

# MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a **Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON) and Eric Robert (Leo Hayes High School, Fredericton, NB).

---

## Mayhem Problems

*Please send your solutions to the problems in this edition by 1 May 2010. Solutions received after this date will only be considered if there is time before publication of the solutions. The Mayhem Staff ask that each solution be submitted on a separate page and that the solver's name and contact information be included with each solution.*

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.*

*The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.*

---

**M420.** *Proposed by the Mayhem Staff.*

Riley is a poor starving university student, but is mathematically astute. He notices that five suppers in residence cost the same as seven lunches. After one week of skipping supper most nights, he notices that five lunches and one supper cost \$48 in total. How much do 16 suppers cost?

**M421.** *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Let  $\lfloor x \rfloor$  be the greatest integer less than or equal to the real number  $x$ . Determine all real numbers  $x$  such that

$$\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{3}{x} \right\rfloor = 4.$$

**M422.** *Proposed by Adnan Arapovic, student, University of Waterloo, Waterloo, ON.*

Prove that

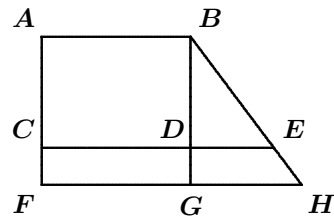
$$\sum_{k=1}^n \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}.$$

**M423.** Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.

The tens digit of a perfect square  $S$  is three greater than the ones digit of  $S$ . Determine all possible remainders when  $S$  is divided by 100.

**M424.** Proposed by Margo Kondratieva, Memorial University of Newfoundland, St. John's, NL.

In the diagram, line segments  $AB$ ,  $CDE$ , and  $FGH$  are parallel. Also, line segments  $ACF$  and  $BDG$  are perpendicular to  $AB$ . Suppose that the area of rectangle  $ABDC$  is  $x$ , the area of rectangle  $CDGF$  is  $y$ , and the area of  $\triangle BDE$  is  $z$ . Determine the area of  $DEHG$  in terms of  $x$ ,  $y$ , and  $z$ .



**M425.** Proposed by Titu Zvonaru, Comănești, Romania.

In  $\triangle ABC$ ,  $\angle BAC = 90^\circ$  and  $I$  is the incentre. The interior bisector of angle  $C$  meets  $AB$  at  $D$ . The line segment through  $D$  perpendicular to  $BI$  intersects  $BC$  at  $E$ . The line segment through  $D$  parallel to  $BI$  meets  $AC$  at  $F$ . Prove that  $E$ ,  $I$ , and  $F$  are collinear.

.....

**M420.** Proposé par l'Équipe de Mayhem.

Richard est un étudiant pauvre et affamé, mais mathématiquement doué. Il a remarqué qu'à la résidence, cinq soupers coûtent le même prix que sept lunchs. Après avoir sauté les soupers presque tous les soirs pendant une semaine, il constate que cinq lunchs et un souper coûtent 48 au total. Combien coûtent 16 soupers ?

**M421.** Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.

Soit  $\lfloor x \rfloor$  le plus grand entier plus petit ou égal au nombre réel  $x$ . Trouver tous les nombres réels tels que

$$\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{3}{x} \right\rfloor = 4.$$

**M422.** Proposé par Adnan Arapovic, étudiant, Université de Waterloo, Waterloo, ON.

Montrer que

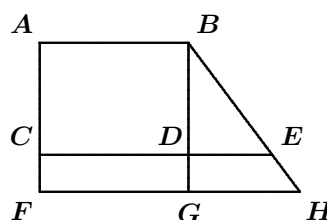
$$\sum_{k=1}^n \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}.$$

**M423.** *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

La différence entre le chiffre des dizaines et celui des unités d'un carré parfait  $S$  est de trois. Trouver tous les restes de la division de  $S$  par 100.

**M424.** *Proposé par Margo Kondratieva, Université Memorial de Terre-Neuve, St. John's, NL.*

Dans la figure ci-contre, les segments de droite  $AB$ ,  $CDE$ , et  $FGH$  sont parallèles. De plus, les segments  $ACF$  et  $BDG$  sont perpendiculaires à  $AB$ . Supposons que les aires respectives des rectangles  $ABDC$ ,  $CDGF$ , et  $\triangle BDE$  sont  $x$ ,  $y$  et  $z$ . Trouver l'aire de  $DEHG$  en fonction de  $x$ ,  $y$  et  $z$ .



