SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3401. [2009 : 42, 44] *Proposed by Tigran Sloyan, Basic Gymnasium of SEUA, Yerevan, Armenia.*

Let $ABCDE$ be a convex pentagon such that $\angle BAC = \angle EAD$ and $\angle BCA = \angle EDA$, and let the lines CB and DE intersect in the point F. Prove that the midpoints of CD, BE, and AF are collinear.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor.

Convexity is not required here. We shall prove that the midpoints of CD, BE, and AF are collinear when ABC and AED are oppositely similar triangles that share the vertex A (and $F = CB \cap DE$). Our argument requires two easy lemmas.

Lemma 1 If $PQRS$ is a quadrilateral for which $\angle PQR = \angle PSR$ and the midpoint M of PR lies between Q and S on QS, then $PQRS$ is a parallelogram.

Proof. Let us call S' the point of the line QM for which M is the midpoint of QS', and show that $S' = S$. Because its diagonals bisect one another, $PQRS'$ is necessarily a parallelogram, whence, $\angle PS'R = \angle PQR = \angle PSR$. But there can only be one point on the ray from Q toward M that can be the vertex of this angle, whence $S' = S$ and $PQRS$ is a parallelogram. ∎

Lemma 2 When all the points P on BC are related by a similarity to all the points P' on $B'C'$ *(that is, $B'C' : BC = B'P' : BP$)*, then the midpoints of PP' are collinear.

Proof. Let B'', C'', P'' be the midpoints of the segments BB', CC', PP'. Translate B', C', P' to B, C_1', P_1' and denote by C_1'', P_1'' the midpoints of CC_1', PP_1'. Because PP_1' cuts the sides BC and BC_1' of triangle BCC_1' proportionally, it follows that $CC_1' \| PP_1'$; consequently, the midpoints C_1'' and P_1'' are collinear with the vertex B. Because $C_1''C''$ is parallel to and half the length of $C_1'C'$, which is parallel and equal to BB', it follows that $BC_1''C''B''$ is a parallelogram. Similarly for $BP_1''P''B''$. Since B, C_1'', P_1'' are collinear, so are B'', C'', P''. ∎

Comment. Lemma 2 is a special case of a classical theorem : *Given two directly similar figures in the plane, the points that divide the line segments joining corresponding points of the two figures in the same ratio form a figure that is directly similar to them.* See, for example, F. G.-M., Exercices de

Géométrie—comprenant l'exposé des méthodes géométriques et 2000 questions résolues, sixième édition, J. De Gigord, Paris, 1920, Paragraph 1146d, pages 473-474, whose proof was used above. In the lemma, our given figures are lines, and the ratio is $1 : 1$. Note that as an immediate consequence of the general theorem, one can continuously transform any figure into any directly similar figure in the plane in such a way that the shape never changes and corresponding points move along straight lines.

We turn now to the given oppositely similar triangles ABC and AED. We assume that the midpoints of CD and BE are distinct; otherwise there is nothing to prove. The lines BC and ED play the roles of BC and $B'C'$ of Lemma 2 — P will move along BC while P' moves along ED in such a way that the triangles ABP and AEP are directly similar; in particular, $\angle APF = \angle AP'F$ for all positions of P. By Lemma 2, the midpoint of PP' moves along the line joining the midpoints of CD and BE. There will be a unique position of P on BC where PP' contains the midpoint of AF. At this position $APFP'$ is a parallelogram by Lemma 1; there the midpoint of AF coincides with the midpoint of PP', and therefore it lies on the line joining the midpoints of CD and BE.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvarez Cubero, Priego de Córdoba, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There was one incorrect submission.

