefore, $x\left(\frac{x}{x+1}\right) = \lfloor x \rfloor$ implies $x^2 = \lfloor x \rfloor (x+1)$. Assume $x = \frac{m}{n}$ where m and n are relatively prime integers and $n > 1$. Then

$$\frac{m^2}{n^2} = \lfloor x \rfloor \left(\frac{m}{n} + 1\right) = \lfloor x \rfloor \left(\frac{m+n}{n}\right) .$$

As a result, $m^2 = \lfloor x \rfloor (m+n) \cdot n$ so that n is a divisor of m^2, a contradiction since m and n are relatively prime and $n > 1$.

Also solved by ARKADY ALT, San Jose, CA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer. Three incorrect solutions were submitted.

Problem of the Month

Ian VanderBurgh

Problems involving averages and their properties appear frequently on contests (*Look at the solution to question 1 of Skoliad on page 5 – Ed.*). This month and next, we will look at a few of these problems, at least one of which uses averages in a very subtle way.

Problem 1 (2008 Small c Contest) The average of three numbers is 13. Two numbers are added to this list so that the average of all five numbers is 17. What is the average of the two new numbers?

(A) 21 (B) 25 (C) 23 (D) 30 (E) 15

One of the things about average problems that I like is that there are really only about $1\frac{1}{2}$ things that you need to know about averages in order to be able to do almost all of such problems. (That's not to say that there isn't a plethora of tricks of the trade that can be useful...)

The first of these $1\frac{1}{2}$ important things is how to calculate an average: add up the given numbers, count the given numbers, and divide the sum by the count to get the average. The extra $\frac{1}{2}$ thing to remember is that the sum of the numbers equals the count times the average. Expressing these facts algebraically, we see that if there are n numbers whose sum is S, then the average, a, satisfies the equation $a = \dfrac{S}{n}$. Rearranging this gives $S = na$. (I concede that occasionally we might use the fact that $n = \dfrac{S}{a}$ as well.)

Let's solve Problem 1 using these properties and then look at our answer to see what we can observe.

Solution to Problem 1. Since the average of the original three numbers is 13, then their sum is $3 \times 13 = 39$. Since the average of all five numbers is 17, then the sum of the five numbers is $5 \times 17 = 85$.

The sum of the additional two numbers equals the sum of all five numbers minus the sum of the original three numbers, or $85 - 39 = 46$. Therefore, the average of these two numbers is $\dfrac{46}{2} = 23$. ■

This problem is particularly nice, in my opinion, because it doesn't require us to use any algebra. Let's look at the data that we have:

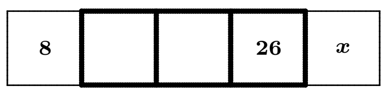
- the average of the first 3 numbers is 13
- the average of all 5 numbers is 17
- the average of the last 2 numbers is 23

Do you notice anything about the position of the overall average relative to the averages of the first and last numbers? You might have noticed that the overall average splits these averages in the ratio $\mathbf{4 : 6}$ which equals $\mathbf{2 : 3}$, which happens to be the ratio of the count of numbers in each partial average (arranged in reverse from what you might quickly guess).

If this rule works in general, then if we had $\mathbf{5}$ numbers with average $\mathbf{22}$ and $\mathbf{3}$ numbers with average $\mathbf{46}$, the average of all $\mathbf{8}$ numbers should split $\mathbf{22}$ and $\mathbf{46}$ in the ratio $\mathbf{3 : 5}$. In other words, the average is $\dfrac{3}{3+5} = \dfrac{3}{8}$ of the way from $\mathbf{22}$ to $\mathbf{46}$, and so equals $22 + \dfrac{3}{8} \times (46 - 22) = 31$. Try solving this problem using the method that we used above to confirm the answer.

Putting this in a more general way, if m numbers have an average of a and n numbers have an average of b with $a < b$, then the average of the $m + n$ numbers splits a and b in the ratio $n : m$ (not $m : n$). Can you prove this? We'll look at another problem next month where this approach is really useful.

Problem 2 (2010 Pascal Contest) In the diagram, each of the five boxes is to contain a number. Each number in a bold outlined box must be the average of the number in the box to the left of it and the number in the box to the right of it. What is the value of x?



(A) 28 (B) 30 (C) 31 (D) 32 (E) 34

Special cases often produce interesting facts. For example, if two numbers x and y have an average of a, then $\dfrac{x+y}{2} = a$ or $x + y = 2a$. Try to use this to solve the following problem algebraically.

Solution to Problem 2. We label the numbers in the empty boxes as y and z, so the numbers in the boxes are thus $8, y, z, 26, x$.

Since the average of z and x is 26, then $x + z = 2(26) = 52$ or $z = 52 - x$. We rewrite the list as $8, y, 52 - x, 26, x$.

Since the average of 26 and y is $52 - x$, then $26 + y = 2(52 - x)$ or $y = 104 - 26 - 2x = 78 - 2x$. We rewrite the list as $8, 78 - 2x, 52 - x, 26, x$.

Since the average of 8 and $52 - x$ is $78 - 2x$, then $8 + (52 - x) = 2(78 - 2x)$ or $60 - x = 156 - 4x$ and so $3x = 96$ or $x = 32$. ∎

Especially while writing a contest, it's very tempting to take the answer that we get and not think about it at all. But let's actually take this a moment to use this answer and go back to the list in terms of x (written as $8, 78 - 2x, 52 - x, 26, x$) and substitute to get the list $8, 14, 20, 26, 32$.

Do you recognize what kind of sequence this list forms? This is an arithmetic sequence. (Look up this term if you've never seen it before.) Do you think that this is a coincidence? (Hint: The answer to this question is almost always no.)

Let's think about this by going back to the list $\mathbf{8, y, z, 26, x}$. Let's avoid using algebra, but we'll keep these labels to make things a little clearer. We are told that $\boldsymbol{y}$ is the average of $\mathbf{8}$ and $\boldsymbol{z}$. The important fact to recognize here is that $\boldsymbol{y}$ is halfway between $\mathbf{8}$ and $\boldsymbol{z}$. In other words, the difference $\boldsymbol{y - 8}$ equals $\boldsymbol{z - y}$. Similarly, $\boldsymbol{z}$ is the average of $\mathbf{26}$ and $\boldsymbol{y}$, so $\boldsymbol{26 - z}$ equals $\boldsymbol{z - y}$. But there is a common difference in these two sentences! (And it's no coincidence that I used the phrase common difference...)

Since there is this common difference, then all three differences must be equal. Since $\mathbf{26 - 8 = 18}$, then each of these differences equals $\mathbf{18 \div 3 = 6}$, and so the numbers in the sequence are $\mathbf{8, 14, 20, 26,} \boldsymbol{x}$. Can you extend this argument another step to explain why $\boldsymbol{x = 32}$?

So what is the connection between averages and arithmetic sequences? An arithmetic sequence is a sequence with the property that each term after the first is the average of the term before and the term after. This is pretty neat, if you've never seen it before. One last thing to think about – the average is sometimes called the *arithmetic* mean. Coincidence?

Adams, Douglas (1952 - 2001) The first nonabsolute number is the number of people for whom the table is reserved. This will vary during the course of the first three telephone calls to the restaurant, and then bear no apparent relation to the number of people who actually turn up, or to the number of people who subsequently join them after the show/match/party/gig, or to the number of people who leave when they see who else has turned up. The second nonabsolute number is the given time of arrival, which is now known to be one of the most bizarre of mathematical concepts, a recipriversexcluson, a number whose existence can only be defined as being anything other than itself. In other words, the given time of arrival is the one moment of time at which it is impossible that any member of the party will arrive. Recipriversexclusons now play a vital part in many branches of math, including statistics and accountancy and also form the basic equations used to engineer the Somebody Else's Problem field. The third and most mysterious piece of nonabsoluteness of all lies in the relationship between the number of items on the bill, the cost of each item, the number of people at the table and what they are each prepared to pay for. (The number of people who have actually brought any money is only a subphenomenon of this field.) "Life, the Universe and Everything." New York: Harmony Books, 1982.

THE OLYMPIAD CORNER

No. 291

R.E. Woodrow

This number we begin by looking at the files of solutions by readers to problems given in the February 2010 number of the *Corner*, and "A" problems proposed but not used at the 2007 IMO in Vietnam, given at [2010 : 18–19].

A2. Let n be a positive integer, and let x and y be positive real numbers such that $x^n + y^n = 1$. Prove that

$$\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)(1-y)}.$$

Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Geupel.

From the identity

$$\frac{1+x^{2k}}{1+x^{4k}} + \frac{(1-x^{3k})(1-x^k)}{x^k(1+x^{4k})} = \frac{1+x^{2k}}{1+x^{4k}} + \frac{(1+x^{4k})-x^k(1+x^{2k})}{x^k(1+x^{4k})} = \frac{1}{x^k}$$

and the premise $0 < x < 1$ we deduce that

$$\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}} < \sum_{k=1}^{n} \frac{1}{x^k} = \frac{1-x^n}{x^n(1-x)}. \qquad (1)$$

Similarly we have

$$\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}} < \frac{1-y^n}{y^n(1-y)}. \qquad (2)$$

The hypothesis $x^n + y^n = 1$ yields

$$\frac{(1-x^n)(1-y^n)}{x^n y^n} = \frac{1-x^n-y^n+x^n y^n}{x^n y^n} = 1. \qquad (3)$$

The desired inequality follows immediately from the relations (1), (2), and (3).

A3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x+f(y)) = f(x+y) + f(y)$$

for all $x, y \in \mathbb{R}^+$. (Here $\mathbb{R}^+$ denotes the set of all positive real numbers.)

Solved by Mohammed Aassila, Strasbourg, France; and Michel Bataille, Rouen, France. We give Bataille's version.

We show that the unique solution is the function $\phi : x \mapsto 2x$.

Since $2(x+2y) = 2(x+y)+2y$ for all $x, y \in \mathbb{R}^+$, this function ϕ is a solution. Conversely, let f be any solution and x, y be any positive real numbers. On the one hand, adding x on both sides of the given equation and taking the images under f, we successively have

$$\begin{aligned} f(x+f(x+f(y))) &= f((x+f(y))+f(x+y)) \\ &= f(2x+y+f(y))+f(x+y) \\ &= f(2x+2y)+f(y)+f(x+y). \end{aligned}$$

On the other hand, using the given equation, we obtain

$$\begin{aligned} f(x+f(x+f(y))) &= f(2x+f(y))+f(x+f(y)) \\ &= f(2x+y)+f(y)+f(x+y)+f(y). \end{aligned}$$

It follows that

$$f(2x+2y) = f(2x+y)+f(y). \qquad (1)$$

Now, suppose that $0 < a < b$. We prove that $f(a) < f(b)$.

- If $b < 2a$, we take $x = \frac{2a-b}{2}$, $y = b-a$ in (1) and obtain $f(b) = f(a)+f(b-a)$, hence $f(b) > f(a)$.
- If $b > 2a$, with $x = \frac{b-2a}{2}$, $y = a$, (1) gives $f(b) = f(a)+f(b-a)$ and $f(b) > f(a)$ again.
- If $b = 2a$, $f(b) = f(2a) = f\left(2(\frac{a}{2}+\frac{a}{2})\right) = f\left(a+\frac{a}{2}\right)+f\left(\frac{a}{2}\right) > f(a)+f\left(\frac{a}{2}\right) > f(a)$. Thus, f is strictly increasing on $(0,\infty)$ and, as such, is injective.

In addition, if a, b are positive and distinct, say $a < b$, then $f(a+b) = f(a)+f(b)$ (for $f\left(a+2\cdot\frac{b-a}{2}\right)+f(a) = f\left(2a+2\cdot\frac{b-a}{2}\right)$).

Lastly, let $y > 0$. Since $f(y) \neq y$ (otherwise $f(x+f(y)) = f(x+y)$ in contradiction with the given functional equation), we may write $f(y+f(y)) = f(y)+f(f(y))$ as well as $f(y+f(y)) = f(2y)+f(y)$. It follows that $f(f(y)) = f(2y)$ and since f is injective, $f(y) = 2y$, as desired.

Next we look at the "C" problems proposed but not used at the 2007 IMO in Vietnam given at [2010: 19–20].

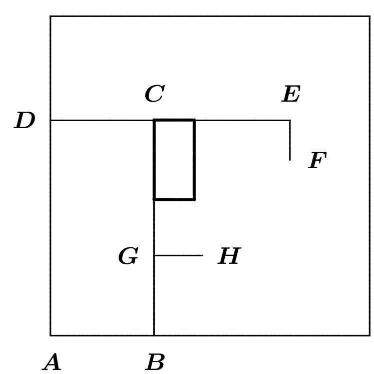
C2. A unit square is dissected into $n > 1$ rectangles such that their sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that one of the rectangles has no point on the boundary of the square.

Solved by Oliver Geupel, Brühl, NRW, Germany.

The proof is by contradiction.

Assume the contrary and consider a counterexample where n is minimal. Let A be one of the vertices of the square, let $\mathcal{E}$ be the locus of the sides of the rectangles of the dissection, and let $ABCD$ be the rectangle that contains the vertex A. Since the square is covered by the rectangles, at least one of the segments BC and DC has an extension in $\mathcal{E}$ beyond the point C. Without loss of generality assume that DC can be extended beyond C where the longest possible extension in $\mathcal{E}$ is up to a point E. Since the line DE intersects the interior of some rectangle, the point E is an interior point of the square. Since the square is covered by the rectangles, a segment EF orthogonal to CE where the points A and F are in the same half-plane relative to the line DE, also belongs to $\mathcal{E}$.

If there were a second rectangle beside $ABCD$ which contains the side BC, then it could be glued together with the rectangle $ABCD$ obtaining a counterexample with $n-1$ rectangles, which contradicts our minimum hypothesis. Hence, there is a segment GH in $\mathcal{E}$ where G is an inner point of BC, and GH is parallel to CE. The rectangle with vertex C and sides on the lines CE and CG is now separated from the boundary of the square by the four segments HG, GC, CE and EF. This is a contradiction which completes the proof.



C5. In the Cartesian coordinate plane let $S_n = \{(x,y) \mid n \leq x < n+1\}$ for each integer n, and paint each region S_n either red or blue. Prove that any rectangle whose side lengths are distinct positive integers may be placed in the plane so that its vertices lie in regions of the same colour.

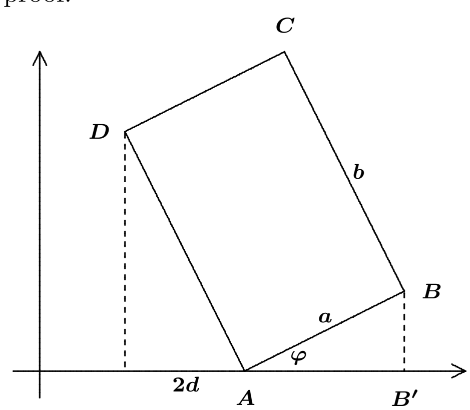
Solved by Oliver Geupel, Brühl, NRW, Germany.

We generalize that any rectangle $ABCD$ with distinct *real* sides $AB = a$ and $BC = b$ may be placed so that its vertices lie in regions of the same colour.

Firstly consider the case where $a \notin \mathbb{Z}$. Without loss of generality, we may assume that S_0 and S_1 have distinct colours. We place $ABCD$ so that A and D have the common x-coordinate $1 - \{a\}$ while B and C have the common x-coordinate $\lceil a \rceil$. We then translate the rectangle with offset $\{a\}$ in the positive x-direction. With this translation, A and D move from S_0 to S_1, while B and D remain in $S_{\lceil a \rceil}$. Consequently, in one of these two positions the vertices A, B, C, D lie in regions of the same colour.

