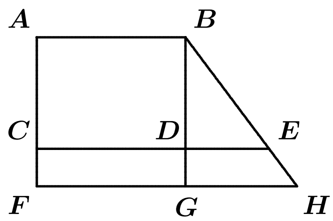
M423. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

La différence entre le chiffre des dizaines et celui des unités d'un carré parfait S est de trois. Trouver tous les restes de la division de S par 100.

M424. *Proposé par Margo Kondratieva, Université Memorial de Terre-Neuve, St. John's, NL.*

Dans la figure ci-contre, les segments de droite AB, CDE, et FGH sont parallèles. De plus, les segments ACF et BDG sont perpendiculaires à AB. Supposons que les aires respectives des rectangles $ABDC$, $CDGF$, et $\triangle BDE$ sont x, y et z. Trouver l'aire de $DEHG$ en fonction de x, y et z.



M425. *Proposé par Titu Zvonaru, Comănești, Roumanie.*

Dans le triangle ABC, $\angle BAC = 90°$ et soit I le centre du cercle inscrit. La bissectrice intérieure de l'angle C coupe AB en D. La droite passant par D et perpendiculaire à BI coupe BC en E. La droite passant par D et parallèle à BI coupe AC en F. Montrer que E, I et F sont colinéaires.

Mayhem Solutions

Last year we received some late solutions that did not appear in the December issue. We therefore acknowledge a correct solution to M383 by Mridul Singh, student, Kendriya Vidyalaya School, Shillong, India, and correct solutions to problems M383, M384, and M386 by Hugo Luyo Sánchez, Pontificia Universidad Católica del Peru, Lima, Peru.

M388. *Proposed by Kyle Sampson, St. John's, NL.*

A sequence is generated by listing (from smallest to largest) for each positive integer n the multiples of n up to and including n^2. Thus, the sequence begins 1, 2, 4, 3, 6, 9, 4, 8, 12, 16, 5, 10, 15, 20, 25, 6, 12, $\ldots$. Determine the 2009^{th} term in the sequence.

Solution by Kristóf Huszár, Valéria Koch Grammar School, Pécs, Hungary.

First, we notice that there are k positive integral multiples of k less than or equal to k^2. If we group the terms of the sequence as the multiples of 1, then the multiples of 2, then the multiples 3, and so on, we notice that the groups have 1 term, then 2 terms, then 3 terms, and so on.

If n is a positive integer, then the sum of the first n positive integers is equal to $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$. Therefore, n^2 is the $\left(\dfrac{n(n+1)}{2}\right)^{\text{th}}$ term of the sequence.

Hence, $63^2 = 3969$ is the $\left(\dfrac{63 \cdot 64}{2}\right)^{\text{th}} = 2016^{\text{th}}$ term. Since 3969 occurs $2016 - 2009 = 7$ terms after the 2009^{th} term, we find that the 2009^{th} term is $63^2 - 7(63) = 3528$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; JACLYN CHANG, student, Western Canada High School, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. One incorrect solution was submitted.

M389. *Proposed by Lino Demasi, student, Simon Fraser University, Burnaby, BC.*

There are 2009 students and each has a card with a different positive integer on it. If the sum of the numbers on these cards is 2020049, what are the possible values for the median of the numbers on the cards?

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Let a be the smallest of the numbers on the cards. The smallest possible sum of the 2009 numbers on the cards is then

$$\begin{aligned} S &= a + (a+1) + (a+2) + \cdots + (a+2008) \\ &= 2009a + \frac{1}{2}(2008)(2009) = 2009a + 2017036\,. \end{aligned}$$

If $a \geq 2$ then $S \geq 2021054 > 2020049$. Therefore, $a = 1$.

Next, consider the sequence of numbers 1, 2, 3, $\ldots$, 2009. The sum of these numbers is 2019045 (which is 1004 less than the desired sum of 2020049). Also, their median is 1005.

In order to obtain the desired sum of 2020049, some of the numbers in this sequence need to be increased. When a certain term in the sequence is increased, then every greater term must be increased as well in order for the terms of the sequence to remain distinct. If a term that is less than 1006 is increased, then every larger term will also have to increase, resulting in an increase of the initial sum 2019045 by at least 1005 (since at least 1005 terms are increased), yielding a new sum of at least $2019045 + 1005 = 2020050$, which is too large. Therefore, only terms that are 1006 or greater may be increased.

When only terms greater than or equal to 1006 are increased, then the median 1005 remains unchanged. Thus, 1005 is the only possible value for the median.

Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. There was one incorrect solution submitted.

M390. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

A Pythagorean triangle is a right-angled triangle with all three sides of integer length. Let a and b be the legs of a Pythagorean triangle and let h be the altitude to the hypotenuse. Determine all such triangles for which

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{h} = 1 .$$

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

Let A be the area of the triangle and c the length of its hypotenuse. Then A equals both $\frac{1}{2}ab$ and $\frac{1}{2}ch$, and so $ab = ch$.