Most of the submitted solutions used coordinates. Geupel found the problem in the 2005 Mathlinks *internet forum,* www.mathlinks.ro/viewtopic.php?t=38041, *where there is a nice synthetic proof from someone who goes by the name of "Armo".*

3402. [2009 : 42, 44] *Proposed by Mihály Bencze, Brasov, Romania.*

Let D and E be the midpoints of the sides AB and AC in triangle ABC, respectively. Prove that CD is perpendicular to BE if and only if

$$5BC^2 \;=\; AC^2 + AB^2\,.$$

I. *Solution by Dung Nguyen Manh, High School of HUS, Hanoi, Vietnam.*

Let $G = BE \cap CD$, and $a = BC$, $b = CA$, $c = AB$, $m_b = BE$, $m_c = CD$. Using the Pythagorean Theorem, its converse, and Stewart's theorem, we have

$$\begin{aligned} CD \perp BE \;&\iff\; BG^2 + CG^2 \;=\; BC^2 \\ &\iff\; \left(\tfrac{2}{3}m_b\right)^2 + \left(\tfrac{2}{3}m_c\right)^2 \;=\; a^2 \\ &\iff\; 4m_b^2 + 4m_c^2 = 9a^2 \\ &\iff\; 2\left(c^2 + a^2\right) - b^2 + 2\left(a^2 + b^2\right) - c^2 \;=\; 9a^2 \\ &\iff\; b^2 + c^2 \;=\; 5a^2\,, \end{aligned}$$

as desired.

II. *Solution by Joe Howard, Portales, NM, USA.*

From Problem 5 of ***CRUX with MAYHEM*** [2003 : 375, 377], in quadrilateral $BCED$ the diagonals CD and BE are perpendicular if and only if $BC^2 + DE^2 = BD^2 + CE^2$. Since D and E are midpoints of their respective sides, the perpendicularity of the two lines is equivalent to

$$BC^2 + \left(\tfrac{1}{2}BC\right)^2 \;=\; \left(\tfrac{1}{2}AB\right)^2 + \left(\tfrac{1}{2}AC\right)^2 \,,$$

or $5BC^2 = AC^2 + AB^2$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; CRISTINEL MORTICI, Valahia University of Târgoviște, Romania (two solutions); JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; ALBERT STADLER, Herrliberg, Switzerland; VASILE TEODOROVICI, Toronto, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Most of the submitted solutions were similar to one of the featured solutions.

3403. [2009 : 42, 44] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

The circles Γ_1 and Γ_2 intersect at P and Q. A line ℓ through P intersects Γ_1 and Γ_2 for the second time at A and B, respectively. The tangents to Γ_1 and Γ_2 at A and B intersect at C. If O is the circumcentre of $\triangle ABC$ determine the locus of O when ℓ rotates about P.

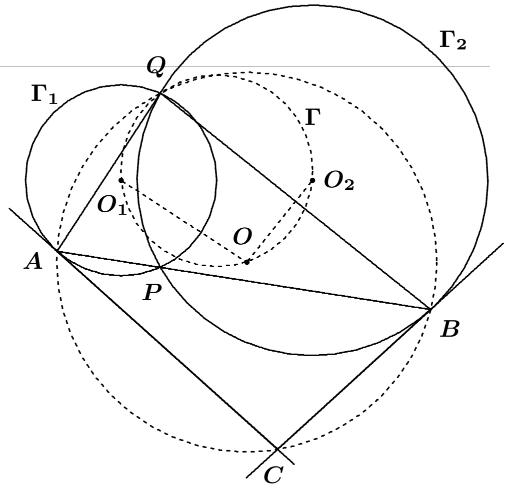
Solution by Michel Bataille, Rouen, France.

Let O_1 and O_2 be the centres of Γ_1 and Γ_2, respectively, and Γ be the circumcircle of the triangle QO_1O_2. We show that the required locus is $\Gamma - \{Q, O_1, O_2\}$. (See the figure on the next page.)

Let σ denote the spiral similarity with centre Q transforming O_1 into O_2. Then, $\sigma(\Gamma_1) = \Gamma_2$ and it follows that $\sigma(A) = B$ (a known property). We exclude the cases when $\triangle ABC$ is degenerate, that is, when ℓ either is the line PQ (in which case $A = B = Q$) or is tangent to Γ_1 or Γ_2 at P (in which case $B = C$ or $A = C$).