It remains to consider $a, b \in \mathbb{Z}$. Let $d = \gcd(a, b)$, and let a_0, a_1, b_0, b_1 be integers such that $a = a_0 d$, $b = b_0 d$, and $a_0 a_1 + b_0 b_1 = 1$. The proof is by contradiction. Suppose $ABCD$ cannot be placed properly. Then S_0 and S_a have distinct colours, and S_0 and S_b have also distinct colours. By induction, S_0 and S_{ua+vb}, where u and v are integers, have distinct colours if and only if $u + v$ is odd. By $b_0 a - a_0 b = 0$, we see that $b_0 - a_0$ is even. Since a_0 and b_0 are coprime, both are odd. Hence, $a_1 + b_1$ is odd, too. Thus, S_0 and $S_{a_1 a + b_1 b} = S_d$ have distinct colours. We conclude that S_0 and S_{2d} have the same colour.

Assume $a < b$, which implies $b_0 \geq 3$. We pitch the rectangle so that the x-coordinates of A and D as well as the x-coordinates of B and C have distance $2d$. It suffices to prove that the we can translate it so that A and B lie in regions of the same colour. If the angle between the x-axis and the line AB is φ and A' and B' are the projections of A and B, respectively, onto the x-axis, then we obtain $\sin \varphi = \dfrac{2d}{b} = \dfrac{2}{b_0}$ and $A'B' = a \cos \varphi = \dfrac{a}{b_0}\sqrt{b_0^2 - 4} \notin \mathbb{Z}$. By the first part of the proof, we can place A' and B' so that they lie in regions of the same colour. This completes the proof.



C7. A convex n-gon P in the plane is given. For every three vertices of P, the triangle determined by them is *good* if all its sides are of unit length. Prove that P has at most $\frac{2}{3}n$ good triangles.

Comment by Mohammed Aassila, Strasbourg, France.

This is not an original problem. It first appeared in J. Pach and R. Pinchasi, *How many unit triangles can be generated by n points in convex position?*, American Math. Monthly 110 (5) 2003 : 40–406.

Next we move to the "G" problems proposed but not used at the 2007 IMO in Vietnam, given at [1020 : 20–21].

G3. Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle of ABC. Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C' respectively. Assume that the points A, B, C, A', B', C' are distinct, and that the lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

Solved by Michel Bataille, Rouen, France.

We denote by Γ the circumcircle of $\triangle ABC$ and by A_0, B_0, C_0 the points of intersection of the lines BB' and CC', CC' and AA', AA' and BB', respectively. We will use areal coordinates relatively to (A, B, C). The equation of the circle Γ is known to be $a^2yz + b^2zx + c^2xy = 0$ where $A = BC$, $b = CA$, $c = AB$. Let $P(x_0, y_0, z_0)$ and $A'(x', y', z')$; note that the line AA' has equation $yz' - zy' = 0$ and that (x_0, y_0, z_0), (x', y', z') are solutions to the system $a^2yz + b^2zx + c^2xy = 0$, $x(y_0 - z_0) = x_0(y - z)$. It readily follows that $\frac{y_0}{z_0}$, $\frac{y'}{z'}$ are solutions of an equation (with unknown U)

$$(c^2x_0)U^2 + \lambda U + (-b^2x_0) = 0$$

for some real λ. From $\frac{y_0}{z_0} \cdot \frac{y'}{z'} = -\frac{b^2}{c^2}$, we deduce $\frac{y'}{-b^2z_0} = \frac{z'}{c^2y_0}$ so that the equation of AA' is $(c^2y_0)y + (b^2z_0)z = 0$. Similarly, the equations of BB' and CC' are $(c^2x_0)x + (a^2z_0)z = 0$ and $(b^2x_0)x + (a^2y_0)y = 0$. Then, it is easily obtained that

$$A_0(-a^2y_0z_0, b^2x_0z_0, c^2x_0y_0), \quad B_0(a^2y_0z_0, -b^2x_0z_0, c^2x_0y_0),$$

$$C_0(a^2y_0z_0, b^2x_0z_0, -c^2x_0y_0).$$

Now, recall that if $M_i(x_i, y_i, z_i)$ with $x_i + y_i + z_i = 1$ for $i = 1, 2, 3$, the ratio $\frac{\text{area}(M_1M_2M_3)}{\text{area}(ABC)}$ is the absolute value of the determinant $\begin{vmatrix} x_1 & y_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$. Here, we have $\frac{\text{area }(A_0B_0C_0)}{\text{area }(ABC)} = |\triangle|$ where

$$\triangle = \frac{a^2y_0z_0}{-a^2y_0z_0 + b^2x_0z_0 + c^2x_0y_0} \cdot \frac{b^2x_0z_0}{a^2y_0z_0 - b^2x_0z_0 + c^2x_0y_0}$$

$$\cdot \frac{c^2x_0y_0}{a^2y_0z_0 + b^2x_0z_0 - c^2x_0y_0} \cdot \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}.$$

Since P is on Γ, $b^2x_0z_0 + c^2x_0y_0 = -a^2y_0z_0$ (for example) so that

$$\triangle = \frac{(a^2y_0z_0)(b^2x_0z_0)(c^2x_0y_0)}{(-2a^2y_0z_0)(-2b^2x_0z_0)(-2c^2x_0y_0)} \cdot 4$$

and

$$\text{area}\ (A_0B_0C_0) \;=\; \frac{1}{2}\text{area}\ (ABC)\,,$$

independent of P.

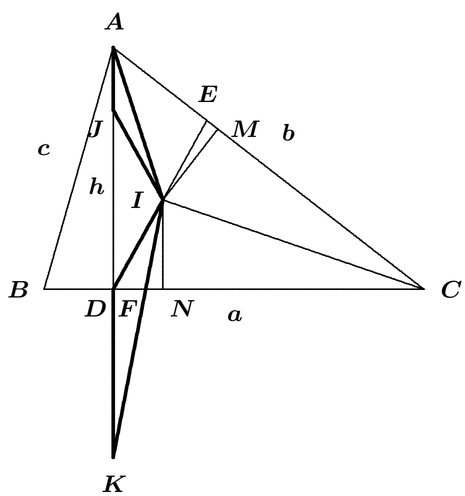
G5. Triangle ABC is acute with $\angle ABC > \angle ACB$, incentre I, and circumradius R. Point D is the foot of the altitude from vertex A, point K lies on line AD such that $AK = 2R$, and D separates A and K. Finally, lines DI and KI meet sides AC and BC at E and F, respectively.

Prove that if $IE = IF$ then $\angle ABC > 3\angle ACB$.

Solved by Oliver Geupel, Brühl, NRW, Germany.

The claim is false. We prove instead that $\angle ABC \leq 3\angle ACB$.

We write $a = BC$, $b = CA$, $c = AB$, $2s = a + b + c$, $h = AD$, $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle ACB$. We denote by r the inradius, by J the point on the segment AD such that $DJ = 2r$, and by M and N the perpendicular projections of I onto AC and BD, respectively.



By standard formulas, we have

$$AI^2 = \frac{r^2}{\sin(A/2)} = \frac{(s-a)(s-b)(s-c)}{s} \cdot \frac{bc}{(s-b)(s-c)} = \frac{(s-a)bc}{s},$$

$$2Rh = bc, \qquad 4Rr = \frac{abc}{[ABC]} \cdot \frac{[ABC]}{s} = \frac{abc}{s}.$$

We obtain $AI^2 = 2R(h - 2r) = AK \cdot AJ$; hence $\dfrac{AI}{AJ} = \dfrac{AK}{AI}$. By $\angle IAJ = \angle KAI$, it follows that $\triangle AIJ \sim \triangle AKI$. Thus, $\angle AIJ = \angle AKI = \angle IKD$. Recognizing the isosceles $\triangle IJD$, we deduce $\angle AJI = 180^\circ - \angle DJI = 180^\circ - \angle IDJ = \angle IDK$.

We obtain $\triangle AIJ \sim \triangle IKD$ and consequently

$$\angle DIK = \angle JAI = \angle BAI - \angle BAD = \frac{\alpha}{2} - (90^\circ - \beta) = \frac{\beta - \gamma}{2}.$$

By $IE = IF$, the triangles EIM and FIN are congruent. The point N is an inner point of the segment CF. Now, if M is between C and E, then $\gamma = 180^\circ - \angle MIN = \angle DIN + \angle EIM > \angle DIK = \dfrac{\beta-\gamma}{2}$; consequently $\beta < 3\gamma$. On the other hand, if the point E is between C and M, or $E = M$, then $\gamma = 180^\circ - \angle MIN = \angle DIK = \dfrac{\beta-\gamma}{2}$ which implies $\beta = 3\gamma$. We are done.

Next we move to the "N" problems proposed but not used at the 2007 IMO in Vietnam, given at [1020 : 21].

N1. Find all pairs (k, n) of positive integers for which $7^k - 3^n$ divides $k^4 + n^2$.

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.

We will demonstrate that there is only one such pair, namely $(k, n) = (2, 4)$. Suppose then that k, n are positive integers such that $(7^k - 3^n) \mid (k^4 + n^2)$; which means that

$$\left\{\begin{array}{l} k^4 + n^2 \;=\; r\cdot(7^k - 3^n), \\ \text{for some nonzero integer } r \\ k, n \in \mathbb{Z}^+. \end{array}\right\} \tag{1}$$

First we show that both k and n must be even. We do so by ruling out the other three possibilities: both k and n being odd, k odd and n even, or (third possibility) k even and n odd.

Possibility 1. both k and n are odd: $k \equiv n \equiv 1 \bmod 2$.

We then have $k^4 \equiv n^2 \equiv 1 \pmod 8$ (the square of an odd integer is congruent to 1 modulo 8. And so,

$$k^4 + n^2 \;\equiv\; 1 + 1 \;\equiv\; 2 \pmod 8\,. \tag{2}$$

Since k and n are both odd positive integers we also have

$$k \;=\; 2m+1, \quad n \;=\; 2l+1; \quad \text{where } m, l \text{ are nonnegative integers.}$$

Thus

$$\begin{aligned} 7^k - 3^n &= 7^{2m+1} - 3^{2l+1} \;\equiv\; (7^2)^m\cdot 7 - (3^2)^l \cdot 3 \\ &\equiv\; 1\cdot 7 - 1\cdot 3 \;\equiv\; 7-3 \;\equiv\; 4 \pmod 8\,. \end{aligned} \tag{3}$$

According to (3), 4 divides $7^k - 3^n$. And by (2), the highest power of 2 dividing $k^4 + n^2$; is $2^1 = 2$.

This renders equation (1) contradictory or impossible. Hence possibility 1 is ruled out.

Possibility 2. k is odd and n is even; $k \equiv 1 \pmod 2$, $n \equiv 0 \pmod 2$.

Clearly $7^k - 3^n \equiv 1 - 1 \equiv 0 \pmod{2}$, while $k^4 + n^2 \equiv 1 + 0 \equiv 1 \pmod{2}$, which renders (1) contradictory modulo 2: the left-hand side is congruent to 0 modulo 2. So this possibility is ruled out as well.

Possibility 3. k is even and n odd; $k \equiv 0 \pmod{2}$, $n \equiv 1 \pmod{2}$.

Same argument as in Possibility 2; the left-hand side is congruent to 1 but the right-hand side is zero modulo 2. So this possibility is eliminated as well.

We conclude that both positive integers k and n must be even:

$$\left\{ \begin{array}{l} k = 2K \quad \text{and} \quad n = 2N, \\ \text{for some positive integers } K \text{ and } N. \end{array} \right\} \tag{4}$$

From (1) and (4) we obtain,

$$16K^4 + 4N^2 \ = \ k^4 + n^2 \ = \ r \cdot (7^K - 3^N)(7^K + 3^N). \tag{5}$$

According to (5), the positive integer $7^K + 3^N$ is a divisor of $k^4 + n^2$. On the other hand,

$$7^5 \ = \ 16807 > 16 \cdot 5^4 \ = \ 10,000$$

(while $7^K < 16K^4$; for $K = 1, 2, 3, 4$). An easy induction shows that $7^K > 16 \cdot K^4$ for $K \geq 5$, we omit the details. Similarly we have $81 = 3^4 > 4 \cdot 4^2 = 64$ (while $3^N < 4N^2$ for $N = 1, 2, 3$). And an easy induction establishes that $3^N > 4N^2$, for $N \geq 4$.

Therefore for $K \geq 5$ and $N \geq 4$ we have $7^K + e^N > 16K^4 + 4N^2$; which implies that

$$(7^K + 3^N) \cdot |7^K - 3^N| \cdot |r| > 16K^4 + 4N^2 \tag{6}$$

since $|r| \cdot |7^K - 3^N|$ is a positive integer.

Clearly (6) condtradicts (5). We have demonstrated that a necessary condition for (5) to hold true is $K \leq 4$ and $N \leq 3$; which means that there is only up to 12 possible pairs (K, N) that may satisfy (5). We form the following table:

	$7^K + 3^N$	$2^4 \cdot K^4 + 2^2 \cdot N^2$
$K = 1, N = 1$	10	$20 = 16 + 4$
$K = 1, N = 2$	16	$32 = 16 + 16$
$K = 1, N = 3$	34	$52 = 16 + 36$
$K = 2, N = 1$	52	$256 + 4 = 260$
$K = 2, N = 2$	58	$256 + 16 = 272$
$K = 2, N = 3$	76	$256 + 36 = 296$
$K = 3, N = 1$	346	$(16)(81) + 4 = 1300$
$K = 3, N = 2$	352	$(16)(81) + 16 = 1312$
$K = 3, N = 3$	370	$1296 + 36 = 1332$
$K = 4, N = 1$	2404	$4096 + 4 = 4100$
$K = 4, N = 2$	2410	$4096 + 16 = 4112$
$K = 4, N = 3$	2428	$4096 + 36 = 4132$

The above table shows that the only pairs (K, N) for which $7^K + 3^N$ divides $2^4 \cdot K^4 + 2^2 \cdot N^2$ are $(K, N) = (1, 1), (1, 2)$. However, the pair $(1, 1)$

does not satisfy (5) since $20 = r \cdot 4 \cdot 10$, is impossible with $r \in \mathbb{Z}$. The other pair, $(K, N) = (1, 2)$ does: $32 = r \cdot (-2)(16)$, satisfied with $r = -1$. Thus $(K, N) = (1, 2)$ is the only pair; and so $(k, n) = (2K, 2N) = (2, 4)$.

N2. Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.

To show that b is the n^{th} power of an integer, it suffices to show that for every prime number p in the prime factorization of b, if p^e is the power of p that appears in the prime factorization of b, then the exponent e is a multiple of n. We set $b = p^e \cdot r$, r a positive integer such that p does not divide r; $(r, p) = 1$, e a positive integer.

We apply the hypothesis of the problem with $k = (p^e)^n = p^{en}$: $k \mid b - a_k^n$ means that there exists a positive integer λ such that

$$b - a_k^n = k \cdot \lambda;$$

$$b - a_k^n = p^{e \cdot n} \cdot \lambda;$$

and since $b = p^e \cdot r$, we obtain

$$\left\{ \begin{array}{c} p^e \cdot r - a_k^n \ = \ p^{e \cdot n} \cdot \lambda, \\ e, r, a_k, \lambda \text{ positive integers such that } (r, p) = 1 \end{array} \right\} \tag{1}$$

Since $n > 1$, it follows that

$$1 \ \leq \ e \ < \ e \cdot n \tag{2}$$

Let p^t, t a positive integer, be the highest power of p which divides a_k:

$$\left\{ \begin{array}{c} a_k \ = \ p^t \cdot b_k, \\ t, b_k \text{ positive integers such that } (b_k, p) \ = \ 1 \end{array} \right\} \tag{3}$$

From (1) and (3) we obtain

$$p^e \cdot r - p^{n \cdot t} \cdot b_k^n = p^{e \cdot n} \cdot \lambda. \tag{4}$$

We claim that (4) implies $e = n \cdot t$. Indeed, if $e \neq n \cdot t$, then either,

Possibility 1. $n \cdot t < e$, or,

Possibility 2. $e < n \cdot t$ holds.