Also,

$$\begin{aligned} 1 &= \frac{1}{a}+\frac{1}{b}+\frac{1}{h} = \frac{bh+ah+ab}{abh} = \frac{ah+bh+ch}{abh} \\ &= \frac{(a+b+c)h}{abh} = \frac{a+b+c}{ab}, \end{aligned}$$

hence $ab = a+b+c$.

By the Pythagorean Theorem, $c = \sqrt{a^2+b^2}$. Since $a+b+c = ab$, we then obtain the equivalent equations

$$\begin{aligned} ab &= a+b+\sqrt{a^2+b^2}, \\ ab-a-b &= \sqrt{a^2+b^2}, \\ (ab-a-b)^2 &= a^2+b^2, \\ a^2b^2+a^2+b^2-2a^2b-2ab^2+2ab &= a^2+b^2, \\ a^2b^2-2a^2b-2ab^2+2ab &= 0, \\ ab(ab-2a-2b+2) &= 0. \end{aligned}$$

Since $ab > 0$, then $ab-2a-2b+2 = 0$, or $b(a-2) = 2a-2$, which implies that $b = \dfrac{2a-2}{a-2} = 2 + \left(\dfrac{2}{a-2}\right)$.

Since a and b are positive integers, then $a-2$ is a positive divisor of 2; that is, $a-2 = 2$ or $a-2 = 1$. If $a-2 = 2$, then $a = 4$, $b = 3$, and $c = 5$. If $a-2 = 1$, then $a = 3$, $b = 4$, and $c = 5$.

There is therefore just one Pythagorean triangle for which $\frac{1}{a}+\frac{1}{b}+\frac{1}{h}=1$, namely the triangle with legs 3 and 4, and hypotenuse 5.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KRISTÓF HUSZÁR, Valéria Koch Grammar School, Pécs, Hungary; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier

University, Waterloo, ON ; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

M391. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Determine all pairs (a, b) of positive integers for which both $\frac{a+1}{b}$ and $\frac{b+2}{a}$ are positive integers.

Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina, modified by the editor.

Let $x = \frac{a+1}{b}$ and $y = \frac{b+2}{a}$, where x and y are positive integers. Rearranging, we obtain $a = bx - 1$ and $b = ay - 2$.

Substituting for a in the second equation, we obtain $b = (bx - 1)y - 2$ and so $y + 2 = bxy - b$ or $b = \frac{y+2}{xy-1}$.

If $x = 1$, then $b = \frac{y+2}{y-1} = \frac{y-1+3}{y-1} = 1 + \frac{3}{y-1}$. Since b and y are positive integers, then $y = 2$ or $y = 4$ (giving $b = 4$ and $b = 2$, respectively).

If $x = 2$, then $b = \frac{y+2}{2y-1}$. Since b is a positive integer, then we must have $y + 2 \geq 2y - 1$ and so $y \leq 3$. Checking $y = 1$, $y = 2$, and $y = 3$ shows that $y = 1$ and $y = 3$ give positive integer values for b (namely $b = 3$ and $b = 1$, respectively).

If $x = 3$, then $b = \frac{y+2}{3y-1}$. Since b is a positive integer, then we must have $y + 2 \geq 3y - 1$ and so $y \leq \frac{3}{2}$. The only possible value of y is $y = 1$, which does not give an integer value for b.

If $x = 4$, then $b = \frac{y+2}{4y-1}$. Since b is a positive integer, then we must have $y + 2 \geq 4y - 1$ and so $y \leq 1$. If $y = 1$, then $b = 1$.

If $x \geq 5$, then $xy - 1 \geq 5y - 1 > y + 2$ for all positive integers y, so $b = \frac{y+2}{xy-1}$ cannot be a positive integer.

We finish by calculating the values of a that go with the values of b to obtain the pairs $(a, b) = (3, 4)$, $(1, 2)$, $(5, 3)$, $(1, 1)$, $(3, 1)$.

Also solved by ANTONIO GODOY TOHARIA, Madrid, Spain ; RICHARD I. HESS, Rancho Palos Verdes, CA, USA ; KRISTÓF HUSZÁR, Valéria Koch Grammar School, Pécs, Hungary ; RICARD PEIRÓ, IES "Abastos", Valencia, Spain ; JOSÉ JAIME SAN JUAN CASTELLANOS, student, Universidad technológica de la Mixteca, Oaxaca, Mexico ; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON ; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

M392. *Proposed by the Mayhem Staff.*

Determine, with justification, the fraction $\frac{p}{q}$, where p and q are positive integers and $q < 1000$, that is closest to, but not equal to, $\frac{19}{72}$.

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON, modified by the editor.

In order to find the desired fraction, we need to minimize the value of

$$\left|\frac{p}{q} - \frac{19}{72}\right| = \frac{|72p - 19q|}{72q},$$

where p and q are positive integers and $q < 1000$.