Since the lines CA and CB are perpendicular to O_1A and O_2B, we also have $\sigma(CA) = CB$.

Thus, we have $\angle(CA, CB) = \theta = \angle(QA, QB) \pmod{\pi}$, where θ is the angle of σ, so that Q is on the circumcircle of $\triangle ABC$. Note that we certainly have $O \neq O_1$, O_2 (if $O = O_1$, say, then $O_1B = O_1A$, hence $B = P$ or Q, which has been excluded) and the lines OO_1 and OO_2 are the perpendicular bisectors of QA and QB, respectively. Thus,



$$\angle(OO_1, OO_2) = \angle(QA, QB)$$
$$= \angle(QO_1, QO_2) \pmod{\pi}$$

and finally, O, Q, O_1, O_2 are concyclic.

Conversely, let $O \neq Q$, O_1, O_2 be any point on the circle (QO_1O_2). Let the perpendicular to OO_1 through Q meet Γ_1 again at A and the perpendicular to OO_2 through Q meet Γ_2 again at B. Then, $\angle(QA, QB) = \angle(OO_1, OO_2) \pmod{\pi}$, hence $\sigma(A) = B$ (since $\sigma(\Gamma_1) = \Gamma_2$) and it follows that A, P, B are collinear on a line ℓ. The circumcentre of $\triangle QAB$ is O (because OO_1 and OO_2 are the perpendicular bisectors of QA and QB; note that $O_1A = O_1Q$ and $O_2B = O_2Q$). Moreover $\sigma(CA) = CB$ (since $\sigma(AO_1) = BO_2$ and $CA \perp AO_1$, $CB \perp BO_2$), hence $\angle(QA, QB) = \angle(CA, CB)$ and the circle (QAB) passes through C. Thus, O is the circumcentre of $\triangle ABC$.

Also solved by RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Geupel noted that problem G4 on the IMO 2002 short list essentially generalizes the given problem (he gave the reference D. Djukić et al., The IMO Compendium, *Springer 2006, pages 319 and 692 for the problem and solution, respectively).*

3404. [2009 : 42, 45] *Proposed by Michel Bataille, Rouen, France.*

Let Q be a cyclic quadrilateral. The perpendiculars to each diagonal through its endpoints form a parallelogram, P. Characterize the centre of P and show that opposite sides of Q intersect on a diagonal of P.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let A, B, C, and D be the vertices of Q in cyclic order.

First, we prove that the centre of P is the circumcentre of Q. The centre of P is the point of intersection of the perpendicular bisectors of the line segments AC and BD. But AC and BD are chords of the circumcircle of Q. Hence, their perpendicular bisectors intersect at the centre of this circle.

It remains to prove that opposite sides of Q intersect on a diagonal of P. Let E, F, G, and H be the vertices of P such that A, B, C, and D lie on HE, EF, FG, and GH, respectively. We show that the lines AD, BC, and EG are concurrent. (The proof that the lines AB, CD, and FH are concurrent is similar.)

Since $\angle AHD = \angle BFC$ and

$$\angle DAH \;=\; 90^\circ - \angle CAD \;=\; 90^\circ - \angle CBD \;=\; \angle CBF\,,$$

we conclude that triangles ADH and BCF are similar. Hence,

$$\frac{DH}{AH} \;=\; \frac{CF}{BF}\,.$$

Similarly,

$$\frac{AE}{DG} \;=\; \frac{BE}{CG}\,.$$

Let the lines AD and BC meet the line EG at points I and J, respectively. Using Menelaus' thereom for $\triangle EGH$ and $\triangle EGF$, we obtain

$$\frac{IE}{IG} \;=\; \frac{DH\cdot AE}{DG\cdot AH} \;=\; \frac{CF\cdot BE}{CG\cdot BF} \;=\; \frac{JE}{JG}\,.$$

Consequently, $I = J$, which shows that the lines AD, BC, and EG are concurrent.