**M425.** *Proposé par Titu Zvonaru, Comănești, Roumanie.*

Dans le triangle  $ABC$ ,  $\angle BAC = 90^\circ$  et soit  $I$  le centre du cercle inscrit. La bissectrice intérieure de l'angle  $C$  coupe  $AB$  en  $D$ . La droite passant par  $D$  et perpendiculaire à  $BI$  coupe  $BC$  en  $E$ . La droite passant par  $D$  et parallèle à  $BI$  coupe  $AC$  en  $F$ . Montrer que  $E$ ,  $I$  et  $F$  sont colinéaires.

---

## Mayhem Solutions

---

Last year we received some late solutions that did not appear in the December issue. We therefore acknowledge a correct solution to M383 by Mridul Singh, student, Kendriya Vidyalaya School, Shillong, India, and correct solutions to problems M383, M384, and M386 by Hugo Luyo Sánchez, Pontificia Universidad Católica del Peru, Lima, Peru.

**M388.** *Proposed by Kyle Sampson, St. John's, NL.*

A sequence is generated by listing (from smallest to largest) for each positive integer  $n$  the multiples of  $n$  up to and including  $n^2$ . Thus, the sequence begins 1, 2, 4, 3, 6, 9, 4, 8, 12, 16, 5, 10, 15, 20, 25, 6, 12, ... Determine the 2009<sup>th</sup> term in the sequence.

*Solution by Kristóf Huszár, Valéria Koch Grammar School, Pécs, Hungary.*

First, we notice that there are  $k$  positive integral multiples of  $k$  less than or equal to  $k^2$ . If we group the terms of the sequence as the multiples of 1, then the multiples of 2, then the multiples 3, and so on, we notice that the groups have 1 term, then 2 terms, then 3 terms, and so on.

If  $n$  is a positive integer, then the sum of the first  $n$  positive integers is equal to  $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ . Therefore,  $n^2$  is the  $\left(\frac{n(n+1)}{2}\right)^{\text{th}}$  term of the sequence.

Hence,  $63^2 = 3969$  is the  $\left(\frac{63 \cdot 64}{2}\right)^{\text{th}} = 2016^{\text{th}}$  term. Since 3969 occurs  $2016 - 2009 = 7$  terms after the  $2009^{\text{th}}$  term, we find that the  $2009^{\text{th}}$  term is  $63^2 - 7(63) = 3528$ .

*Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; JACLYN CHANG, student, Western Canada High School, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. One incorrect solution was submitted.*

**M389.** *Proposed by Lino Demasi, student, Simon Fraser University, Burnaby, BC.*

There are 2009 students and each has a card with a different positive integer on it. If the sum of the numbers on these cards is 2020049, what are the possible values for the median of the numbers on the cards?

*Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.*

Let  $a$  be the smallest of the numbers on the cards. The smallest possible sum of the 2009 numbers on the cards is then

$$\begin{aligned} S &= a + (a + 1) + (a + 2) + \cdots + (a + 2008) \\ &= 2009a + \frac{1}{2}(2008)(2009) = 2009a + 2017036. \end{aligned}$$

If  $a \geq 2$  then  $S \geq 2021054 > 2020049$ . Therefore,  $a = 1$ .

Next, consider the sequence of numbers 1, 2, 3, ..., 2009. The sum of these numbers is 2019045 (which is 1004 less than the desired sum of 2020049). Also, their median is 1005.

In order to obtain the desired sum of 2020049, some of the numbers in this sequence need to be increased. When a certain term in the sequence is increased, then every greater term must be increased as well in order for the terms of the sequence to remain distinct. If a term that is less than 1006 is increased, then every larger term will also have to increase, resulting in an increase of the initial sum 2019045 by at least 1005 (since at least 1005 terms are increased), yielding a new sum of at least  $2019045 + 1005 = 2020050$ , which is too large. Therefore, only terms that are 1006 or greater may be increased.

When only terms greater than or equal to 1006 are increased, then the median 1005 remains unchanged. Thus, 1005 is the only possible value for the median.

Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. There was one incorrect solution submitted.

**M390.** Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

A Pythagorean triangle is a right-angled triangle with all three sides of integer length. Let  $a$  and  $b$  be the legs of a Pythagorean triangle and let  $h$  be the altitude to the hypotenuse. Determine all such triangles for which

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{h} = 1.$$

*Solution by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let  $A$  be the area of the triangle and  $c$  the length of its hypotenuse. Then  $A$  equals both  $\frac{1}{2}ab$  and  $\frac{1}{2}ch$ , and so  $ab = ch$ .