If Possibility 1 holds then, by (2) we have:

$$1 \ \leq \ n \cdot t \ < \ e \ , \ e \cdot n. \tag{5}$$

And so (4) implies that

$$p^{(e - n \cdot t)} \cdot r - b_k^n \ = \ p^{c \cdot n - n \cdot t} \cdot \lambda, \tag{6}$$

which implies by (5) and (6) that $p \mid b_k^n$; (since p is prime) $p \mid b_k$, contrary to (3).

If possibility 2 holds, then by (2),

$$1 \;\leq\; e < \min\{e \cdot n, n \cdot t\} \;=\; m \tag{7}$$

And so, (4) implies,

$$r - p^{n \cdot t - e} \cdot b_k^n \;=\; p^{e \cdot n - e} \cdot \lambda \tag{8}$$

Thus (7) and (8) imply that $p \mid r$, contrary to (1).

We have proved that $e = n \cdot t$; and so b must be the n^{th} power of an integer.

N4. For every integer $k \geq 2$, prove that 2^{3k} divides the number

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}}$$

but 2^{3k+1} does not.

Comment by Mohammed Aassila, Strasbourg, France.

This is not an original problem. It appeared first in D.B. Fuchs and M.B. Fuchs, *Arithmetic of binomial coefficients*, KVANT 6 (1970).

N5. Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ and every prime p, the number $f(m+n)$ is divisible by p if and only if $f(m) + f(n)$ is divisible by p. ($\mathbb{N}$ is the set of all positive integers.)

Solved by Oliver Geupel, Brühl, NRW, Germany.

The identity $f(n) = n$ is such a function. We prove that it is the unique solution. Suppose that f is any solution.

We show by contradiction that f is injective. Assume the contrary. Consider the equivalence relation on $\mathbb{N}$ defined by $m \sim n$ if and only if $f(m)$ and $f(n)$ have the same prime divisors. By assumption, there are numbers $m < n$ such that $f(m) = f(n)$. For every $k \in \mathbb{N}$ and every prime p, we have

$$p \mid f(m+k) \quad \Leftrightarrow \quad p \mid f(m)+f(k) \quad \Leftrightarrow \quad p \mid f(n)+f(k) \quad \Leftrightarrow \quad p \mid f(n+k).$$

Hence, for every $k \in \mathbb{N}$ it holds $m + k \sim n + k$, i.e. each integer $s > n$ is equivalent to $s - (n - m)$. Let p be a prime such that the least number s with the property $p \mid f(s)$ is greater than n. We have $s \sim s - (n - m)$, but $p \nmid f(s - (n - m))$, a contradiction. This completes the proof that f is injective.

We show by contradiction that

$$f(1) \;=\; 1. \tag{1}$$

Suppose that there were a prime p such that $p \mid f(1)$. Then, for every $n \in \mathbb{N}$, we would have $p \left| \sum_{k=1}^{n} f(1) \right.$ and, by Mathematical Induction, $p \mid f(n)$. Therefore, f is not surjective, a contradiction, which completes the proof of (1).

We prove that for every $m \in \mathbb{N}$ it holds

$$|f(m+1) - f(m)| = 1 \tag{2}$$

Suppose contrariwise that for any number $m \in \mathbb{N}$ there is a prime p such that $p \mid f(m+1) - f(m)$. Since f is surjective, there is a number $n \in \mathbb{N}$ such that $p \mid f(m) + f(n)$. We obtain $p \mid f(m+n)$ and $p \mid f(m+1) + f(n)$; hence $p \mid f(m+n+1)$. Thus, $p \mid f(1)$, which contradicts (1). This proves that $f(m+1) = f(m) + 1$.

From (1) and (2) and the injectivity of f, it follows by Mathematical Induction that $f(n) = n$ for every $n \in \mathbb{N}$.

Next we turn to solutions to problems of the Bundeswettbewerb Mathematik 2006 given at [2010 : 22].

1. A circle is divided into $2n$ congruent sectors, n of them coloured black and the remaining n sectors coloured white. The white sectors are numbered clockwise from 1 to n, starting anywhere. Afterwards, the black sectors are numbered counter clockwise from 1 to n, again starting anywhere.

Prove that there exist n consecutive sectors having the numbers from 1 to n.

Comment by Mohammed Aassila, Strasbourg, France.

This problem appeared in the 20$^{\text{th}}$ Tournament of the Towns, Spring 1999, A-level, problem 4. The author is V. Proizvolov.

2. Let $\mathbb{Q}^+$ (resp. $\mathbb{R}^+$) denote the set of positive rational (resp. real) numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{R}^+$ that satisfy

$$f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)} \qquad \text{for all } x, y \in \mathbb{Q}^+ .$$

Solved by Michel Bataille, Rouen, France; and by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Bataille's solution.

We show that the only solution is $x \mapsto \dfrac{1}{x^2}$.

Let f satisfy the given functional equation, denoted by (E) in what follows. With $x = y = 1$, (E) readily gives $f(2) = \frac{1}{4}$. Let $a = f(1)$. With $y = 1$ in (E), we obtain

$$f(x+1) = \frac{f(x)}{(2x+1)f(x) + a}, \tag{1}$$

which successively yields $f(3) = f(2+1) = \dfrac{1}{4a+5}$ and $f(4) = f(3+1) = \dfrac{1}{4a^2+5a+7}$.

However, from (E) with $x = y = 2$, we also deduce $f(4) = \frac{1}{16}$. Comparing with (1), we see that $4a^2 + 5a - 9 = 0$, and, since $a > 0$, it follows that $a = f(1) = 1$.

Now, (1) rewrites as $\dfrac{1}{f(x+1)} = \dfrac{1}{f(x)} + 2x + 1$ and an easy induction shows that

$$\frac{1}{f(x+n)} = \frac{1}{f(x)} + 2nx + n^2 \tag{2}$$

for all positive integers n and rationals x. Thus, $f(n) = \dfrac{1}{n^2}$ for all n in $\mathbb{N}$ ($x = 1$ in (2)) and

$$\frac{1}{f(\frac{1}{n} + n)} = \frac{1}{f(1/n)} + 2 + n^2.$$

Comparing with the result given by (E) with $x = n$ and $y = \frac{1}{n}$, we have

$$\left(f\left(\frac{1}{n}\right)\right)^2 - \left(n^2 - \frac{1}{n^2}\right) f\left(\frac{1}{n}\right) - 1 = 0$$

and $f\left(\frac{1}{n}\right) = n^2$ follows.

Lastly, if $m, n \in \mathbb{N}$, taking $x = m$ and $y = \frac{1}{n}$ in (E) and using (2) lead to

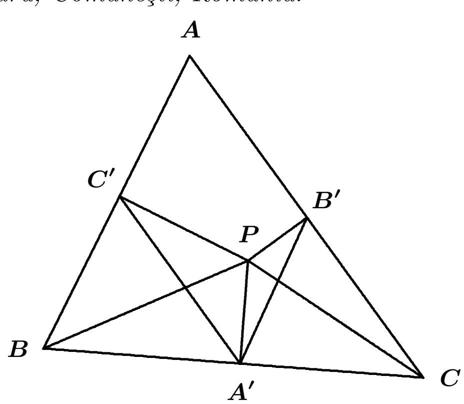
$$\frac{1}{m^2} + n^2 + \frac{2m}{n} f\left(\frac{m}{n}\right) = f\left(\frac{m}{n}\right)\left(\frac{1}{n^2} + \frac{2m}{n} + m^2\right)$$

and a short calculation gives $f\left(\frac{m}{n}\right) = \left(\frac{n}{m}\right)^2$. As a result, $f(x) = \dfrac{1}{x^2}$ for all rational numbers x.

Conversely, it is easily checked that $x \mapsto \dfrac{1}{x^2}$ satisfies (E) for all rational numbers x, y.

3. The point P lies inside the acute-angled triangle ABC and C', A' and B' are the feet of the perpendiculars from P to AB, BC, CA. Find all positions of P such that $\angle BAC = \angle B'A'C'$ and $\angle CBA = \angle C'B'A'$.

Solved by Titu Zvonaru, Comăneşti, Romania.



Since the quadrilaterals $PA'BC'$ and $PA'CB'$ are cyclic, we have

$$\begin{aligned}\angle BAC &= \angle B'A'C' = \angle B'A'P + \angle PA'C' \\ &= \angle PCA + \angle PBA.\end{aligned}$$

It results that

$$\begin{aligned}\angle BPC &= 180^\circ - \angle PBC - \angle PCB \\ &= 180^\circ - (\angle ABC - \angle PBA) - (\angle BCA - \angle PCA) \\ &= 180^\circ - \angle ABC - \angle BCA + \angle PBA + \angle PCA \\ &= \angle BAC + \angle BAC = 2\angle BAC,\end{aligned}$$

hence $\angle BPC = 2\angle BAC$.

We deduce that the point P lies on an arc of a circle which passes through B and C.

Similarly, we deduce that the point P lies on an arc of a circle which passes through A and C.

It follows that there exists at most one point P satisfying the given condition.

Since it is easy to see that the circumcentre satisfies the problem (the medial triangle is similar to the given triangle), we conclude that the desired point is the circumcentre.

And next we look at solutions for problems of the Bundeswettebwerb Mathematik 2007 given at [2010 : 22].

1. Show that one can distribute the integers from 1 to 4014 on the vertices and the midpoints of the sides of a regular 2007-gon so that the sum of the three numbers along any side is constant.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Starting at any vertex number the vertices counterclockwise from v_1 to v_{2007}. Setting $v_{2008} = v_1$, label as m_k the midpoint of side $v_k v_{k+1}$. For $k = 1, 2, \ldots, 2007$, assign the integer $2k - 1$ to m_k. For $k = 1, 2, \ldots, 1004$, assign the integer $2k$ to $v_{2009-2k}$, and for $k = 1005, 1006, \ldots, 2007$, assign the integer $2k$ to $v_{4016-2k}$. Then m_j has the value $2j - 1$, v_{2j} has the value $4016 - 2j$, and v_{2j+1} has the value $2008 - 2j$. Hence, the sum on side $v_{2j}m_{2j}v_{2j+1}$ is $(4016 - 2j) + (4j - 1) + (2008 - 2j) = 6023$, and the sum on side $v_{2j-1}m_{2j-1}v_{2j}$ is $(20080(2j - 2)) + (4j - 3) + (4016 - 2j) = 6023$.

2. Each positive integer is coloured either red or green so that

(a) The sum of three (not necessarily different) red numbers is red.

(b) The sum of three (not necessarily different) green numbers is green.

(c) There is at least one green number and one red number.

Find all colourings that satisfy these conditions.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and by Titu Zvonaru, Comăneşti, Romania. We give Zvonaru's response.

Let R be the set of red numbers and G be the set of green numbers.

Let a, b be two positive integers such that $a \neq b$, $a \in R$, $b \in G$. It is easy to see that

$$a + a + a = 3a, \quad a + a + 3a = 5a, \quad a + a + 5a = 7a, \ldots \in R$$

and

$$b + b + b = 3b, \quad b + b + 3b = 5b, \quad b + b + 5b = 7b, \ldots \in G.$$

It results that

$$R \text{ and } G \text{ are infinite sets.} \tag{1}$$

To make a choice, we assume that $1 \in R$: It follows that R contains all odd positive integers.

If we suppose that the even integer $2k \in R$, then an easy induction shows that every integer t, with $t \geq 2k$ belongs to R.

Since $1 \in R$ and $2k \in R$, then $1 + 1 + 2k = 2(k + 1) \in R$ and so on.

We deduce that $\{t, t \geq 2k\} \subset R$, hence the set G is finite — a contradiction with (1).

It results that we have two possibilities:

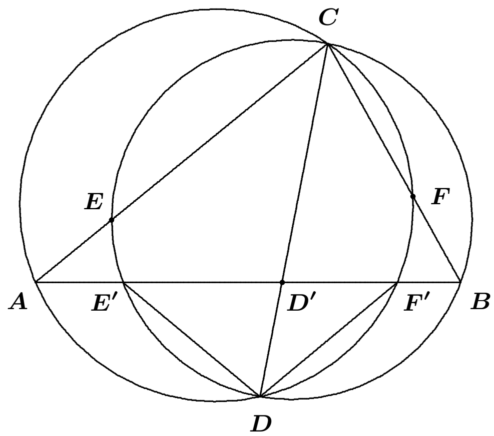
- (i) R is the set of all odd positive integers; G is the set of all even positive integers.

- (ii) R is the set of all even positive integers; G is the set of all odd positive integers.

3. In triangle ABC the points E and F lie in the interiors of sides AC and BC (respectively) so that $|AE| = |BF|$. Furthermore, the circle through A, C and F and the circle through B, C and E intersect in a point $D \neq C$.

Prove that the line CD is the bisector of $\angle ACB$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by Michel Bataille, Rouen, France. We give the solution of Amengual Covas.



Let the circle through $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{E}$ intersect $\boldsymbol{AB}$ at $\boldsymbol{E'} \neq \boldsymbol{B}$.

Let the circle through $\boldsymbol{A}$, $\boldsymbol{C}$, and $\boldsymbol{F}$ intersect $\boldsymbol{AB}$ at $\boldsymbol{F'} \neq \boldsymbol{A}$.

We denote by $\boldsymbol{D'}$ the point where the line $\boldsymbol{CD}$ intersects $\boldsymbol{AB}$.

Observing that $\boldsymbol{E}$, $\boldsymbol{E'}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are concyclic, as are $\boldsymbol{F}$, $\boldsymbol{F'}$, $\boldsymbol{A}$ and $\boldsymbol{C}$, we have $\boldsymbol{AE \cdot AC = AE' \cdot AB}$ and $\boldsymbol{BF \cdot BC = BF' \cdot BA}$ implying $\dfrac{\boldsymbol{AE \cdot AC}}{\boldsymbol{BF \cdot BC}} = \dfrac{\boldsymbol{AE' \cdot AB}}{\boldsymbol{BF' \cdot BA}}$ which simplifies to

$$\frac{\boldsymbol{AC}}{\boldsymbol{BC}} = \frac{\boldsymbol{AE'}}{\boldsymbol{BF'}} \tag{1}$$

because $\boldsymbol{AE = BF}$ by hypothesis.

Also, since $\boldsymbol{E}$, $\boldsymbol{E'}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are concyclic, as are $\boldsymbol{F}$, $\boldsymbol{F'}$, $\boldsymbol{A}$, and $\boldsymbol{C}$, then $\boldsymbol{DD' \cdot D'C = BD' \cdot D'E'}$ and $\boldsymbol{DD' \cdot D'C = AD' \cdot D'F'}$ implying $\boldsymbol{AD' \cdot D'F' = BD' \cdot D'E'}$.

Hence

$$\begin{aligned}\frac{\boldsymbol{AD'}}{\boldsymbol{BD'}} &= \frac{\boldsymbol{D'E'}}{\boldsymbol{D'F'}} \\ &= \frac{\boldsymbol{AD' - D'E'}}{\boldsymbol{BD' - D'F'}} \\ &= \frac{\boldsymbol{AE'}}{\boldsymbol{BF'}} \\ &= \frac{\boldsymbol{AC}}{\boldsymbol{BC}} \quad \text{by (1).}\end{aligned}$$

By the converse of the internal angle bisector theorem, then, $\boldsymbol{CD'}$ is the bisector of $\angle \boldsymbol{ACB}$. That is, the line $\boldsymbol{CD}$ bisects $\angle \boldsymbol{ACB}$.