If we assume that parallel lines intersect at infinity, then the result remains true when one or both pairs of opposite sides of Q are parallel (Q is then an isosceles trapezium or rectangle, respectively). If opposite sides of Q are parallel, then they are also parallel to a diagonal of P, in which case they both intersect the diagonal at infinity.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3405. [2009 :42, 45] *Proposed by Michel Bataille, Rouen, France.*

Find the minimum value of

$$|\cos\alpha| + |\cos\beta| + |\cos\gamma| + |\cos(\alpha-\beta)| + |\cos(\beta-\gamma)| + |\cos(\gamma-\alpha)|\,,$$

where α, β, and γ are real numbers.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We prove that the minimum value is 2. Suppose that x, y, and z are real numbers such that $x+y+z=0$. Then using the triangle inequality as well as the trigonometric addition formulas, we obtain

$$\begin{aligned} |\cos x| + |\cos y| + |\cos z| & \\ \geq\ & |\cos x| + |\cos y \sin z + \sin y \cos z| \\ =\ & |\cos x| + |\sin(y+z)| \\ \geq\ & |\cos x \cos(y+z) - \sin x \sin(y+z)| \\ =\ & |\cos(x+y+z)| = 1 \end{aligned}$$

Choosing successively $(-\alpha, \beta, \alpha - \beta)$, $(-\beta, \gamma, \beta - \gamma)$, $(-\gamma, \alpha, \gamma - \alpha)$, and $(\alpha - \beta, \beta - \gamma, \gamma - \alpha)$ for (x, y, z), yields

$$\begin{aligned} |\cos\alpha| + |\cos\beta| + |\cos(\alpha-\beta)| &\geq 1\,, \\ |\cos\beta| + |\cos\gamma| + |\cos(\beta-\gamma)| &\geq 1\,, \\ |\cos\gamma| + |\cos\alpha| + |\cos(\gamma-\alpha)| &\geq 1\,, \\ |\cos(\alpha-\beta)| + |\cos(\beta-\gamma)| + |\cos(\gamma-\alpha)| &\geq 1\,. \end{aligned}$$

By adding up, we conclude that

$$|\cos\alpha| + |\cos\beta| + |\cos\gamma| + |\cos(\alpha-\beta)| + |\cos(\beta-\gamma)| + |\cos(\gamma-\alpha)| \geq 2\,.$$

The minimum is achieved when $(\alpha, \beta, \gamma) = \left(0, \frac{\pi}{2}, \frac{3\pi}{2}\right)$, among other values.

Also solved by ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer

3406. [2009 : 43, 45] *Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find

$$\lim_{n\to\infty} \ln\left[\frac{1}{2^n}\prod_{k=1}^{n}\left(2+\frac{k}{n^2}\right)\right]\,.$$

Independent solutions by Michel Bataille, Rouen, France and Alberto Arenas Gómez, student, University of La Rioja, Logroño, Spain.

Let $A_n = \ln\left[\frac{1}{2^n}\prod_{k=1}^{n}\left(2+\frac{k}{n^2}\right)\right]$. Then we have

$$A_n = \ln\left(\prod_{k=1}^{n}\left(1+\frac{k}{2n^2}\right)\right) = \sum_{k=1}^{n}\ln\left(1+\frac{k}{2n^2}\right)\,.$$

Using the known inequalities $\frac{x}{1+x} \leq \ln(1+x) \leq x$ valid for positive x, and setting $x = \frac{k}{2n^2}$, we obtain

$$\frac{k}{2n^2+k} = \frac{\frac{k}{2n^2}}{1+\frac{k}{2n^2}} \leq \ln\left(1+\frac{k}{2n^2}\right) \leq \frac{k}{2n^2}\,. \tag{1}$$