Also,

$$\begin{aligned} 1 &= \frac{1}{a} + \frac{1}{b} + \frac{1}{h} = \frac{bh + ah + ab}{abh} = \frac{ah + bh + ch}{abh} \\ &= \frac{(a + b + c)h}{abh} = \frac{a + b + c}{ab}, \end{aligned}$$

hence  $ab = a + b + c$ .

By the Pythagorean Theorem,  $c = \sqrt{a^2 + b^2}$ . Since  $a + b + c = ab$ , we then obtain the equivalent equations

$$\begin{aligned} ab &= a + b + \sqrt{a^2 + b^2}, \\ ab - a - b &= \sqrt{a^2 + b^2}, \\ (ab - a - b)^2 &= a^2 + b^2, \\ a^2b^2 + a^2 + b^2 - 2a^2b - 2ab^2 + 2ab &= a^2 + b^2, \\ a^2b^2 - 2a^2b - 2ab^2 + 2ab &= 0, \\ ab(ab - 2a - 2b + 2) &= 0. \end{aligned}$$

Since  $ab > 0$ , then  $ab - 2a - 2b + 2 = 0$ , or  $b(a - 2) = 2a - 2$ , which implies that  $b = \frac{2a - 2}{a - 2} = 2 + \left(\frac{2}{a - 2}\right)$ .

Since  $a$  and  $b$  are positive integers, then  $a - 2$  is a positive divisor of 2; that is,  $a - 2 = 2$  or  $a - 2 = 1$ . If  $a - 2 = 2$ , then  $a = 4$ ,  $b = 3$ , and  $c = 5$ . If  $a - 2 = 1$ , then  $a = 3$ ,  $b = 4$ , and  $c = 5$ .

There is therefore just one Pythagorean triangle for which  $\frac{1}{a} + \frac{1}{b} + \frac{1}{h} = 1$ , namely the triangle with legs 3 and 4, and hypotenuse 5.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KRISTÓF HUSZÁR, Valéria Koch Grammar School, Pécs, Hungary; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier

University, Waterloo, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

**M391.** Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

Determine all pairs  $(a, b)$  of positive integers for which both  $\frac{a+1}{b}$  and  $\frac{b+2}{a}$  are positive integers.

*Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina, modified by the editor.*

Let  $x = \frac{a+1}{b}$  and  $y = \frac{b+2}{a}$ , where  $x$  and  $y$  are positive integers. Rearranging, we obtain  $a = bx - 1$  and  $b = ay - 2$ .

Substituting for  $a$  in the second equation, we obtain  $b = (bx - 1)y - 2$  and so  $y + 2 = bxy - b$  or  $b = \frac{y+2}{xy-1}$ .

If  $x = 1$ , then  $b = \frac{y+2}{y-1} = \frac{y-1+3}{y-1} = 1 + \frac{3}{y-1}$ . Since  $b$  and  $y$  are positive integers, then  $y = 2$  or  $y = 4$  (giving  $b = 4$  and  $b = 2$ , respectively).

If  $x = 2$ , then  $b = \frac{y+2}{2y-1}$ . Since  $b$  is a positive integer, then we must have  $y + 2 \geq 2y - 1$  and so  $y \leq 3$ . Checking  $y = 1$ ,  $y = 2$ , and  $y = 3$  shows that  $y = 1$  and  $y = 3$  give positive integer values for  $b$  (namely  $b = 3$  and  $b = 1$ , respectively).

If  $x = 3$ , then  $b = \frac{y+2}{3y-1}$ . Since  $b$  is a positive integer, then we must have  $y + 2 \geq 3y - 1$  and so  $y \leq \frac{3}{2}$ . The only possible value of  $y$  is  $y = 1$ , which does not give an integer value for  $b$ .

If  $x = 4$ , then  $b = \frac{y+2}{4y-1}$ . Since  $b$  is a positive integer, then we must have  $y + 2 \geq 4y - 1$  and so  $y \leq 1$ . If  $y = 1$ , then  $b = 1$ .

If  $x \geq 5$ , then  $xy - 1 \geq 5y - 1 > y + 2$  for all positive integers  $y$ , so  $b = \frac{y+2}{xy-1}$  cannot be a positive integer.