4. Let $\boldsymbol{a}$ be a positive integer. How many nonnegative integers $\boldsymbol{x}$ satisfy

$$\left\lfloor \frac{\boldsymbol{x}}{\boldsymbol{a}} \right\rfloor = \left\lfloor \frac{\boldsymbol{x}}{\boldsymbol{a+1}} \right\rfloor ?$$

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comǎneşti, Romania. We give the solution of Bataille.

We show that the required number is $\frac{a(a+1)}{2}$.

By long division, we may write any nonnegative integer x as $qa(a+1)+r$ for some nonnegative integers q and r such that $r < a(a+1)$. Using the fact that $\lfloor m+u \rfloor = m + \lfloor u \rfloor$ for $m \in \mathbb{Z}$ and $u \in \mathbb{R}$, the proposed equation becomes

$$q + \left\lfloor \frac{r}{a} \right\rfloor = \left\lfloor \frac{r}{a+1} \right\rfloor . \qquad (1)$$

Now, $\frac{r}{a} \geq \frac{r}{a+1}$, hence $\left\lfloor \frac{r}{a} \right\rfloor \geq \left\lfloor \frac{r}{a+1} \right\rfloor$ and (1) cannot be satisfied if $q \geq 1$. Thus, the desired number is also the number of elements $r \in A$ with $A = \{0, 1, 2, \dots, a(a+1)-1\}$ such that

$$\left\lfloor \frac{r}{a} \right\rfloor = \left\lfloor \frac{r}{a+1} \right\rfloor . \qquad (2)$$

Any $r \in A$ satisfies $ka \leq r < (k+1)a$ for some unique $k \in \{0, 1, 2, \dots, a\}$ and then $\left\lfloor \frac{r}{a} \right\rfloor = k$. Observing that such an r satisfies $\frac{r}{a+1} < \frac{(k+1)a}{a+1} < k+1$, we see that this r is a solution if and only if $\frac{r}{a+1} \geq k$, that is $r \geq ka+k$. As a result, solutions r with $ka \leq r < (k+1)a$ exist if and only if $k \leq a-1$, in which case the solutions are $ka+k,\ ka+k+1,\ \dots,\ ka+a-1$. Thus, for each $k \in \{0, 1, 2, \dots, a-1\}$, we obtain $(a-1)-(k-1) = a-k$ solutions and no other solution exists. In conclusion the number of solutions is

$$(a-0)+(a-1)+\cdots+(a-(a-2))+(a-(a-1)) = \frac{a(a+1)}{2}.$$

Next we move to the March 2010 number of the *Corner* and solutions to problems of the Republic of Moldova Selection tests for BMO 2007 and IMO 2007 given at [2010 : 81–83].

1. In triangle ABC the points M, N and P are the midpoints of the sides BC, AC and AB, respectively. The lines AM, BN and CP intersect the circumcircle of ABC at A_1, B_1 and C_1, respectively. Prove that the area of the triangle ABC does not exceed the sum of the areas of the triangles BA_1C, AB_1C and AC_1B.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Michel Bataille, Rouen, France; by Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India; and by Titu Zvonaru, Comǎneşti, Romania. We give the solution of De.

Notation: $[XYZ]$ = Area of the triangle XYZ. m_a = median on the side of a triangle with length a.

We see at once that

$$\frac{[BA_1C]}{[ABC]} = \frac{MA_1}{AM}, \quad \frac{[AB_1C]}{[ABC]} = \frac{NB_1}{BN}, \quad \frac{[AC_1B]}{[ABC]} = \frac{PC_1}{CP}.$$

Let $BC = a$, $CA = b$ and $AB = c$. Chords AA_1 and BC of the circumcircle of triangle ABC intersect at M. Therefore $AM \cdot MA_1 = BM \cdot MC$. Also M is the midpoint of BC. Therefore $BM = MC = \frac{1}{2}a$ and hence $MA_1 = \dfrac{a^2}{4AM}$. Thus

$$\frac{MA_1}{AM} = \frac{a^2}{4AM^2} = \left(\frac{a}{2m_a}\right)^2.$$

Similarly we can show that $\dfrac{NB_1}{BN} = \left(\dfrac{b}{2m_b}\right)^2$ and $\dfrac{PC_1}{CP} = \left(\dfrac{c}{2m_c}\right)^2$. Therefore

$$\frac{[BA_1C]}{[ABC]} + \frac{[AB_1C]}{[ABC]} + \frac{[AC_1B]}{[ABC]} = \frac{1}{4}\left(\frac{a^2}{m_a^2} + \frac{b^2}{m_b^2} + \frac{c^2}{m_c^2}\right) \ldots \qquad (1)$$

Recall that

$$\begin{aligned} a^2 &= \frac{4}{9}(2(m_b^2 + m_c^2) - m_a^2) \\ b^2 &= \frac{4}{9}(2(m_c^2 + m_a^2) - m_b^2) \\ c^2 &= \frac{4}{9}(2(m_a^2 + m_b^2) - m_c^2). \end{aligned}$$

Using these in (1) we obtain

$$\frac{1}{4}\left(\frac{a^2}{m_a^2} + \frac{b^2}{m_b^2} + \frac{c^2}{m_c^2}\right) = \frac{1}{9}\left(2\left(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - 3\right) \ldots, \qquad (2)$$

where

$$x = \frac{m_a^2}{m_b^2}, \quad y = \frac{m_b^2}{m_c^2}, \quad \text{and} \quad z = \frac{m_c^2}{m_a^2}.$$

Now, $2(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}) - 3 \geq 9$ because $x + \frac{1}{x} \geq 2$, $y + \frac{1}{y} \geq 2$ and $z + \frac{1}{z} \geq 2$. Thus from (1) and (2) we can conclude that

$$[BA_1C] + [AB_1C] + [AC_1B] \geq [ABC].$$

2. Let p be a prime number, $p \neq 2$, and $m_1, m_2, \ldots, m_p$ positive consecutive integers, and σ a permutation of the set $A = \{1, 2, \ldots, p\}$. Prove that the set A contains 2 distinct numbers k and l such that p divides $m_k \cdot m_{\sigma(k)} - m_l \cdot m_{\sigma(l)}$.

Solved by Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India.

The residue classes modulo p of the p consecutive positive integers is $C=\{0,1,2,\ldots,p-1\}$. Let $m_k\equiv 0 \pmod p$ for some $k\in A$ and $\sigma(k)\neq k$. Then there exists some $l\in A$, $l\neq k$ such that $\sigma(l)=k$. Thus we obtain two distinct positive integers m and l such that $p\mid(m_k m_{\sigma(k)}-m_l m_{\sigma(l)})$.

Now suppose for some positive integer $j\in A$ we have $m_j\equiv 0 \pmod p$ and $\sigma(j)=j$. Consider the set $A'=A-\{j\}$.

Claim. There exist distinct positive integers k and l in A' such that $p\mid(m_k m_{\sigma(k)}-m_l m_{\sigma(l)})$.

To prove it assume on the contrary that no such pair of positive integers exists. Then $\{m_r m_{\sigma(r)} \pmod p : r\in A'\}=\{1,2,\ldots,p-1\}$. Now observe that

$$\prod_{r\in A'} m_r m_{\sigma(r)} \equiv (p-1)! \pmod p \ldots. \tag{1}$$

Again observe that

$$\prod_{r\in A'} m_r m_{\sigma(r)} \equiv \left(\prod_{r\in A'} m_r\right)\left(\prod_{r\in A'} m_{\sigma(r)}\right) \equiv ((p-1)!)^2 \pmod p)\ldots. \tag{2}$$

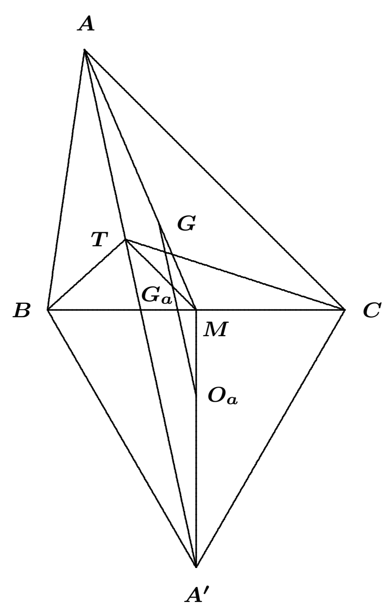
From (1) and (2) we conclude that $(p-1)!\equiv 1 \pmod p$ and this contradicts Wilson's Theorem. Thus our assumption is wrong and the claim is correct.

3. Inside the triangle ABC there exists a point T such that

$$m(\angle ATB) = m(\angle BTC) = m(\angle CTA) = 120^\circ.$$

Prove that the Euler lines of the triangles ATB, BTC and ATC are concurrent.

Solved by Titu Zvonaru, Comăneşti, Romania.



Let M be the midpoint of BC, and let G_a be the centroid of the triangle BTC. Let A' be the point on the other side of BC as A such that $\triangle BA'C$ is equilateral.

It is known that the point T lies on AA'. Since $\angle BTC+\angle BA'C=180^\circ$, the quadrilateral $BA'CT$ is cyclic; it results that the circumcentre O_a of $\triangle BA'C$ is the circumcentre of $\triangle BTC$. We deduce that O_aG_a is the Euler line of $\triangle BTC$. Since $\dfrac{G_aM}{G_aT}=\dfrac{1}{3}=\dfrac{MO_a}{O_aA'}$, we obtain that $O_aG_a\|A'T$.

Now, denoting $G=O_aG_a\cap AM$, by similarity, it results that $\dfrac{GM}{GA}=\dfrac{O_aM}{O_aA'}=\dfrac{1}{3}$, hence G is the centroid of $\triangle ABC$.

Similarly, we deduce that the Euler lines of $\triangle CTA$ and $\triangle ATB$ pass through the centroid of $\triangle ABC$, hence the three Euler lines are concurrent.

5. Determine the smallest positive integers m and k such that:

(a) there exist $2m+1$ consecutive positive integers whose sum of cubes is a perfect cube;

(b) there exist $2k+1$ consecutive positive integers whose sum of squares is a perfect square.

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We use Manes' solution.

For part (a), $m = 1$ since there exist three consecutive positive integers whose cubes sum to a perfect cube; namely $3^3 + 4^3 + 5^3 = 6^3$. For part (b), we will show that $k = 5$.

Let $s(n, 2k+1) = n^2 + (n+1)^2 + \cdots + (n+2k)^2$ be the sum of the squares of $2k+1$ consecutive integers, the smallest of which is n. If $k = 1$, then

$$s(n-1,3) \;=\; (n-1)^2 + n^2 + (n+1)^2 \;=\; 3n^2 + 2 \;\equiv\; 2 \pmod{3}\,.$$

Therefore, $s(n-1,3)$ is not a perfect square since x^2 is not congruent to 2 modulo 3 for any integer x.

If $k = 2$, then

$$\begin{aligned} s(n-2,5) &= (n-2)^2 + (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 \\ &= 5(n^2+2) \;\equiv\; 2 \text{ or } 3 \pmod{4}\,. \end{aligned}$$

Thus, $s(n-2,5)$ is not a perfect square since $x^2 \equiv 0$ or $1 \pmod{4}$ for every integer x.

If $k = 3$, assume that $s(n-3,7) = r^2$ for some integer r. This equation reduces to $7(n^2+4) = r^2$. Hence, 7 divides r so that $r = 7t$ for some integer t. Therefore, $n^2 + 4 = 7t^2$ and so, $n^2 + 4 \equiv 0 \pmod{7}$ or $n^2 \equiv 3 \pmod{7}$, a contradiction since 3 is not a quadratic residue of 7. Thus, $s(n-3,7)$ is not a perfect square.

If $k = 4$, assume that $s(n-4,9) = r^2$ for some integer r. This equation reduces to $3(3n^2+20) = r^2$. Therefore, 3 divides r so that $r^2 = 9t^2$ for some integer t, whence $3n^2 + 20 = 3t^2$. Thus, 3 divides 20, a contradiction that shows $s(n-4,9)$ is not a perfect square.

If $k = 5$, assume that $s(n-5,11) = m^2$ for some integer m. Then $s(n-5,11) = 11(n^2+10) = m^2$ implies that 11 divides m so that $m^2 = 11^2t^2$ for some integer t. Therefore $n^2 + 10 = 11t^2$ or $n^2 - 1 \equiv 0 \pmod{11}$. Thus, $n \equiv \pm 1 \pmod{11}$, and so $n = 11j \pm 1$ for some integer j. Then

$$\begin{aligned} s(n-5,11) &= 11(n^2+10) \;=\; 11[(11j \pm 1)^2 + 10] \\ &= 11^2[11j^2 \pm 2j + 1] \;=\; 11^2[10j^2 + (j \pm 1)^2]. \end{aligned}$$

The problem now reduces to finding the smallest value of j so that $10j^2+(j\pm1)^2$ is a perfect square. The value $j=1$ is easily dispensed with. However, for $j=2$, $10j^2+(j+1)^2=7^2$ and $n=11j+1=23$. Accordingly, $s(18,11)=77^2$ or

$$18^2+19^2+20^2+\cdots+28^2 \;=\; 77^2.$$

6. Let I be the incenter of triangle ABC and let R be the circumradius. Prove that $AI+BI+CI \;\le\; 3R$.

Solved by Arkady Alt, San Jose, CA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Titu Zvonaru, Cománeşti, Romania. We give the version of Amengual Covas.

We write a, b, c respectively for the lengths of the sides BC, CA, AB, and $s=\frac{a+b+c}{2}$ for the semiperimeter. Let r be the radius of the incircle.

Note that $\cos\frac{A}{2}$ may be expressed as $\frac{s-a}{AI}$, and also as $\sqrt{\frac{s(s-a)}{bc}}$. Equating these and solving for AI, we get $AI=\sqrt{\frac{bc(s-a)}{s}}$, with symmetric results for BI and CI.

Applying the Cauchy's inequality $\vec{u}\cdot\vec{v} \;\le\; \|\vec{u}\|\cdot\|\vec{v}\|$ with $\vec{u}=\left(\sqrt{bc},\sqrt{ca},\sqrt{ab}\right)$ and $\vec{v}=\left(\sqrt{\frac{s-a}{s}},\sqrt{\frac{s-b}{s}},\sqrt{\frac{s-c}{s}}\right)$, we now get

$$\begin{aligned} AI+BI+CI &= \sqrt{\frac{bc(s-a)}{s}}+\sqrt{\frac{ca(s-b)}{s}}+\sqrt{\frac{ab(s-c)}{s}} \\ &\le \sqrt{ab+bc+ca} \end{aligned} \qquad (1)$$

Using the relations $ab+bc+ca=r^2+4Rr+s^2$, $r\le\frac{R}{2}$ and $s\le\frac{3\sqrt{3}}{2}R$, we have

$$ab+bc+ca\le\left(\frac{R}{2}\right)^2+4R\cdot\frac{R}{2}+\left(\frac{3\sqrt{3}}{2}R\right)^2=9R^2 \qquad (2)$$

By (1) and (2), we obtain the desired inequality. Since equality in (1) and (2) holds if and only if $a=b=c$, it holds in the required inequality if and only if $\triangle ABC$ is equilateral.

