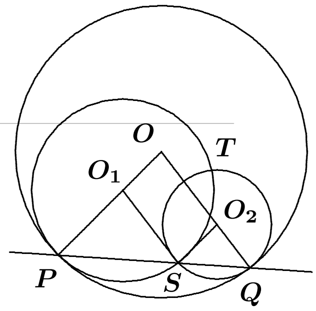
Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain and Geoffrey A. Kandall, Hamden, CT, USA (independently).

Let O be the centre of the large circle of radius r. Let O_1 be the centre of the smaller circle of radius a tangent to the large circle at point P, and let O_2 be the centre of the smaller circle of radius b tangent to the large circle at point Q.



Since the circles centred at O_1 and O_2 are tangent to the large circle, then O, O_1, P are collinear, as are O, O_2, Q.

Triangle OPQ is isosceles with $OP = OQ$, triangle O_1PS is isosceles with $O_1P = O_1S$, and triangle O_2QS is isosceles with $O_2Q = O_2S$ (since each of these triangles has two radii of one of the circles as sides). Therefore, $\angle OPQ = \angle OQP$, $\angle O_1PS = \angle O_1SP$, and $\angle O_2QS = \angle O_2SQ$.

Since P, S, and Q are collinear, then

$$\angle PSO_1 = \angle O_1PS = \angle OPQ = \angle PQO,$$

which tells us that O_1S and OQ are parallel. Similarly,

$$\angle QSO_2 = \angle O_2QS = \angle OQP = \angle QPO,$$

which tells us that O_2S and OP are parallel. Therefore, quadrilateral OO_1SO_2 is a parallelogram.

Thus, $OO_1 = SO_2$. But $SO_2 = b$ and $OO_1 = OP - O_1P = r - a$, and so $r - a = b$, or $r = a + b$.

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Problem of the Month

Ian VanderBurgh

To start the new year of Problems of the Month, we'll look at a problem that relies on a concept that we learn early on – addition – but requires us to think in some fairly deep ways to come up with a complete solution and total understanding of what is going on.

Since we've had a short break since the last issue, we should start with a warm-up problem. Your task is to compute the following sum :

$$2 + 22 + 222 + 2222 + 22222 + 222222 + 2222222 .$$

But before you start, there are two rules : no calculators are allowed, and you have to compute the sum aloud.

If you're as out of practice on this sort of thing as some of us are, this isn't all that easy. We can rewrite the sum first in a form that makes it easier to compute :

$$\begin{array}{rrrrrrrr} & & & & & & & 2 \\ & & & & & & 2 & 2 \\ & & & & & 2 & 2 & 2 \\ & & & & 2 & 2 & 2 & 2 \\ & & & 2 & 2 & 2 & 2 & 2 \\ & & 2 & 2 & 2 & 2 & 2 & 2 \\ + & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ \hline \end{array}$$

Here's an attempt to do this in words :

> The sum in the units' column is 14. Put down the 4; carry the 1. The sum in the tens' column is 12 plus the carry of 1 gives 13. Put down the 3; carry the 1. The sum in the hundreds' column is 10 plus the carry of 1 gives 11. Put down the 1; carry the 1. The sum in the thousands' column is 8 plus the carry of 1 gives 9. Put down the 9; no carry. The sum in the ten thousands' column is 6. Put down the 6. The sum in the hundred thousands' column is 4. Put down the 4. The sum in the millions' column is 2. Put down the 2. The final sum is thus 2469134.

That's a bit of a workout, isn't it? We should clarify the role of the digit and carry. If the sum in a column is 14, we write this as $14 = 10(1) + 4$; the units digit (the 4) is the digit that we write down, while the quotient when dividing by 10 (the 1) is the carry. (The units digit is also the remainder when we divide the sum by 10.) Let's have a look at our Problem of the Month, then.

Problem (2009 Fryer Contest) The addition shown below, representing $2{+}22{+}222{+}2222{+}\cdots$, has 101 rows and the last term consists of 101 2's :

$$\begin{array}{rrrrrrrr} & & & & & & & 2 \\ & & & & & & 2 & 2 \\ & & & & & 2 & 2 & 2 \\ & & & & 2 & 2 & 2 & 2 \\ & & & & \vdots & & & \vdots \\ & & 2 & 2 & \cdots\ 2 & 2 & 2 & 2 \\ + & 2 & 2 & 2 & \cdots\ 2 & 2 & 2 & 2 \\ \hline & & & & \cdots & C & B & A \end{array}$$

(a) Determine the value of the ones digit A .

(b) Determine the value of the tens digit B and the value of the hundreds digit C.

(c) Determine the middle digit of the sum.

This problem looks pretty scary at first glance. Despite this, at least (a) and (b) can be answered exactly as in our warm-up problem. Let's do these parts and then think a bit about part (c).

Solution to (a) and (b) We proceed exactly as we did above. The units' column consists of 101 copies of the digit 2. Therefore, the sum in the units' column is $101 \times 2 = 202$. We put down the 2 and carry 20.

The tens' column consists of 100 copies of the digit 2 plus the carry of 20. Therefore, the sum in the tens' column is $100 \times 2 + 20 = 220$. We put down the 0 and carry 22.

The hundreds' column consists of 99 copies of the digit 2 plus the carry of 22. Therefore, the sum in the hundreds' column is $99 \times 2 + 22 = 220$. We put down the 0 and carry 22.