We finish by calculating the values of  $a$  that go with the values of  $b$  to obtain the pairs  $(a, b) = (3, 4), (1, 2), (5, 3), (1, 1), (3, 1)$ .

*Also solved by ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KRISTÓF HUSZÁR, Valéria Koch Grammar School, Pécs, Hungary; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; JOSÉ JAIME SAN JUAN CASTELLANOS, student, Universidad tecnológica de la Mixteca, Oaxaca, Mexico; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.*

**M392.** Proposed by the Mayhem Staff.

Determine, with justification, the fraction  $\frac{p}{q}$ , where  $p$  and  $q$  are positive integers and  $q < 1000$ , that is closest to, but not equal to,  $\frac{19}{72}$ .

*Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON, modified by the editor.*

In order to find the desired fraction, we need to minimize the value of

$$\left| \frac{p}{q} - \frac{19}{72} \right| = \frac{|72p - 19q|}{72q},$$

where  $p$  and  $q$  are positive integers and  $q < 1000$ .

To do this, we attempt to make the numerator of this difference as small as possible, while at the same time keeping the denominator as large as possible, hence by making  $q$  as large as possible.

To minimize the numerator, we try to make  $72p - 19q$  equal to 1 or  $-1$ . Consider  $72p - 19q$  modulo 19. Since  $72 \equiv -4 \pmod{19}$ , then  $72p - 19q \equiv -4p \pmod{19}$ , so we try to find  $p$  such that  $-4p \equiv \pm 1 \pmod{19}$ . Solving this congruence, we obtain  $p \equiv 14 \pmod{19}$ , or  $p \equiv 5 \pmod{19}$ , and so  $p = 14 + 19k$  for some integer  $k \geq 0$  or  $p = 5 + 19k$  for some integer  $k \geq 0$ .

In the first case,  $72p - 19q = 1$ , so  $q = \frac{72p - 1}{19} = 53 + 72k$ ; since  $q < 1000$ , then  $k \leq 13$ ; when  $k = 13$ ,  $q = 989$ . In the second case,  $72p - 19q = -1$ , so  $q = \frac{72p + 1}{19} = 19 + 72k$ ; since  $q < 1000$ , then  $k \leq 13$ ; when  $k = 13$ ,  $q = 955$ .

In either of these cases, the difference is equal to  $\frac{1}{72q}$ , and so is minimized when  $q$  is maximized. Thus, in these cases, the minimum possible difference occurs when  $q$  is as large as possible, or  $q = 989$  (and so  $k = 13$  and  $p = 261$ ). This difference is  $\frac{1}{72 \cdot 989}$ .

It is not possible to achieve a smaller difference when  $|72p - 19q| \geq 2$  and  $q < 1000$ , since this difference would always be at least  $\frac{2}{72 \cdot 1000}$  which is larger than the difference that we have already found.

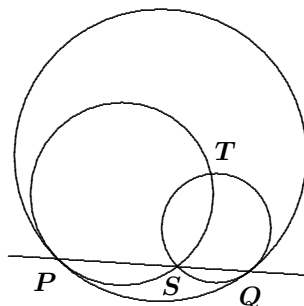
Therefore, the fraction closest to, but not equal to  $\frac{19}{72}$  under the given conditions is  $\frac{p}{q} = \frac{261}{989}$ .

*Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and KRISTÓF HUSZÁR, Valéria Koch Grammar School, Pécs, Hungary; One incorrect solution was also submitted.*

*A very similar problem appeared in the 2006 Canadian Open Mathematics Challenge (problem 4(a), Part B).*

### **M393.** Proposed by the Mayhem Staff.

Inside a large circle of radius  $r$  two smaller circles of radii  $a$  and  $b$  are drawn, as shown, so that the smaller circles are tangent to the larger circle at  $P$  and  $Q$ . The smaller circles intersect at  $S$  and  $T$ . If  $P$ ,  $S$ , and  $Q$  are collinear (that is, they lie on the same straight line), prove that  $r = a + b$ .



*Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain and Geoffrey A. Kandall, Hamden, CT, USA (independently).*

Let  $O$  be the centre of the large circle of radius  $r$ . Let  $O_1$  be the centre of the smaller circle of radius  $a$  tangent to the large circle at point  $P$ , and let  $O_2$  be the centre of the smaller circle of radius  $b$  tangent to the large circle at point  $Q$ .