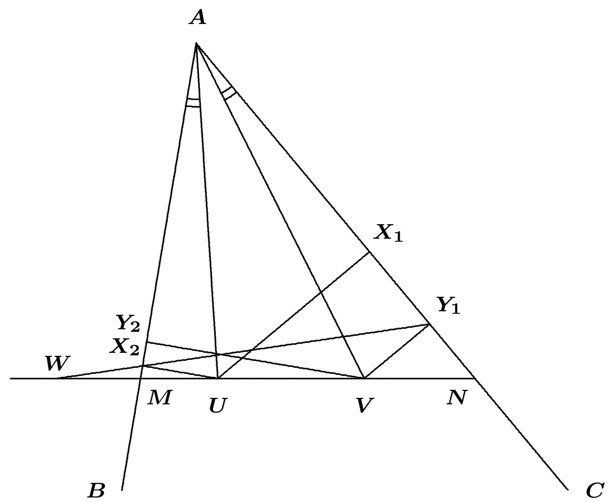
Comment. Also solved, by using the Erdős-Mordell inequality, on page 38 of the book Experiences in Problem Solving: A W. J. Blundon Commemorative, Atlantic Provinces Council on the Sciences, Canada, (1994).

7. Let U and V be two points inside the angle BAC such that

$$m(\angle BAU) \;=\; m(\angle CAV)\,.$$

Denote projections from U and V on the angle sides AC, AB as X_1, X_2 and Y_1, Y_2 respectively. Let W be the intersection of the lines X_2Y_1 and X_1Y_2. Prove that U, V, W are collinear.

Solved by Titu Zvonaru, Comănești, Romania.



Let the lines UV and AB intersect at M, and the lines UV and AC intersect at N. We denote

$$\alpha = \angle BAC, \quad \gamma = \angle ANM, \quad \beta = \angle AMN, \quad \varphi = \angle BAV = \angle CAV.$$

Suppose that the lines X_2Y_1 and MN intersect at the point W. We will prove that the points W, Y_2, X_1 are collinear.

By Menelaus' theorem we obtain:

$$\frac{WM}{WN}\cdot\frac{Y_1N}{Y_1A}\cdot\frac{X_2A}{X_2N} = 1 \Leftrightarrow \frac{WM}{WN}\cdot\frac{VN\cos\gamma}{AV\cos\varphi}\cdot\frac{AV\cos\varphi}{MU\cos\beta} = 1$$
$$\Leftrightarrow \frac{WM}{WN} = \frac{AV\cdot MU\cos\beta}{AU\cdot VN\cos\gamma} \qquad (1)$$

By the converse of Menelaus' theorem, we have to prove that

$$\frac{WM}{WN}\cdot\frac{X_1N}{X_1A}\cdot\frac{Y_2A}{Y_2M} = 1 \Leftrightarrow \frac{WM}{WN}\cdot\frac{UN\cos\gamma}{AU\cos(\alpha-\varphi)}\cdot\frac{AV\cos(\alpha-\varphi)}{MV\cos\beta}$$
$$\Leftrightarrow \frac{WM}{WN} = \frac{AU\cdot MV\cdot\cos\beta}{AV\cdot UN\cdot\cos\gamma}.$$

By (1), it suffices to prove that

$$\frac{AV\cdot MU}{AU\cdot VN} = \frac{AU\cdot MV}{AV\cdot UN} \Leftrightarrow \frac{AU^2}{AV^2} = \frac{MU\cdot UN}{NV\cdot VM},$$

which is Steiner's Theorem with respect to isogonal cevians.

Here a proof: We denote by $[XYZ]$ the area of $\triangle XYZ$. We have

$$\begin{aligned}\frac{MU}{NV}\cdot\frac{UN}{VM} &= \frac{[AMU]}{[AVM]}\cdot\frac{[AUN]}{[ANV]}\\ &= \frac{AM\cdot AU\cdot\sin\varphi}{AN\cdot AV\sin(\alpha-\varphi)}\cdot\frac{AU\cdot AN\sin(\alpha-\varphi)}{AV\cdot AM\cdot\sin\gamma}\\ &= \frac{AU^2}{AV^2}.\end{aligned}$$

9. Let $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ be real numbers in the interval $[0,1]$. Let $S = a_1^3 + a_2^3 + \cdots + a_n^3$. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{2n+1+S-a_i^3} \leq \frac{1}{3}.$$

Solved by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and by Titu Zvonaru, Cománeşti, Romania. We give the solution of Díaz-Barrero.

Since a_i, $1 \leq i \leq n$, lie in the interval $[0,1]$, then $1 - a_i^3 \geq 0$ for $1 \leq i \leq n$, and

$$\sum_{i=1}^{n} \frac{a_i}{2n+1+S-a_i^3} \leq \sum_{i=1}^{n} \frac{a_i}{2n+S}.$$

So, it will suffice to prove that

$$\sum_{i=1}^{n} \frac{a_i}{2n+S} \leq \frac{1}{3}$$

or equivalently,

$$3(a_1 + a_2 + \ldots + a_n) \leq 2n + S = \sum_{i=1}^{n}(1+1+a_i^3)$$

which trivially holds on account of AM-GM inequality. Indeed,

$$\sum_{i=1}^{n}(1+1+a_i^3) \geq \sum_{i=1}^{n} 3\sqrt[3]{1\cdot 1\cdot a_i^3} = 3\sum_{i=1}^{n} a_i.$$

Equality holds when $a_1 = a_2 = \ldots = a_n = 1$, and we are done.

10. Find all polynomials f with integer coefficients, such that $f(p)$ is a prime for every prime p.

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

Note that the polynomial $f(x) = x$ and the constant polynomials $f(x) = c$ where c is a prime satisfy the requirements in the problem. They are the only such polynomials that do.

Assume that f is a polynomial with integer coefficients such that if p is a prime, then $f(p)$ is a prime and also assume that $f(x) \neq x$ and $f(x) \neq c$. Then there exists a prime π such that $\gcd(\pi, f(\pi)) = 1$ since otherwise p divides $f(p)$ for all primes p and it would follow that either $f(x) = 0$ or $f(p) = p$ so that $f(x) = x$, both contradictions. By Dirichlet's Theorem, there exist infinitely many integers n_i such that $n_i f(\pi) + \pi$ is a prime, say $n_i f(\pi) + \pi = q_i$, $i = 1, 2, 3, \ldots$. Since $q_i - \pi$ divides $f(q_i) - f(\pi)$ we get that $f(\pi)$ divides $f(q_i)$ for each $i \geq 1$. Hence $f(x) = f(\pi)$, a contradiction. As a result, the only polynomials f with integer coefficients for which $f(p)$ is a prime for every prime p are $f(x) = x$ and $f(x) = c$ where c is a prime.

11. Let ABC be a triangle with $a = BC$, $b = AC$, $c = AB$, inradius r and circumradius R. Let r_A, r_B and r_C be the radii of the excircles of the triangle ABC. Prove that

$$a^2\left(\frac{2}{r_A} - \frac{r}{r_B r_C}\right) + b^2\left(\frac{2}{r_B} - \frac{r}{r_C r_A}\right) + c^2\left(\frac{2}{r_C} - \frac{r}{r_B r_A}\right) = 4(R+3r)\,.$$

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give Kandall's write-up.

Let $K = [ABC]$, $s = \frac{a+b+c}{2}$. It is well known (Heron) that

$$\begin{aligned} 16K^2 &= (a+b+c)(-a+b+c)(a-b+c)(a+b-c) \\ &= -(a^4+b^4+c^4) + 2(a^2b^2+b^2c^2+a^2c^2) \end{aligned}$$

and $K = rs = r_A(s-a) = r_B(s-b) = r_C(s-c) = \dfrac{abc}{4R}$.

Let

$$X = a^2\left(\frac{2}{r_A} - \frac{r}{r_B r_c}\right) = b^2\left(\frac{2}{r_B} - \frac{r}{r_C r_A}\right) + c^2\left(\frac{2}{r_C} - \frac{r}{r_B r_A}\right).$$

We have

$$\begin{aligned} a^2\left(\frac{2}{r_A} - \frac{r}{r_B r_C}\right) &= a^2\left(\frac{2(s-b)}{K} - \frac{K}{s}\cdot\frac{s-b}{K}\cdot\frac{s-c}{K}\right) \\ &= \frac{a^2}{sK}\left(2\cdot\frac{a+b+c}{2}\cdot\frac{a-b+c}{2}\right. \\ &\qquad \left. - \frac{a-b+c}{2}\cdot\frac{a+b-c}{2}\right) \\ &= \frac{1}{4sK}(-3a^4 + 3a^2b^2 + 3a^2c^2 + 2a^2bc). \end{aligned}$$

Similarly,

$$b^2\left(\frac{2}{r_B}-\frac{r}{r_Cr_A}\right) = \frac{1}{4sK}(-3b^4+3a^2b^2+3b^2c^2+2ab^2c),$$
$$c^2\left(\frac{2}{r_C}-\frac{r}{r_Br_A}\right) = \frac{1}{4sK}(-3c^4+3a^2c^2+3b^2c^2+2abc^2).$$

Consequently,

$$\begin{aligned} X &= \frac{1}{4Sk}(-3(A^4+B^4+C^4) \\ &\qquad +6(A^2B^2+B^2C^2+A^2C^2)+2ABC(A+B+C)) \\ &= \frac{1}{4(\frac{K}{r})K}\left(3\cdot 16K^2+2\cdot 4RK\cdot 2\frac{K}{r}\right) \\ &= 4(R+3r), \end{aligned}$$

as required.

12. Consider n distinct points in the plane $n \geq 3$, arranged such that the number $r(n)$ of segments of length l is maximized. Prove that $r(n) \leq \frac{n^2}{3}$.

Solved by Oliver Geupel, Brühl, NRW, Germany.

We will apply Turán's Theorem: Let G be a simple graph with n vertices which does not contain a complete subgraph containing p vertices. Let r be the remainder of n modulo p. Then the number of edges of G is not greater than

$$f(n,p) = \frac{(p-2)n^2-r(p-1-r)}{2(p-1)}.$$

Consider the graph G whose vertices are the n given points and where two vertices P and Q are connected by an edge if and only if $PQ = l$. Clearly, G does not contain a complete subgraph with 4 vertices. By Turán's Theorem, the number of edges of G is not greater than

$$f(n,4) = \frac{2n^2-r(3-r)}{6} \leq \frac{n^2}{3}.$$

Remark. Problem A-6 of the 49$^{\text{th}}$ William Lowell Putnam Competition (1978) asked for proving the inequality $r(n) < 2n^{3/2}$. This is a sharper upper bound for each $n \geq 36$. See: *The William Lowell Putnam Mathematical Competition problems and solutions: 1965-1984*, ed. by G.L. Alexanderson, L.F. Klosinski, and L.C. Larson, MAA, 1986, p. 104f.

14. Let $b_1, b_2, \ldots, b_n$ ($n \geq 1$) be nonnegative real numbers at least one of which is positive. Prove that $P(X) = X^n - b_1X^{n-1} - \cdots - b_{n-1}X - b_n$, has a single positive root p, which is simple, and that the absolute value of each root of $P(X)$ is not greater than p.

Solved by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Oliver Geupel, Brühl, NRW, Germany. We give the solution by Díaz-Barrero.

We consider the continuous function $A : (0, +\infty) \to \mathbb{R}$ defined by

$$A(x) = \frac{b_1}{x} + \frac{b_2}{x^2} + \ldots + \frac{b_n}{x^n} - 1$$

Since $A'(x) = -\frac{b_1}{x^2} - \frac{2b_2}{x^3} - \ldots - \frac{nb_n}{x^{n+1}} < 0$ for all $x > 0$, then A is a decreasing continuous function. Furthermore, $\lim_{x\to+\infty} A(x) = -1$ and $\lim_{x\to 0^+} A(x) = +\infty.$ Therefore, on account of Bolzano's theorem the equation $A(x) = -\frac{P(x)}{x^n} = 0$ has only one positive root, say $p,$ which is a zero of polynomial P. On the other hand, from $A'(p) < 0$ follows $P'(p) > 0$ and p is a simple zero.

To see that all the zeros of P have modulus less than or equal to p we argue by contradiction. Assume that x_0 is a zero of P and let $|x_0| = \alpha$ with $\alpha > p$. Then $A(\alpha) < A(p) = 0$ and $P(\alpha) > 0.$ On the other hand from $x_0^n = b_n + b_{n-1}x_0 + \ldots + b_1 x_0^{n-1}$ we have

$$|x_0^n| = |b_n + b_{n-1}x_0 + \ldots + b_1 x_0^{n-1}| \le b_n + b_{n-1}|x_0| + \ldots + b_1|x_0^{n-1}|$$

from which follows

$$P(\alpha) = \alpha^n - b_1\alpha^{n-1} - \ldots - b_{n-1}\alpha - b_n < 0$$

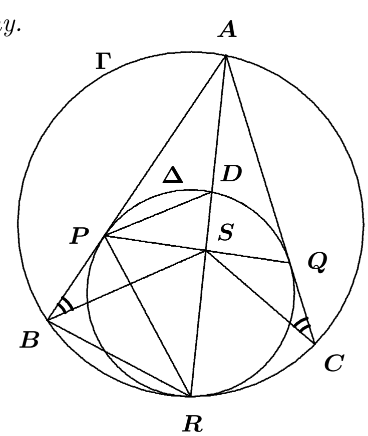
Contradiction, and we are done.

Comment. The first part of the statement also follows immediately applying the well-known Descartes Rule of Signs.

15. A circle is tangent to the sides AB and AC of the triangle ABC and to its circumcircle at P, Q and R respectively. If $PQ \cap AR = \{S\}$ prove that $m(\angle SBA) = m(\angle SCA)$.

Solved by Oliver Geupel, Brühl, NRW, Germany.

Let Γ be the circle through A, B, and C, and let Δ be the circle through P, Q, and R. Let D be the second intersection of Δ and the line AR. Since AP is tangent to Δ, we have $\angle PDR = \angle BPR$. Since Δ and Γ have a common tangent t at R, it holds $\angle DPR = \angle(AR, t) = \angle ABR = \angle PBR$. Hence, the triangles DPR and PBR are similar. It follows that $\angle ARP = \angle PRD = \angle BRP$, that is, the line RP is the internal bisector of $\angle R$ in $\triangle ABR$.



Therefore,

$$\frac{AP}{BP} = \frac{AR}{BR}.$$

Similarly,

$$\frac{AQ}{CQ} = \frac{AR}{CR}.$$

By $AP = AQ$, we obtain

$$\frac{BP}{BR} = \frac{CQ}{CR}.$$

By

$$\frac{PS}{QS} = \frac{\sin\angle PAS}{\sin\angle QAS} = \frac{\sin\angle BAR}{\sin\angle CAR} = \frac{BR}{CR},$$

we deduce that

$$\frac{BP}{CQ} = \frac{PS}{QS}.$$

By $\angle BPS = \angle CQS$, it follows that $\triangle BPS \sim \triangle CQS$. Consequently,

$$\angle SBA = \angle SBP = \angle SCQ = \angle SCA,$$

which completes the proof.

16. Prove that there are infinitely many primes p for which there exists a positive integer n such that p divides $n! + 1$ and n does not divide $p - 1$.

Comment by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

A proof is by Paul Erdős, (c.f. p. 558 of G.E. Hardy and M.V. Subbarao, "A modified Problem of Pillai and Some Related Questions", The American Math. Monthly, Vol 109, pp. 554–559).

BOOK REVIEWS

Amar Sodhi

Alex's Adventures in Numberland
by Alex Bellos
Bloomsbury Publishing, 2010
ISBN-13: 978-0747597162, hardcover, 448 pages, $30.95
Reviewed by ***Bruce Shawyer****, Memorial University of Newfoundland, St. John's, NL*

Alex's Adventures in Numberland is a book about mathematics aimed at readers of all abilities. It was published in March 2010 in the UK by Bloomsbury. In the US the book appeared in June with the title Here's Looking at Euclid, published by Free Press. The editions have different covers. I have the UK edition (448 pages) and have not seen the US edition (320 + xi pages). Both are available in Canada via the usual retailers.