We can stop at this point, since we have determined the hundreds, tens, and units digits of the sum. Therefore, $A = 2$, $B = 0$, and $C = 0$. ∎

Great – that was much less scary than it looked like it could be. Now we need to try to tackle (c), which actually *is* quite scary.

One approach would be to proceed by "brute force" and work our way systematically column by column from the units' column towards the left. Of course, we don't need to go all of the way to the leftmost column, since we can stop when we get to the middle digit of the sum. Which digit will this be ? In order to answer this, we need to know how many digits the final sum has. How many digits do you think that it has ? My best guess is 101 digits, since it seems pretty unlikely that the single 2 in the leftmost column is going to have enough of a carry from the column to its right to create two-digit sum in this leftmost column. How do we know for sure that this is correct ?

If we knew this for sure, then the middle digit would be the 51st digit, since there would be 50 digits to its left and 50 digits to its right. Now, this 51^{st} column consists of 51 copies of the digit 2, so its sum is 102 plus whatever carry comes from the column to the right. The column to the right consists of 52 copies of 2, so its sum is 104 plus the carry from the column to its right, whose sum is at least 106 (that is, 106 from the 2's plus the carry). This is getting complicated !

Let's try this again with a bit of agreement on our terminology. We'll denote the leftmost column $C1$ and the rightmost column $C101$; we label the columns in between in the logical way. We also use s_n to represent the sum in the n^{th} column, including the carry.

We saw above that the sum of the digits in $C51$ is 102, in $C52$ is 104, and in $C53$ is 106. Therefore, $s_{53} \geq 106$. (We haven't included any carry here from $C54$.) Therefore, the carry from $C53$ to $C52$ is at least 10, so

$s_{52} \geq 104 + 10 = 114$. Therefore, the carry from $C52$ to $C51$ is at least 11, so $s_{51} \geq 102 + 11 = 113$.

If $s_{51} = 113$, then the middle digit is 3, and we're done. But is it actually the case that $s_{51} = 113$? Could it be bigger?

If s_{51} was at least 114, then the carry from $C52$ to $C51$ would be at least $114 - 102 = 12$, which would mean that $s_{52} \geq 120$. If $s_{52} \geq 120$, then the carry from $C53$ to $C52$ would be at least $120 - 104 = 16$, so $s_{53} \geq 160$. If $s_{53} \geq 160$, then carry from $C54$ to $C53$ would be at least $160 - 106 = 54$, so $s_{54} \geq 540$. If $s_{54} \geq 540$, then the carry from $C55$ to $C54$ would be at least $540 - 108 = 432$, which is getting just plain silly, given that in parts (a) and (b) the carries that we got from the "largest columns" were 22 only.

So it seems pretty clear that s_{51} should be 113, so the middle digit should be 3.

Now, I don't know about you, but I'm just about convinced. However, I'm not sure if "the middle digit should be 3" is all that rigorous and "just plain silly" counts as a solid mathematical proof. So we should prove some restriction on the carries. Let's do this, and also write out a cohesive solution to part (c). We'll use a bit of algebraic notation to simplify things.

Solution to (c) We label the columns as above and let s_n be the sum in the n^{th} column, including the carry from the $(n+1)^{\text{th}}$ column; we denote this carry by c_{n+1}. Column n consists of n copies of the digit 2, so $s_n = 2n + c_{n+1}$.

From our solution to (a) and (b), we know that $c_{101} = 20$ and that $c_{100} = c_{99} = 22$. Let's argue that $c_n \leq 22$ for all n with $1 \leq n \leq 101$.

We use an informal backwards induction. Suppose that $c_{n+1} \leq 22$. (We know that this is true for $n = 98, 99, 100$.) Then $s_n = 2n + c_{n+1}$ is at most $2(101) + 22 = 224$ and so $c_n \leq 22$. Thus, if $c_{n+1} \leq 22$, then $c_n \leq 22$. Since $c_{101} \leq 22$, then we can carry this chain along to show that $c_n \leq 22$ for all n with $1 \leq n \leq 101$.

We can use this to show that the sum has exactly 101 digits. For the sum to have more than 101 digits, we would need to have $s_1 \geq 10$. But $s_1 = 2 + c_2$, so this would mean that $c_2 \geq 8$ and so $s_2 \geq 80$. But $s_2 = 4 + c_3$, so this would mean that $c_3 \geq 76$, which is not possible. Therefore, the sum has exactly 101 digits.

Finally, we can determine the 51^{st} digit. We know that $s_{51} = 102 + c_{52}$ and $s_{52} = 104 + c_{53}$ and $s_{53} = 106 + c_{54}$. Since $0 \leq c_{54} \leq 22$, then $106 \leq s_{53} \leq 128$. Thus, $10 \leq c_{53} \leq 12$.

Since $10 \leq c_{53} \leq 12$, then $114 \leq s_{52} \leq 116$. Thus, $c_{52} = 11$, which means that $s_{51} = 113$, and so the 51^{st} (that is, the middle) digit of the sum is 3. ■

Let's make a couple of observations to finish this off. First, a little bit of algebra and notation helped to save us a large number of words and convoluted explanations. Second, a relatively simple topic like addition gave us a problem that requires some pretty high-level thinking. To me, one of the great beauties of mathematics is that simplicity and complexity can be so completely interwoven.