Since the circles centred at  $O_1$  and  $O_2$  are tangent to the large circle, then  $O$ ,  $O_1$ ,  $P$  are collinear, as are  $O$ ,  $O_2$ ,  $Q$ .

Triangle  $OPQ$  is isosceles with  $OP = OQ$ , triangle  $O_1PS$  is isosceles with  $O_1P = O_1S$ , and triangle  $O_2QS$  is isosceles with  $O_2Q = O_2S$  (since each of these triangles has two radii of one of the circles as sides). Therefore,  $\angle OPQ = \angle OQP$ ,  $\angle O_1PS = \angle O_1SP$ , and  $\angle O_2QS = \angle O_2SQ$ .

Since  $P$ ,  $S$ , and  $Q$  are collinear, then

$$\angle PSO_1 = \angle O_1PS = \angle OPQ = \angle PQQ,$$

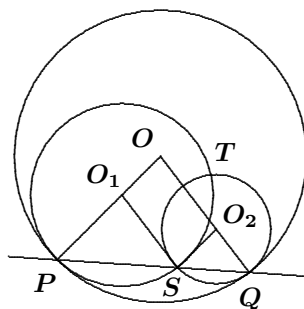
which tells us that  $O_1S$  and  $OQ$  are parallel. Similarly,

$$\angle QSO_2 = \angle O_2QS = \angle OQP = \angle QPO,$$

which tells us that  $O_2S$  and  $OP$  are parallel. Therefore, quadrilateral  $OO_1SO_2$  is a parallelogram.

Thus,  $OO_1 = SO_2$ . But  $SO_2 = b$  and  $OO_1 = OP - O_1P = r - a$ , and so  $r - a = b$ , or  $r = a + b$ .

*Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; ANTONIO GODOY TOHARIA, Madrid, Spain; KRISTÓF HUSZÁR, Valéria Koch Grammar School, Pécs, Hungary; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.*



## Problem of the Month

Ian VanderBurgh

To start the new year of Problems of the Month, we'll look at a problem that relies on a concept that we learn early on – addition – but requires us to think in some fairly deep ways to come up with a complete solution and total understanding of what is going on.





- (a) Determine the value of the ones digit  $A$ .
- (b) Determine the value of the tens digit  $B$  and the value of the hundreds digit  $C$ .
- (c) Determine the middle digit of the sum.

This problem looks pretty scary at first glance. Despite this, at least (a) and (b) can be answered exactly as in our warm-up problem. Let's do these parts and then think a bit about part (c).

**Solution to (a) and (b)** We proceed exactly as we did above. The units' column consists of 101 copies of the digit 2. Therefore, the sum in the units' column is  $101 \times 2 = 202$ . We put down the 2 and carry 20.

The tens' column consists of 100 copies of the digit 2 plus the carry of 20. Therefore, the sum in the tens' column is  $100 \times 2 + 20 = 220$ . We put down the 0 and carry 22.

The hundreds' column consists of 99 copies of the digit 2 plus the carry of 22. Therefore, the sum in the hundreds' column is  $99 \times 2 + 22 = 220$ . We put down the 0 and carry 22.

We can stop at this point, since we have determined the hundreds, tens, and units digits of the sum. Therefore,  $A = 2$ ,  $B = 0$ , and  $C = 0$ . ■

Great – that was much less scary than it looked like it could be. Now we need to try to tackle (c), which actually *is* quite scary.

One approach would be to proceed by “brute force” and work our way systematically column by column from the units' column towards the left. Of course, we don't need to go all of the way to the leftmost column, since we can stop when we get to the middle digit of the sum. Which digit will this be? In order to answer this, we need to know how many digits the final sum has. How many digits do you think that it has? My best guess is 101 digits, since it seems pretty unlikely that the single 2 in the leftmost column is going to have enough of a carry from the column to its right to create two-digit sum in this leftmost column. How do we know for sure that this is correct?

If we knew this for sure, then the middle digit would be the 51st digit, since there would be 50 digits to its left and 50 digits to its right. Now, this 51<sup>st</sup> column consists of 51 copies of the digit 2, so its sum is 102 plus whatever carry comes from the column to the right. The column to the right consists of 52 copies of 2, so its sum is 104 plus the carry from the column to its right, whose sum is at least 106 (that is, 106 from the 2's plus the carry). This is getting complicated!

Let's try this again with a bit of agreement on our terminology. We'll denote the leftmost column  $C1$  and the rightmost column  $C101$ ; we label the columns in between in the logical way. We also use  $s_n$  to represent the sum in the  $n^{\text{th}}$  column, including the carry.