The US edition of the book only contains a brief bibliography. Thanks to the wonders of the internet, the author has been able to make available on his WEB site (alexbellos.com) the complete chapter-by-chapter bibliography, with comments and suggested further reading. It is enhanced with links!

Here are some of the difference between the two editions, gleaned from the WEB site:

1. The UK edition has a cartoon preceding each chapter.

2. Both the British and American publishers felt that their respective titles worked best for their respective audiences.

3. The American one is slightly shorter (the section on the British 50p piece is omitted, for example, since the shape means nothing in Arkansas or Wyoming) and it has less diagrams overall.

4. The British version also has a 12-page colour plate section.

The author, Alex Bellos (with degrees in Mathematics and Philosophy) is a writer, broadcaster, football (soccer) lover and self-proclaimed math geek, with a colourful career including several books and short films.

The book consists of mathematics (real mathematics) for the lay person. But yet, it is mathematics for the mathematics student and for the mathematics teacher at all levels. It covers a very broad range of topics, which are all related in an engaging narrative style.

For example, Bellos describes how Yorkshire shepherds count (his list agrees with the song by Jake Thackray); why business card origami is abhorrent to the Japanese; a mnemonic for the digits of $\boldsymbol{\pi}$ which is a remarkable modernist pastiche of Poe's The Raven; and why we most commonly $\boldsymbol{x}$ as the name of a variable. We meet fanatics, crackpots, anthropologists and gurus as well as a few mathematicians (such as Aitken, Brahmagupta, Cantor, Descartes, Euler and Fibonacci).

All of the mathematics (with one exception) is well explained and correct. It is a pity that he has an error in the penultimate paragraph of the book. Here, he describes infinite cardinals, and makes two claims: first, that the number of curves in the plane is larger than $\boldsymbol{c}$, the cardinality of the continuum (this is false if curves must be continuous; there are only $\boldsymbol{c}$ of them); and second, that nobody has been able to come up with a larger set (Cantor proved that the set of subsets of any set is larger than the original set - perhaps Alex meant a larger "naturally-occurring" set).

Despite this, we have a book well worth reading. As I read it, I thought that it might well form the basis for an elementary course on the History of Mathematics. Now there is something that every student of mathematics should study. This book puts much of mathematics into context, and so, will encourage students to want to know more.

Editor's note: the North American edition, Here's Looking at Euclid: A Surprising Excursion Through the Astonishing World of Math (ISBN 978-1-4165-8825-2) retails at $32.99.

Einstein, Albert (1879-1955) A human being is a part of the whole, called by us "Universe", a part limited in time and space. He experiences himself, his thoughts and feelings as something separated from the rest, a kind of optical delusion of his consciousness. This delusion is a kind of prison for us, restricting us to our personal desires and to affection for a few persons nearest to us. Our task must be to free ourselves from this prison by widening our circle of compassion to embrace all living creatures and the whole of nature in its beauty. Nobody is able to achieve this completely, but the striving for such achievement is in itself a part of the liberation and a foundation for inner security. In H. Eves "Mathematical Circles Adieu", Boston: Prindle, Weber and Schmidt, 1977.

PROBLEMS

Solutions to problems in this issue should arrive no later than ***1 August 2010****. An asterisk* $(\star)$ *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

Note: As ***CRUX with MAYHEM*** is running behind schedule, we will accept solutions past the posted due date. Solutions will be accepted until we process them for publication. Currently we are delayed by about four months. Check the CMS website, **cms.math.ca/crux**, for our status in processing problems.

3601. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Suppose that b is a positive real number such that there are exactly two integers strictly between b and $2b$, and exactly two integers strictly between $2b$ and b^2. Find all possible values of b.

3602. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Prove that if $a_i > 0$ for $i = 1, 2, 3, 4$, then

$$\sum_{\text{cyclic}} \frac{1}{a_i^2 + a_{i+1}^2 + a_{i+2}^2} \geq \frac{12}{(a_1 + a_2 + a_3 + a_4)^2}$$

3603. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let ABC be a given triangle and $0 < \lambda < \frac{1}{2}$. Let D and E be points on AB such that $AD = BE = \lambda \cdot AB$, and F, G points on AC such that $AF = CG = \lambda \cdot AC$. Let $BF \cap CE = H$ and $BG \cap CD = I$. Show that

i) $HI \parallel BC$ and

ii) $HI = \dfrac{1 - 2\lambda}{\lambda^2 - \lambda + 1} BC$.

3604. *Proposed by Michel Bataille, Rouen, France.*

Evaluate

$$\lim_{n \to \infty} \frac{\int_0^1 (x^2 - x - 2)^n dx}{\int_0^1 (4x^2 - 2x - 2)^n dx}.$$

3605. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + 1$ be a real polynomial with positive coefficients and having all its zeros real. Prove that

$$\sqrt[n]{A(1)A(2)\cdots A(n)} \geq (n+1)!$$

3606. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Let ABC be a triangle with $\angle A = 20^\circ$. Let BD be the angle bisector of $\angle ABC$ with D on AC. If $AD = DB + BC$, determine $\angle B$.

3607. *Proposed by George Miliakos, Sparta, Greece.*

Let $c_1 = 9$, $c_2 = 15$, $c_3 = 21$, $c_4 = 25$, $\ldots$, where c_n is the n^{th} composite odd integer. Evaluate

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}}\,.$$

3608. *Proposed by Michel Bataille, Rouen, France.*

Let

$$f(x) \;=\; \frac{e^{1/x}-1}{e^{1/(x+1)}-1}\,.$$

(a) Show that for all $x \in (0,\infty)$,

$$f(x) \;>\; \sqrt{\frac{x+1}{x}}\,.$$

(b) ★ Prove or disprove:

$$f(x) \;<\; \sqrt{\frac{x+1}{x-1}}$$

for all $x \in (1,\infty)$.

3609. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let r be a real number. and let D, E, and F be points on the sides BC, CA, and AB of a triangle ABC with

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = r.$$

The cevians AD, BE, and CF bound a triangle PQR whose area we denote by $[PQR]$. Find the value of r for which the ratio of the areas, $\frac{[DEF]}{[PQR]}$ equals 4.

3610. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let $S = \{2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, \ldots\}$ be the set of positive integers whose only prime divisors are 2 or 3. Let $a_1 = 2, a_2 = 3, \ldots$, be the elements of S, with $a_1 < a_2 < \ldots$.

(i) Determine $\displaystyle\sum_{i=1}^{\infty} \left(\frac{1}{a_i}\right)$.

(ii) ★ For each positive integer n, let $s(n)$ be the sum of all its divisors including 1 and n itself. Prove $\frac{s(n)}{n} < 3$ for all members of S.

3611. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Given x, y, and z are positive integers such that

$$\frac{x(y+1)}{x-1}, \frac{y(z+1)}{y-1}, \text{ and, } \frac{z(x+1)}{z-1}$$

are positive integers. Find the smallest positive integer N such that $xyz \le N$.

3612. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Find all nonconstant polynomials P such that $P(\{x\}) = \{P(x)\}$, for all $x \in \mathbb{R}$, where $\{a\}$ denotes the fractional part of a.

. . . .

3601. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

On suppose que b est un nombre réel positif tel qu'il existe exactement deux entiers strictement compris entre b et $2b$, de même qu'exactement deux entiers strictement compris entre $2b$ et b^2. Trouver toutes les valeurs possibles de b.

3602. *Proposé par Pham Van Thuan, Université de Science de Hanoï, Hanoï, Vietnam.*

Montrer que si $a_i > 0$ pour $i = 1, 2, 3, 4$, alors

$$\sum_{\text{cyclique}} \frac{1}{a_i^2 + a_{i+1}^2 + a_{i+2}^2} \ge \frac{12}{(a_1 + a_2 + a_3 + a_4)^2}$$

3603. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit ABC un triangle et $0 < \lambda < \frac{1}{2}$. Soit D et E deux points sur AB tels que $AD = BE = \lambda \cdot AB$, F et G deux points sur AC tels que $AF = CG = \lambda \cdot AC$. Soit $BF \cap CE = H$ et $BG \cap CD = I$. Montrer que

i) $HI \parallel BC$ et

ii) $HI = \dfrac{1-2\lambda}{\lambda^2 - \lambda + 1} BC$.

3604. *Proposé par Michel Bataille, Rouen, France.*

Calculer

$$\lim_{n\to\infty} \frac{\int_0^1 (x^2 - x - 2)^n dx}{\int_0^1 (4x^2 - 2x - 2)^n dx}.$$

3605. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit $A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$ un polynôme réel à coefficients positifs n'ayant que des racines réelles. Montrer que

$$\sqrt[n]{A(1)A(2)\cdots A(n)} \geq (n+1)!$$

3606. *Proposé par Václav Konečný, Big Rapids, MI, É-U.*

Soit ABC un triangle avec $\angle A = 20^\circ$. Soit BD la bissectrice de l'angle au sommet B avec D sur AC. Si $AD = DB + BC$, trouver $\angle B$.

3607. *Proposé par George Miliakos, Sparte, Grèce.*

Soit $c_1 = 9$, $c_2 = 15$, $c_3 = 21$, $c_4 = 25$, $\ldots$, où c_n désigne le n^{e} entier impair non premier. Calculer

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}}.$$

3608. *Proposé par Michel Bataille, Rouen, France.*

Soit

$$f(x) \;=\; \frac{e^{1/x}-1}{e^{1/(x+1)}-1}\,.$$

(a) Montrer que pour tout $x \in (0,\infty)$,

$$f(x) \;>\; \sqrt{\frac{x+1}{x}}\,.$$

(b) ★ Trouver si oui ou non, on a

$$f(x) \;<\; \sqrt{\frac{x+1}{x-1}}$$

pour tous les $x \in (1,\infty)$

3609. *Proposé par Panagiote Ligouras, École Secondaire Léonard de Vinci, Noci, Italie.*

Soit r un nombre réel et D, E et F des points sur les côtés BC, CA et AB d'un triangle ABC avec

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = r.$$

Les céviennes AD, BE et CF limitent un triangle PQR dont on désigne l'aire par $[PQR]$. Trouver la valeur de r pour laquelle le rapport $\frac{[DEF]}{[PQR]}$ des aires est égal à 4.

3610. *Proposé par Peter Y. Woo, Université Biola, La Mirada, CA, É-U.*

Soit $S = \{2,3,4,6,8,9,12,16,18,24,27,\ldots\}$ l'ensemble des entiers positifs dont les seuls diviseurs premiers sont 2 ou 3. Notons $a_1 = 2, a_2 = 3, \ldots$ les éléments de S, avec $a_1 < a_2 < \ldots$.

(i) Trouver $\displaystyle\sum_{i=1}^{\infty}\left(\frac{1}{a_i}\right)$.

(ii) ★ Pour chaque entier positif n, soit $s(n)$ la somme de tous ses diviseurs, y compris 1 et n lui-même. Montrer que $\frac{s(n)}{n} < 3$ pour tous les éléments de S.

3611. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

On donne trois entiers positifs x, y et z tels que

$$\frac{x(y+1)}{x-1}, \frac{y(z+1)}{y-1} \text{ et } \frac{z(x+1)}{z-1}$$

sont des entiers positifs. Trouver le plus petit entier positif N tel que $xyz \leq N$.

3612. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Trouver tous les polynômes non constants P tels que $P(\{x\}) = \{P(x)\}$ pour tout $x \in \mathbb{R}$, où $\{a\}$ désigne la partie fractionnaire de a.

Just a reminder, it makes it easier for us if problem proposals and solutions are sent to us in electronic format. Material sent in TEX or LATEX is preferred, but we will also accept pdf, Microsoft Word as well as handwritten material (mailed or scanned).

When sending electronic solutions, please name the files in a meaningful way to identify yourself and the problem. For example, if I was to submit a solution to problem 3603 from this issue, I would name it **February_3603_Godin.tex**.

Please place each solution on its own separate sheet(s) with the problem number, your name and affiliation on each page. Multiple solutions on one page means we have to do lots of photocopying and it increases the chances that something will get overlooked or misfiled.

As always no problem is ever closed. We always accept new solutions and generalizations to past problems. Also, in the last issue [2010 : 545, 547], Chris Fisher published a list of unsolved problems from *Crux*. Below is a sample of one of these unsolved problems:

609★. [1981 : 49; 1982 : 27-28] *Proposed by Ian June L. Garces, Ateneo de Manila University, The Philippines.*

$A_1B_1C_1D_1$ is a convex quadrilateral inscribed in a circle and M_1, N_1, P_1, Q_1 are the mid-points of sides B_1C_1, C_1D_1, D_1A_1, A_1B_1, respectively. The chords A_1M_1, B_1N_1, C_1P_1, D_1Q_1 meet the circle again in A_2, B_2, C_2, D_2, respectively. Quadrilateral $A_3B_3C_3D_3$ is formed from $A_2B_2C_2D_2$ as the latter was formed from $A_1B_1C_1D_1$, and the procedure is repeated indefinitely. Prove that quadrilateral $A_nB_nC_nD_n$ "tends to" a square as $n \to \infty$.

What happens if $A_1B_1C_1D_1$ is not convex?

Enjoy!

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3501. [2010 : 44, 46] *Proposed by Hassan A. ShahAli, Tehran, Iran.*

Let $\mathbb{N}$ be the set of positive integers, E the set of all even positive integers, and O the set of all odd positive integers. A set $S \subseteq \mathbb{N}$ is *closed* if $x + y \in S$ for all distinct x, $y \in S$, and *unclosed* if $x + y \notin S$ for all distinct x, $y \in S$. Prove that if $\mathbb{N}$ is partitioned into A and B, where A is closed and nonempty, and B is unclosed and infinite, then $A = E$ and $B = O$.

Solution by Harry Sedinger, St. Bonaventure University, St. Bonaventure, New York, USA.

We prove first that $1 \in B$.

Assume by contradiction that $1 \in A$. Chose $m, n \in B$ then $m + n \in A$. But since $1 \in A$, it follows that all integers greater than $m + n$ are in A, which contradicts B infinite.

We prove now that $2 \in A$. Again assume by contradiction that $2 \in B$. Chose $n > 2, n \in B$. Then $3 \in A, n + 1 \in A, n + 2 \in A$ and hence $n + 4, n + 5 \in A$. Then $2n + 5, 2n + 6, 2n + 7 \in A$, and since $3 \in A$, it follows that A contains all the integers greater than $2n + 5$, which contradicts B infinite.

Now we show that $3 \in B$. Assume by contradiction $3 \in A$. Then $5 \in A, 7 \in A, 8 \in A$, and then, since $2 \in A$, it follows that A contains all the integers greater than 7. Again this contradicts B infinite.

Next we prove that neither A nor B contains two consecutive integers.

If $n, n + 1$ are in A, then $n \geq 4$, and since $2 \in A$, it follows that A contains all the integers greater than n, a contradiction.

Assume now that B contains two consecutive integers $n, n + 1$. Since B is infinite, there exists $m \in B; m > n + 1$. But then A contains the consecutive integers $m + n, m + n + 1$, a contradiction.

Hence $1 \in B, 2 \in A$ and neither A nor B contains two consecutive integers. Then it follows by induction that $2k - 1 \in B, 2k \in A$ for all $k \geq 1$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW, Germany; JOHAN GUNARDI, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia; MICHAEL JOSEPHY, Universidad de Costa Rica, San Pedro, Costa Rica; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect solution.