To do this, we attempt to make the numerator of this difference as small as possible, while at the same time keeping the denominator as large as possible, hence by making q as large as possible.

To minimize the numerator, we try to make $72p - 19q$ equal to 1 or -1. Consider $72p - 19q$ modulo 19. Since $72 \equiv -4 \pmod{19}$, then $72p - 19q \equiv -4p \pmod{19}$, so we try to find p such that $-4p \equiv \pm 1 \pmod{19}$. Solving this congruence, we obtain $p \equiv 14 \pmod{19}$, or $p \equiv 5 \pmod{19}$, and so $p = 14 + 19k$ for some integer $k \geq 0$ or $p = 5 + 19k$ for some integer $k \geq 0$.

In the first case, $72p - 19q = 1$, so $q = \dfrac{72p - 1}{19} = 53 + 72k$; since $q < 1000$, then $k \leq 13$; when $k = 13$, $q = 989$. In the second case, $72p - 19q = -1$, so $q = \dfrac{72p + 1}{19} = 19 + 72k$; since $q < 1000$, then $k \leq 13$; when $k = 13$, $q = 955$.

In either of these cases, the difference is equal to $\dfrac{1}{72q}$, and so is minimized when q is maximized. Thus, in these cases, the minimum possible difference occurs when q is as large as possible, or $q = 989$ (and so $k = 13$ and $p = 261$). This difference is $\dfrac{1}{72 \cdot 989}$.

It is not possible to achieve a smaller difference when $|72p - 19q| \geq 2$ and $q < 1000$, since this difference would always be at least $\dfrac{2}{72 \cdot 1000}$ which is larger than the difference that we have already found.

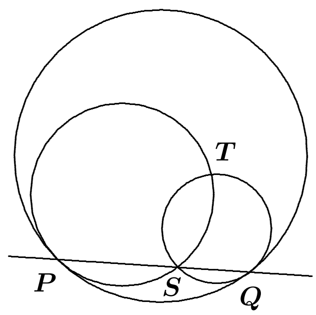
Therefore, the fraction closest to, but not equal to $\dfrac{19}{72}$ under the given conditions is $\dfrac{p}{q} = \dfrac{261}{989}$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and KRISTÓF HUSZÁR, Valéria Koch Grammar School, Pécs, Hungary; One incorrect solution was also submitted.

A very similar problem appeared in the 2006 Canadian Open Mathematics Challenge (problem 4(a), Part B).

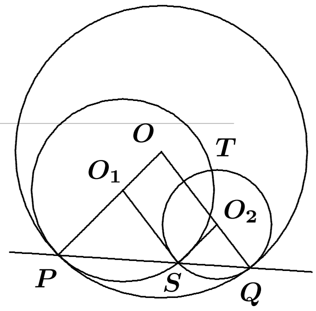
M393. *Proposed by the Mayhem Staff.*

Inside a large circle of radius r two smaller circles of radii a and b are drawn, as shown, so that the smaller circles are tangent to the larger circle at P and Q. The smaller circles intersect at S and T. If P, S, and Q are collinear (that is, they lie on the same straight line), prove that $r = a + b$.



Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain and Geoffrey A. Kandall, Hamden, CT, USA (independently).

Let O be the centre of the large circle of radius r. Let O_1 be the centre of the smaller circle of radius a tangent to the large circle at point P, and let O_2 be the centre of the smaller circle of radius b tangent to the large circle at point Q.



Since the circles centred at O_1 and O_2 are tangent to the large circle, then O, O_1, P are collinear, as are O, O_2, Q.

Triangle OPQ is isosceles with $OP = OQ$, triangle O_1PS is isosceles with $O_1P = O_1S$, and triangle O_2QS is isosceles with $O_2Q = O_2S$ (since each of these triangles has two radii of one of the circles as sides). Therefore, $\angle OPQ = \angle OQP$, $\angle O_1PS = \angle O_1SP$, and $\angle O_2QS = \angle O_2SQ$.

Since P, S, and Q are collinear, then

$$\angle PSO_1 = \angle O_1PS = \angle OPQ = \angle PQO\,,$$

which tells us that O_1S and OQ are parallel. Similarly,

$$\angle QSO_2 = \angle O_2QS = \angle OQP = \angle QPO\,,$$

which tells us that O_2S and OP are parallel. Therefore, quadrilateral OO_1SO_2 is a parallelogram.

Thus, $OO_1 = SO_2$. But $SO_2 = b$ and $OO_1 = OP - O_1P = r - a$, and so $r - a = b$, or $r = a + b$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; ANTONIO GODOY TOHARIA, Madrid, Spain; KRISTÓF HUSZÁR, Valéria Koch Grammar School, Pécs, Hungary; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

Problem of the Month

Ian VanderBurgh

To start the new year of Problems of the Month, we'll look at a problem that relies on a concept that we learn early on – addition – but requires us to think in some fairly deep ways to come up with a complete solution and total understanding of what is going on.