We saw above that the sum of the digits in  $C51$  is 102, in  $C52$  is 104, and in  $C53$  is 106. Therefore,  $s_{53} \geq 106$ . (We haven't included any carry here from  $C54$ .) Therefore, the carry from  $C53$  to  $C52$  is at least 10, so

$s_{52} \geq 104 + 10 = 114$ . Therefore, the carry from  $C52$  to  $C51$  is at least 11, so  $s_{51} \geq 102 + 11 = 113$ .

If  $s_{51} = 113$ , then the middle digit is 3, and we're done. But is it actually the case that  $s_{51} = 113$ ? Could it be bigger?

If  $s_{51}$  was at least 114, then the carry from  $C52$  to  $C51$  would be at least  $114 - 102 = 12$ , which would mean that  $s_{52} \geq 120$ . If  $s_{52} \geq 120$ , then the carry from  $C53$  to  $C52$  would be at least  $120 - 104 = 16$ , so  $s_{53} \geq 160$ . If  $s_{53} \geq 160$ , then carry from  $C54$  to  $C53$  would be at least  $160 - 106 = 54$ , so  $s_{54} \geq 540$ . If  $s_{54} \geq 540$ , then the carry from  $C55$  to  $C54$  would be at least  $540 - 108 = 432$ , which is getting just plain silly, given that in parts (a) and (b) the carries that we got from the "largest columns" were 22 only.

So it seems pretty clear that  $s_{51}$  should be 113, so the middle digit should be 3.

Now, I don't know about you, but I'm just about convinced. However, I'm not sure if "the middle digit should be 3" is all that rigorous and "just plain silly" counts as a solid mathematical proof. So we should prove some restriction on the carries. Let's do this, and also write out a cohesive solution to part (c). We'll use a bit of algebraic notation to simplify things.

**Solution to (c)** We label the columns as above and let  $s_n$  be the sum in the  $n^{\text{th}}$  column, including the carry from the  $(n+1)^{\text{th}}$  column; we denote this carry by  $c_{n+1}$ . Column  $n$  consists of  $n$  copies of the digit 2, so  $s_n = 2n + c_{n+1}$ .

From our solution to (a) and (b), we know that  $c_{101} = 20$  and that  $c_{100} = c_{99} = 22$ . Let's argue that  $c_n \leq 22$  for all  $n$  with  $1 \leq n \leq 101$ .

We use an informal backwards induction. Suppose that  $c_{n+1} \leq 22$ . (We know that this is true for  $n = 98, 99, 100$ .) Then  $s_n = 2n + c_{n+1}$  is at most  $2(101) + 22 = 224$  and so  $c_n \leq 22$ . Thus, if  $c_{n+1} \leq 22$ , then  $c_n \leq 22$ . Since  $c_{101} \leq 22$ , then we can carry this chain along to show that  $c_n \leq 22$  for all  $n$  with  $1 \leq n \leq 101$ .

We can use this to show that the sum has exactly 101 digits. For the sum to have more than 101 digits, we would need to have  $s_1 \geq 10$ . But  $s_1 = 2 + c_2$ , so this would mean that  $c_2 \geq 8$  and so  $s_2 \geq 80$ . But  $s_2 = 4 + c_3$ , so this would mean that  $c_3 \geq 76$ , which is not possible. Therefore, the sum has exactly 101 digits.

Finally, we can determine the 51<sup>st</sup> digit. We know that  $s_{51} = 102 + c_{52}$  and  $s_{52} = 104 + c_{53}$  and  $s_{53} = 106 + c_{54}$ . Since  $0 \leq c_{54} \leq 22$ , then  $106 \leq s_{53} \leq 128$ . Thus,  $10 \leq c_{53} \leq 12$ .

Since  $10 \leq c_{53} \leq 12$ , then  $114 \leq s_{52} \leq 116$ . Thus,  $c_{52} = 11$ , which means that  $s_{51} = 113$ , and so the 51<sup>st</sup> (that is, the middle) digit of the sum is 3. ■

Let's make a couple of observations to finish this off. First, a little bit of algebra and notation helped to save us a large number of words and convoluted explanations. Second, a relatively simple topic like addition gave us a problem that requires some pretty high-level thinking. To me, one of the great beauties of mathematics is that simplicity and complexity can be so completely interwoven.