3502. [2010 : 44, 46] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find all real solutions of the following system of equations

$$\begin{aligned} x_1^2 + \sqrt{x_2^2 + 21} &= \sqrt{x_2^2 + 77}\,, \\ x_2^2 + \sqrt{x_3^2 + 21} &= \sqrt{x_3^2 + 77}\,, \\ \cdots \quad & \cdots \quad \cdots \\ x_n^2 + \sqrt{x_1^2 + 21} &= \sqrt{x_1^2 + 77}\,. \end{aligned}$$

Similar solutions by Michel Bataille, Rouen, France; and Titu Zvonaru, Cománeşti, Romania. We give Bataille's write up.

Let $u_i = x_i^2$ and $f(x) = \sqrt{x+77} - \sqrt{x+21}$. Then, the equations are:

$$u_1 = f(u_2)\,;\; u_2 = f(u_3)\,;\; \dots;\; u_{n-1} = f(u_n)\,;\; u_n = f(u_1)\,.$$

We observe that $f([0,\infty)) \subset [0,\infty)$ and $f(4) = 4$. In addition $f'(x) = \frac{1}{2}\left(\frac{1}{\sqrt{x+77}} - \frac{1}{\sqrt{x+21}}\right)$, hence

$$|f'(x)| < \frac{1}{2\sqrt{x+21}} \le \frac{1}{2\sqrt{21}}\,.$$

Let $k = \frac{1}{2\sqrt{21}}$ and f^m denote the composition $f \circ f \circ f \circ \dots \circ f$. Then $k < 1$ and it follows by induction that

$$|(f^m)'(x)| < k^m\,,$$

for all $x \in [0,\infty)$ and m positive integer.

Let $1 \le i \le n$. Then $u_i \in [0,\infty)$ and $f^n(u_i) = u_i$. We show that $u_i = 4$.

Assume by contradiction that $u_i \ne 4$. Then by the Mean Value Theorem, there exists a c between u_i and 4 so that

$$|u_i - 4| = |(f^n)(u_i) - (f^n)(4)| = |(f^n)'(c)|\,|u_i - 4| \le k^n\,|u_i - 4|\,.$$

But this contradicts $k < 1$.

This shows that $u_i = 4$ for all i. Conversely $u_1 = u_2 = .. = u_n = 4$ is obviously a solution for our system.

Thus, all the real solutions of the system are $(\pm 2, \pm 2, \dots, \pm 2)$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; and the proposer. One incomplete solution was submitted.

3503. [2010 : 44, 47] *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Given a triangle and the midpoints of its sides, with the use of a straight edge and only three uses of a pair of compasses, bisect all three angles of the triangle.

Solution by Mohammed Aassila, Strasbourg, France.

Soit ABC le triangle donné ; I, J et K les milieux respectifs de BC, CA et AB. On va utiliser une **seule fois** le compas.

On trace le cercle de centre K et de rayon $KA = KB$. Soit E le point d'intersection de la droite KJ avec ce cercle. Comme $KE = KB$, alors E est sur la bissectrice de l'angle ABC car KBE triangle isocèle et KJ parallèle à BC.

De même, soit F le point d'intersection du cercle avec la droite KI, alors on a $KF = KA$, et donc F est sur la bissectrice de l'angle BAC car KAF triangle isocèle et KI est parallèle à AC.

Ces deux bissectrices se coupent en H, centre du cercle inscrit et donc sur la troisième bissectrice CH.

Les trois bissectrices sont donc AH, BH et CH.

Also solved by the following readers, with the number of uses of the compass indicated in parentheses: MICHEL BATAILLE, Rouen, France (2); OLIVER GEUPEL, Brühl, NRW, Germany (2); JOHAN GUNARDI, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia (3); GEOFFREY A. KANDALL, Hamden, CT, USA (2); VÁCLAV KONEČNÝ, Big Rapids, MI, USA (2); MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA (2); EDMUND SWYLAN, Riga, Latvia (1); PETER Y. WOO, Biola University, La Mirada, CA, USA (2); TITU ZVONARU, Comăneşti, Romania (2); and the proposer (3).

Swylan was the only other solver who used the compass just once. Bataille, the Missouri State University Problem Solving Group, and Woo all noted that by the Poncelet–Steiner theorem just one use of the compass suffices to carry out the construction.

3504. *Proposed by Mariia Rozhkova, Kiev, Ukraine.*

Given triangle ABC, set $Q = a\cos^2 A + b\cos^2 B + c\cos^2 C$, and let ABC have area S and circumradius R. Prove that

(a) $Q \geq \dfrac{S}{R}$, with equality if and only if ABC is equilateral.

(b) $Q \leq \dfrac{S\sqrt{2}}{R}$ if ABC is not obtuse, with equality if and only if ABC is an isosceles right triangle.

Solution to part (a) by Michel Bataille, Rouen, France; solution to part (b) by the proposer, modified by the editor.

(a) Let $a = BC$, $b = CA$, $c = AB$ and let I and H be the incentre and orthocentre of $\triangle ABC$. Since $2sI = aA + bB + Cc$, we have

$2s\overrightarrow{HI} = a\overrightarrow{HA} + b\overrightarrow{HB} + c\overrightarrow{HC}$, and so

$$\begin{aligned} 4s^2HI^2 &= a^2HA^2 + b^2HB^2 + c^2HC^2 \\ &\quad + 2ab\overrightarrow{HA}\cdot\overrightarrow{HB} + 2bc\overrightarrow{HB}\cdot\overrightarrow{HC} + 2ca\overrightarrow{HC}\cdot\overrightarrow{HA} \\ &= a^2HA^2 + b^2HB^2 + c^2HC^2 + ab(HB^2 + HA^2 - c^2) \\ &\quad + bc(HC^2 + HB^2 - a^2) + ca(HA^2 + HC^2 - b^2) \\ &= 2s(aHA^2 + bHB^2 + cHC^2 - abc)\,. \end{aligned}$$

Now, if A' is the midpoint of BC, then $HA^2 = (2OA')^2 = 4R^2\cos^2 A$ and similar results hold for HB^2 and HC^2. It follows that $2sHI^2 = 4R^2 \cdot Q - abc$ and

$$Q = \frac{abc}{4R^2} + \frac{sHI^2}{2R^2}\,.$$

Since $\dfrac{abc}{4R^2} = \dfrac{S}{R}$, we see that $Q \geq \dfrac{S}{R}$, with equality if and only if $H = I$, that is, if and only if ABC is equilateral.

(b) Substituting the well-known relations $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$, $S = 2R^2\sin A\sin B\sin C$ into the inequality and canceling $2R$ from each side yields the equivalent inequality

$$\sin A\cos^2 A + \sin B\cos^2 B + \sin C\cos^2 C \leq \sqrt{2}\sin A\sin B\sin C\,.$$

Assume that $A \geq B \geq C$ and set $\phi = \frac{1}{2}(B + C)$, $\psi = \frac{1}{2}(B - C)$. Then from the hypotheses we have $0 \leq \psi \leq \dfrac{\pi}{4} \leq \phi \leq \dfrac{\pi}{3}$, and

$$A = \pi - 2\phi\,, \qquad B = \phi + \psi\,, \qquad C = \phi - \psi\,.$$

In terms of ϕ and ψ the desired inequality takes the form

$$\begin{aligned} &\sin 2\phi\cos^2 2\phi + \cos^2(\phi+\psi)\sin(\phi+\psi) + \cos^2(\phi-\psi)\sin(\phi-\psi) \\ &\quad \leq \sqrt{2}\sin 2\phi\sin(\phi+\psi)\sin(\phi-\psi)\,. \end{aligned}$$

In this inequality make the replacements $\sin 2\phi = 2\sin\phi\cos\phi$, $\cos 2\phi = \cos^2\phi - \sin^2\phi$, $\cos(\phi\pm\psi) = \cos\phi\cos\psi \mp \sin\phi\sin\psi$, and $\sin(\phi\pm\psi) = \sin\phi\cos\psi \pm \cos\phi\sin\psi$, then expand each side and cancel like terms, then cancel a common term $\sin\phi > 0$ from each side, then apply the identities $\sin^2 x = 1 - \cos^2 x$ for $x = \phi, \psi$. This yields the equivalent inequality

$$\begin{aligned} &\cos^3\phi - (1-\cos^2\phi)\cos\phi + \cos^2\phi\cos^3\psi \\ &+ (1-\cos^2\phi)\cos\psi(1-\cos^2\psi) - 2\cos^2\phi(1-\cos^2\psi)\cos\psi \\ \leq\ &\sqrt{2}(1-\cos^2\phi)\cos\phi\cos^2\psi - \sqrt{2}\cos^3\phi(1-\cos^2\psi) \end{aligned}$$

Set $x = \cos\phi$, $y = \cos\psi$, so $x \in \left[\dfrac{1}{2}, \dfrac{\sqrt{2}}{2}\right]$ and $y \in I = \left[\dfrac{\sqrt{2}}{2}, 1\right]$. Making these substitutions and simplifying yields the equivalent inequality

$$x^3(2+\sqrt{2}) + 4x^2y^3 - 3x^2y - \sqrt{2}xy^2 - x - y^3 + y \leq 0\,.$$

Let $t = x\sqrt{2}$. Then t and y are in the interval I, and we need to prove that

$$f(t,y) = \frac{2+\sqrt{2}}{2\sqrt{2}} + 2t^2y^3 - \frac{3}{2}t^2y - ty^2 - \frac{t}{\sqrt{2}} - y^3 + y \leq 0\,.$$

We have

$$f(1,y) = y^3 - y^2 - \frac{1}{2}y + \frac{1}{2} = (y-1)\left(y^2 - \frac{1}{2}\right) \leq 0\,, \qquad y \in I\,,$$

with equality only for $y = 1$ and $y = \frac{\sqrt{2}}{2}$.

Next we prove that $f(t,y) \leq f(1,y)$.

Let $g(y) = 4y^3 - 2y^2 - 3y + 1$ and $h(y) = 4y^3 - 3y$. We then have

$$2[f(t,y) - f(1,y)] = (t-1)\left[t^2(\sqrt{2}+1) + t(\sqrt{2}+1) + th(y) + g(y)\right]\,.$$

Since $t - 1 \leq 0$, it suffices to prove that

$$t^2(\sqrt{2}+1) + t(\sqrt{2}+1) + th(y) + g(y) > 0\,. \qquad (1)$$

Note that (1) follows from

$$t(\sqrt{2}+1) + g(y) \geq 1\,, \qquad (2)$$

$$t(\sqrt{2}+1) + h(y) \geq 1\,, \qquad (3)$$

since the left side of (1) is t times the left side of (3) plus the left side of (2).

The quadratic equation $g'(y) = 0$ has roots $y = \frac{1 \pm \sqrt{10}}{2}$ and $\frac{1+\sqrt{10}}{2} < \frac{\sqrt{2}}{2}$, so $g(y)$ is strictly increasing on I and $g(y) \geq g\left(\frac{\sqrt{2}}{2}\right) = \frac{-\sqrt{2}}{2}$. Also, $t \geq \frac{\sqrt{2}}{2}$, so that $t(\sqrt{2}+1) \geq 1 + \frac{\sqrt{2}}{2}$, and (2) follows.

Furthermore, $h(y) \geq g(y)$ for $y \in I$, since $h(y) = g(y) + (2y^2 - 1)$ and $2y^2 - 1 \geq 0$ on I, so (3) follows from (2).

This completes the proof of the inequality in part (b).

Equality holds in $f(t,y) - f(1,y) \leq 0$ only for $t = 1$, and equality holds in $f(1,y) \leq 0$ only for $y = 1$ and $y = \frac{\sqrt{2}}{2}$. These cases correspond to a triangle with $A = \frac{\pi}{2}$ and $B = C = \frac{\pi}{4}$, or to a degenerate triangle with $A = B = \frac{\pi}{2}$ and $C = 0$.

Part (a) *also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; SCOTT BROWN, Auburn University, Montgomery, AL, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; and the proposer. One incomplete solution to part* (a)*, one incorrect solution to part* (a)*, and three incomplete solutions to part* (b) *were received.*

The proposer said she was influenced by ***Crux*** *problem 3167, which asked to show that* $a\cos^3 A + b\cos^3 B + c\cos^3 C \leq abc/4R^2$ *holds for non-obtuse triangles* ABC*. She indicated that the inequality in part* (a) *occurs in a different form on p. 14 of V.P. Soltan and I. Majdan's book* Identities and Inequalities in a Triangle*, Kishinev, 1982 (Russian), although the proof there is of a general nature and different from the one she constructed.*

3505. [2010 : 45, 47, 107, 109] *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

The circles Γ_1 and Γ_2 have a common centre O, and Γ_1 lies inside Γ_2. The point $A \neq O$ lies inside Γ_1; a ray not parallel to AO that starts at A intersects Γ_1 and Γ_2 at the points B and C, respectively. Let tangents to corresponding circles at the points B and C intersect at the point D. Let E be a point on the line BC such that DE is perpendicular to BC. Prove that $AB = EC$ if and only if OA is perpendicular to BC.

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

Let A' be the projection of O on the line $\ell = BC$. A and A' lie on the same side of B on ℓ (because they are interior points of Γ_1), so we have

$$OA \perp \ell \Leftrightarrow A = A' \Leftrightarrow AB = A'B. \tag{1}$$

Since $\angle OBD = \angle OCD = 90^\circ$, B and C lie on the circle whose diameter is OD, which we will denote by Γ_3; let M be its centre. Since $MB = MC$, the projection of M on ℓ is the midpoint N of BC. The lines OA', MN, and DE are parallel (they are all perpendicular to ℓ). Since $OM = MD$ it follows that $A'N = NE$. Hence, $A'B = A'N - BN = NE - NC = EC$. That is,

$$A'B = EC. \tag{2}$$

Using (2) and (1) we conclude that $AB = EC$ if and only if $AB = A'B$ if and only if $OA \perp \ell$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; RICHARD EDEN, student, Purdue University, West Lafayette, IN, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHAN GUNARDI, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia; JOHN G. HEUVER, Grande Prairie, AB; JOEL SCHLOSBERG, Bayside, NY, USA; MIHAÏ STOËNESCU, Bischwiller, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.

3506. [2010 : 45, 47] *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Prove that $Q(n) + Q\left(n^2\right) + Q\left(n^3\right)$ is a perfect square for infinitely many positive integers n that are not divisible by 10, where $Q(n)$ is the sum of the digits of n.

Solution by the Missouri State University Problem Solving Group, Springfield, MO, USA.

More generally, we will show that the following result holds: If $Q(n, b)$ denotes the sum of the digits of n in base b, then $Q(n, b) + Q(n^2, b) + Q(n^3, b)$

is a perfect square for infinitely many positive integers n that are not divisible by b.

Let a be the square-free part of $b-1$, $k = a\ell^2$ with $\ell \in \mathbb{N}$, and $n = b^k - 1$. Now n consists of k $b-1$'s when written in base b and hence $Q(n) = k(b-1)$. Using the binomial theorem, it is easy to see that the base b representation of n^2 consists of $k-1$ $b-1$'s, one $b-2$, $k-1$ 0's, and one 1 (hence $Q(n^2, b) = k(b-1)$) and n^3 consists of $k-1$ $b-1$'s, one $b-3$, $k-1$ 0's, one 2, and k $b-1$'s (hence $Q(n^3, b) = 2k(b-1)$). Therefore $Q(n, b) + Q(n^2, b) + Q(n^3, b) = 4k(b-1) = 4a(b-1)\ell^2$, but since a is the square-free part of $b-1$, $a(b-1)$ is a perfect square and we're done.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHAN GUNARDI, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PETER HURTHIG, Columbia College, Vancouver, BC; MICHAEL JOSEPHY, Universidad de Costa Rica, Costa Rica; R. LAUMEN, Deurne, Belgium; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

The other submitted solutions were similar and they can be summarized by the following sequences $n_k = 2 \times 10^k + 7$ $(k \geq 2)$, $n_k = 7 \times 10^k + 2$ $(k \geq 2)$, $n_k = 10^{k^2} - 1$ $(k \geq 1)$, $n_k = 10^k + 17$ $(k \geq 4)$, *and* $n_k = 18 \times 10^k + 18$ $(k \geq 4)$.