For each $k = 1, 2, \ldots, n$ we have

$$1 + \frac{k}{2n^2} \le 1 + \frac{n}{2n^2} = \frac{2n+1}{2n}$$

and (1) yields

$$\frac{2n}{2n+1}\left(\frac{k}{2n^2}\right) \le \ln\left(1 + \frac{k}{2n^2}\right) \le \frac{k}{2n^2}\,.$$

Summing over k and using the identity $\sum\limits_{k=1}^{n} k = \dfrac{n(n+1)}{2}$ yields

$$\frac{n+1}{2(2n+1)} \le A_n \le \frac{1}{4}\cdot\frac{n+1}{n}\,.$$

Finally, by the Squeeze Theorem, we have that $\lim\limits_{n\to\infty} A_n = \dfrac{1}{4}$.

Also solved by ARKADY ALT, San Jose, CA, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There was one incomplete solution submitted.

Stan Wagon, Macalester College, St. Paul, MN, USA, submitted a computer generated solution.

3407. [2009 : 43, 45] *Proposed by Roy Barbara, Lebanese University, Fanar, Lebanon.*

Let S be a set of positive integers containing the integer 2007 and such that

(a) If x, $y \in S$ and $x \neq y$, then $|x - y| \in S$, and

(b) If $x \in S$, then $\left(x^3 - 1007x + 3007\right) \in S$.

Prove that S is the set of all positive integers.

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Since $2007 \in S$, we have that $2007^3 - 1007 \cdot 2007 + 3007 \in S$. By subtracting 2007 enough times, we find that $1000 \in S$.

Thus, by property (a), we also find that $1007 = 2007 - 1000 \in S$ and that $7 = 1007 - 1000 \in S$.

Since $6 = 1000 - 7 \cdot 142$, we also obtain $6 \in S$. Thus $1 = 7 - 6 \in S$.

We define $x_1 = 2007$ and $x_{i+1} = x_i^3 - 1007x_i + 3007$. Then

$$\lim_{i\to\infty} x_i = \infty\,.$$

[*Ed.: Since* $x_1 = 2007$, *it is obvious that* $x_{i+1} > x_i > 2007$, *hence* x_i *is an increasing sequence of integers.*]

Now let n be any positive integer. We know that there exists an i such that $x_i \in S$ and $x_i \geq n$. Then, by subtracting 1 from x_i enough times, we find $n \in S$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; JOEL SCHLOSBERG, Bayside, NY, USA; DIGBY SMITH, Mount Royal College, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; VASILE TEODOROVICI, Toronto, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Hess, Smith, and the proposer pointed out that part (b) contained a minor error, since for $x = 4, 5, \ldots, 30$ *the cubic* $x^3 - 1007x + 3007$ *takes negative integer values. This can be remedied by rephrasing (b) as: If* $x \in S$, *then* $\left|x^3 - 1007x + 3007\right| \in S$.

3408. [2009 : 43, 45] *Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia.*

Let $\{c_i\}_{i=1}^{\infty}$ be a sequence of distinct positive integers, and let $|q| < 1$. Prove that the inequality

$$\frac{\displaystyle\sum_{i=1}^{\infty} c_i q^{c_i}}{1 + \displaystyle\sum_{i=1}^{\infty} q^{c_i}} \leq \frac{q}{1-q}$$

holds for all such sequences $\{c_i\}_{i=1}^{\infty}$ if and only if $q \in \left[0, \frac{1}{2}\right]$.

Solution by Michel Bataille, Rouen, France, modified by the editor.