3507. [2010 : 45, 47] *Proposed by Pham Huu Duc, Ballajura, Australia.*

Let a, b, and c be positive real numbers. Prove that

$$\sqrt{\frac{a(b+c)}{a^2+bc}} + \sqrt{\frac{b(c+a)}{b^2+ca}} + \sqrt{\frac{c(a+b)}{c^2+ab}} \leq \sqrt{2(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right)}\,.$$

Solution by Joe Howard, Portales, NM, USA, modified by the editor.

By the Cauchy-Schwarz inequality, we have that

$$\left(\sum_{\text{cyc}} \sqrt{\frac{a(b+c)}{a^2+bc}}\right)^2 \leq 2(a+b+c)\left(\sum_{\text{cyc}} \frac{a}{a^2+bc}\right)$$

Thus, it suffices to show that

$$\sum_{\text{cyc}} \frac{a}{a^2+bc} \leq \sum_{\text{cyc}} \frac{1}{a+b}$$

Without loss of generality, let $a \geq b \geq c$. Then $(a-c)(b-c) \geq 0$, so $c^2 + ab \geq ac + bc$ and $\frac{c}{c^2+ab} \leq \frac{c}{ac+bc} = \frac{1}{a+b}$. Therefore, it now suffices to show that

$$\frac{a}{a^2+bc} + \frac{b}{b^2+ac} \leq \frac{1}{b+c} + \frac{1}{a+c}$$

Since this inequality holds for $a = b$, we can assume that $a > b \geq c$. This simplifies to

$$a^2c^2 + b^2c^2 + a^3b + ab^3 + ab^2c + a^2bc \leq 2abc^2 + a^4 + b^4 + a^3c + b^3c$$

$$c^2(a-b)^2 + b^2c(a-b) + b^3(a-b) \leq a^3(a-b) + a^2c(a-b)$$

Since $a - b > 0$, we obtain

$$c^2(a-b) \leq a^3 - b^3 + c(a^2 - b^2)$$

$$c^2(a-b) \leq (a-b)(a^2+ab+b^2) + c(a-b)(a+b)$$

$$c^2 \leq a^2 + ab + b^2 + c(a+b)$$

The result follows.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.

3508. [2010 : 45, 47] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, c, d be nonnegative real numbers such that $a+b+c+d = 4$. Prove that

$$a\sqrt{bc} + b\sqrt{cd} + c\sqrt{da} + d\sqrt{ab} \;\leq\; 2\left(1 + \sqrt{abcd}\right).$$

Solution by the proposer.

Let (x, y, z, t) be a permutation of (a, b, c, d) such that $x \geq y \geq z \geq t$. We clearly have $\sqrt{x} \geq \sqrt{y} \geq \sqrt{z} \geq \sqrt{t}$ and

$$\sqrt{xyz} \;\geq\; \sqrt{xyt} \;\geq\; \sqrt{xzt} \;\geq\; \sqrt{yzt},$$

and therefore, by the Rearrangement Inequality, we have

$$\sqrt{x}\sqrt{xyz} + \sqrt{y}\sqrt{xyt} + \sqrt{z}\sqrt{xzt} + \sqrt{t}\sqrt{yzt}$$
$$\geq\; \sqrt{a}\sqrt{abc} + \sqrt{b}\sqrt{bcd} + \sqrt{c}\sqrt{cda} + \sqrt{d}\sqrt{dab}.$$

It remains to prove that

$$\sqrt{x}\sqrt{xyz} + \sqrt{y}\sqrt{xyt} + \sqrt{z}\sqrt{xzt} + \sqrt{t}\sqrt{yzt} \;\leq\; 2(1 + \sqrt{abcd}),$$

or

$$(\sqrt{xy} + \sqrt{zt})(\sqrt{xz} + \sqrt{yt}) \;\leq\; 2(1 + \sqrt{xyzt}).$$

Since $uv \leq \frac{1}{2}\left(u^2 + v^2\right)$, it is enough to prove that

$$(\sqrt{xy} + \sqrt{zt})^2 + (\sqrt{xz} + \sqrt{yt})^2 \;\leq\; 4(1 + \sqrt{xyzt}),$$

or

$$xy + zt + xz + yt \;\le\; 4,$$

which is equivalent to $(x+t)(y+z) \le 4$. This is clearly true by the AM–GM Inequality, since $x+y+z+t=4$, and we are done.

There were 2 incomplete solutions.

3509. [2010 : 45, 48] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. For each positive real number k, find the maximum value of

$$\left(a^2b+k\right)\left(b^2c+k\right)\left(c^2a+k\right).$$

The proposer's submitted solution is distributed among several other solutions to several other problem proposals, while all of the other submitted solutions to this problem were either incomplete or incorrect.

The editor has therefore elected to leave this problem open until a correct and complete "one piece" solution is received.

3510. [2010 : 45, 48] *Proposed by Cosmin Pohoaţă, Tudor Vianu National College, Bucharest, Romania.*

Let d be a line exterior to a given circle Γ with centre O. Let A be the orthogonal projection of O on the line d, M be a point on Γ, and X, Y be the intersections of Γ, d with the circle Γ' of diameter AM. Prove that the line XY passes through a fixed point as M moves about Γ.

I. *Solution by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia, modified by the editor.*

When M lies on OA the lines OA and XY coincide, so that a fixed point would necessarily lie on OA. For any position of M on Γ off OA let Q denote the intersection of XY and OA. We must prove that the position of Q is independent of the choice of M. Let P be the centre of Γ'; we first show that $OPQX$ is cyclic. To that end, note that $\angle OPM$ is both an exterior angle of ΔAOP and half the apex angle of the isosceles triangle PXM. Therefore,

$$\angle OAP + \angle POA = \angle OPM = 90^\circ - \angle AMX. \tag{1}$$

Also, because $MY \perp YA$ and $YA \perp OA$, it follows that $MY || OA$ and, thus, $\angle OAP = \angle YMA$. From (1) therefore,

$$\angle OAP + \angle POA - \angle OAP = 90^\circ - \angle AMX - \angle YMA$$

that is,

$$\angle POA = 90^\circ - \angle YMX. \tag{2}$$

But $\angle POQ = \angle POA$ and, because the angle at the center P of Γ' is twice the corresponding inscribed angle at M, $\angle YMX = \frac{1}{2}\angle YPX$. Note also that $90^\circ - \frac{1}{2}\angle YPX = \angle PXY = \angle PXQ$. Consequently, equation (2) becomes $\angle POQ = \angle PXQ$, and we conclude that O, P, Q, X are concyclic, as desired. This now implies that $\angle OQP = \angle OXP$; moreover, because triangles PXM and OMX are isosceles we have $\angle OXP = \angle PMO$. Define R to be the point where the line parallel to PQ through M meets OA. We have $\angle ORM = \angle OQP = \angle PMO = \angle AMO$. Therefore, triangles MRO and AMO are similar and we have $\frac{OR}{OM} = \frac{OM}{OA}$; hence

$$OR = \frac{OM^2}{OA},$$

so that R is a fixed point. But because P is the midpoint of AM and $PQ||MR$, Q must be the midpoint of AR. We conclude that line XY passes through the midpoint of the fixed segment AR as M moves about Γ.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

We do not need the assumption that d is exterior to Γ so long as it is neither tangent to Γ nor passing through O. Let p be the power of A with respect to Γ. Note that p is negative if A is inside Γ. Let i be the transformation defined by $i(U) = V$ if and only if A, U, V are collinear and as signed lengths, $AU \cdot AV = p$. [*Editor's comment.* Zhou called this transformation an inversion. When A is outside Γ, then i is indeed an inversion in the circle with centre A that is orthogonal to Γ; when A is inside Γ, i is the commutative product of a halfturn about A and inversion in the circle with centre A whose diameter is the chord intercepted from d by Γ.] Note that d and Γ are invariant under i. Let $M' = i(M), X' = i(X)$, and $Y' = i(Y)$. Then $i(\Gamma')$ is the line passing through M', X', Y' and $i(XY)$is the circle passing through A, X', Y'. Since $AX \perp MX$, we have $AM' \perp M'X'$, whence MX' is a diameter of Γ. Let B be the point symmetrical with A about O. Then the triangles OBX' and OAM are congruent, which implies that $\angle OBP = \angle OAM$. This last angle equals the directed angle between the lines $Y'A$ and $Y'X'$ (because corresponding sides are perpendicular). We conclude that B is on the circle through A, X', and Y', whence $i(B)$ is the fixed point through which XY passes.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece(2 solutions); MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comăneşti, Romania; and the proposer. There was one incorrect submission.

The final equation of solution I implies that R is the inverse of A with respect to Γ, something Woo proved in his solution using projective geometry. Many of the other solutions easily solved the problem with the help of coordinates or trigonometry.

3511. [2010 : 46, 48] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, and d be nonnegative real numbers. Prove that

$$\prod_{\text{cyclic}} \left(a^2+b^2+c^2\right) \;\le\; \frac{1}{64}(a+b+c+d)^8.$$

Solution by George Apostolopoulos, Messolonghi, Greece.

We write $f(a,b,c,d) = \prod_{\text{cyclic}} \left(a^2+b^2+c^2\right)$, and without loss of generality we assume that $a \ge b \ge c \ge d$.

Since

$$b^2+c^2+d^2 \le \left(b+\frac{c+d}{2}\right)^2,$$
$$c^2+d^2+a^2 \le \left(a+\frac{c+d}{2}\right)^2,$$
$$a^2+b^2+c^2 \le \left(a+\frac{c+d}{2}\right)^2 + \left(b+\frac{c+d}{2}\right)^2,$$

and

$$a^2+c^2+d^2 \le \left(a+\frac{c+d}{2}\right)^2 + \left(b+\frac{c+d}{2}\right)^2,$$

we obtain $f(a,b,c,d) \le f\left(a+\frac{c+d}{2}, b+\frac{c+d}{2}, 0, 0\right)$. Let $x = a + \frac{c+d}{2}$ and $y = b+\frac{c+d}{2}$, so that $x+y = a+b+c+d$.

We now need to prove that $\left(x^2+y^2\right)^2 x^2y^2 \le \frac{1}{64}(x+y)^8$. However, this inequality follows from an application of the AM–GM inequality:

$$\left(x^2+y^2\right)xy = \frac{1}{2}\left(x^2+y^2\right)(2xy) \le \frac{1}{2}\left(\frac{x^2+y^2+2xy}{2}\right)^2 = \frac{1}{8}(x+y)^4,$$

and the proof is complete.

Equality holds precisely when two of a, b, c, d are equal and the remaining two are zero.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece (second solution); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OLIVER GEUPEL, Brühl, NRW, Germany; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3512. [2010 : 46, 48] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let α be a real number and let $p \geq 1$. Find

$$\lim_{n\to\infty} \prod_{k=1}^{n} \frac{n^p + (\alpha-1)k^{p-1}}{n^p - k^{p-1}}.$$

Solution by Albert Stadler, Herrliberg, Switzerland.

Let $p_n = \prod_{k=1}^{n} \frac{n^p + (\alpha-1)k^{p-1}}{n^p - k^{p-1}}$. Then

$$\begin{aligned} \ln p_n &= \sum_{k=1}^{n} \ln\left(\frac{1 + (\alpha-1)\left(\frac{k}{n}\right)^{p-1} \cdot \frac{1}{n}}{1 - \left(\frac{k}{n}\right)^{p-1} \cdot \frac{1}{n}} \right) \\ &= \sum_{k=1}^{n} \ln\left(1 + (\alpha-1)\left(\frac{k}{n}\right)^{p-1} \cdot \frac{1}{n}\right) \\ &\quad - \sum_{k=1}^{n} \ln\left(1 - \left(\frac{k}{n}\right)^{p-1} \cdot \frac{1}{n}\right). \end{aligned} \tag{1}$$

For each $k = 1, 2, \ldots, n$, we have, as $n \to \infty$

$$\ln\left(1 + (\alpha-1)\left(\frac{k}{n}\right)^{p-1} \cdot \frac{1}{n}\right) = (\alpha-1)\left(\frac{k}{n}\right)^{p-1} \cdot \frac{1}{n} + O\left(\frac{1}{n^2}\right), \tag{2}$$

$$\ln\left(1 - \left(\frac{k}{n}\right)^{p-1} \cdot \frac{1}{n}\right) = -\left(\frac{k}{n}\right)^{p-1} \cdot \frac{1}{n} + O\left(\frac{1}{n^2}\right). \tag{3}$$

From (1), (2), and (3) we have

$$\ln p_n = \alpha \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{p-1} \cdot \frac{1}{n} + O\left(\frac{1}{n^2}\right). \tag{4}$$

Note that the sum $\sum_{n=1}^{n} S_n = \left(\frac{k}{n}\right)^{p-1} \cdot \frac{1}{n}$ is a Riemann sum for the function $f(x) = x^{p-1}$ over the interval $[0,1]$. Hence,

$$\lim_{n\to\infty} S_n = \int_0^1 x^{p-1}\,dx = \frac{1}{p}. \tag{5}$$

From (4) and (5) we have $\lim_{n\to\infty} \ln p_n = \frac{\alpha}{p}$, hence $\lim_{n\to\infty} p_n = e^{\alpha/p}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; ANASTASIOS KOTRONIS, Heraklion, Greece; and the proposer. Two incorrect solutions were submitted.

3513. [2010 : 46, 48] *Proposed by Hassan A. ShahAli, Tehran, Iran.*

Let α and β be positive real numbers, and r be a positive rational number. Prove that there exist infinitely many integers m and n such that

$$\frac{\lfloor m\alpha \rfloor}{\lfloor n\beta \rfloor} = r,$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

Solution by Oliver Geupel, Brühl, NRW, Germany, modified by the editor.

We show that the statement is false. It is well known (see D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom, *The USSR Olympiad Problem Book: selected problems and theorems of elementary mathematics*, Dover, New York, 1993, Problem 108) that if α and β are positive irrational numbers such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$, then all positive integers appear with no duplications in the two sequences $\lfloor \alpha \rfloor$, $\lfloor 2\alpha \rfloor$, $\lfloor 3\alpha \rfloor$, $\dots$ and $\lfloor \beta \rfloor$, $\lfloor 2\beta \rfloor$, $\lfloor 3\beta \rfloor$, $\dots$. [*Ed.: These are known in the literature as complementary Beatty sequences.*]

If we take $r = 1$, $\alpha = \sqrt{2}$, and $\beta = 2 + \sqrt{2}$, then clearly $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, so there are no positive integers m and n such that $\frac{\lfloor m\alpha \rfloor}{\lfloor n\beta \rfloor} = 1$. It is also clear that $m \neq 0$, $n \neq 0$, and if m and n are of opposite signs, then $\lfloor m\alpha \rfloor \neq \lfloor n\beta \rfloor$.

Finally, suppose $m < 0$ and $n < 0$. Using the trivial fact that $\lfloor x \rfloor + \lfloor -x \rfloor = -1$ for all reals x which are not integers, we see that the sequences $\lfloor -\alpha \rfloor$, $\lfloor -2\alpha \rfloor$, $\lfloor -3\alpha \rfloor$, $\dots$ and $\lfloor -\beta \rfloor$, $\lfloor -2\beta \rfloor$, $\lfloor -3\beta \rfloor$, $\dots$ contain all nonnegative integers with no duplications. Hence it is again impossible for $\lfloor m\alpha \rfloor = \lfloor n\beta \rfloor$ to hold, and our proof is complete.

Two incorrect solutions were submitted.