First, suppose that $q \in \left[0, \frac{1}{2}\right]$ and let $\{c_i\}_{i=1}^{\infty}$ be a sequence of distinct positive integers. The inequality is equivalent to

$$(1-q)\sum_{i=1}^{\infty} c_i q^{c_i} \;\leq\; q + \sum_{i=1}^{\infty} q^{c_i+1}\,. \qquad (1)$$

Both sides are 0 if $q = 0$, so we suppose that $0 < q \leq \frac{1}{2}$. Henceforth, we will also suppose that $c_1 < c_2 < c_3 < \cdots$, since the series involved are absolutely convergent and therefore may be rearranged. Rewriting the left side of (1) as $c_1 q^{c_1} + \sum_{i=1}^{\infty}\left(c_{i+1}q^{c_{i+1}} - c_i q^{c_i+1}\right)$, we see it suffices to prove

(a) $c_1 q^{c_1} \leq q$ and (b) $c_{i+1}q^{c_{i+1}} - c_i q^{c_i+1} \leq q^{c_i+1} \quad (i = 1, 2, \ldots)$.

Now, (a) is equivalent to $q^{c_1-1} \le \frac{1}{c_1}$, which holds since $q^{c_1-1} \le \frac{1}{2^{c_1-1}}$ and $2^{n-1} \ge n$ for each positive integer n. As for inequality (b), we rewrite it as

$$q^{c_{i+1}-c_i-1} \;\le\; \frac{c_i+1}{c_{i+1}}\,.$$

We have $c_{i+1} - c_i - 1 \ge 0$, so it suffices to prove that

$$\begin{aligned}\frac{1}{2^{c_{i+1}-c_i-1}} &\le \frac{c_i+1}{c_{i+1}}\,, \quad \text{or}\\ \frac{c_{i+1}}{2^{c_{i+1}}} &\le \frac{c_i+1}{2^{c_i+1}}\,.\end{aligned}$$

The latter holds because $c_{i+1} \ge c_i + 1 \ge 2 > \frac{1}{\ln 2}$, and $f(x) = \frac{x}{2^x}$ is decreasing on the interval $\left[\frac{1}{\ln 2}, \infty\right)$.

Next, suppose $-1 < q < 0$. Let $c_i = 2i$, for each $i = 1, 2, \ldots$. Observe that the left-hand side of the original inequality is positive, while the right-hand side of the original inequality is negative, a contradiction. Thus, the inequality does not hold in this case for all admissible sequences $\{c_i\}_{i=1}^{\infty}$.

Lastly, suppose $\frac{1}{2} < q < 1$. Let $c_i = i+1$ for each $i = 1, 2, \ldots$. Then, $c_1q^{c_1} = 2q^2 > q$. Observing that $c_{i+1} = c_i + 1$ for each i, we deduce that

$$c_1q^{c_1} + \sum_{i=1}^{\infty} c_{i+1}q^{c_{i+1}} \;>\; q + \sum_{i=1}^{\infty}(c_i+1)q^{c_i+1}\,,$$

and so (1) does not hold. This completes the proof.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; KEE-WAI LAU, Hong Kong, China; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There were two incomplete solutions submitted.

3409. [2009 : 43, 45] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, c, and d be positive real numbers. Prove that

$$\begin{aligned}&\frac{ab+bc+ca}{a^3+b^3+c^3} + \frac{ab+bd+da}{a^3+b^3+d^3} + \frac{ac+cd+da}{a^3+c^3+d^3} + \frac{bc+cd+db}{b^3+c^3+d^3}\\ &\quad\le\; \min\left\{ \frac{a^2+b^2}{(ab)^{3/2}} + \frac{c^2+d^2}{(cd)^{3/2}},\ \frac{a^2+c^2}{(ac)^{3/2}} + \frac{b^2+d^2}{(bd)^{3/2}},\ \frac{a^2+d^2}{(ad)^{3/2}} + \frac{b^2+c^2}{(bc)^{3/2}} \right\}.\end{aligned}$$

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

By the AM–GM Inequality, $a^3+b^3+c^3 \geq 3abc$, so that

$$\frac{ab+bc+ca}{a^3+b^3+c^3} \;\leq\; \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\,,$$

with analogous inequalities holding for the other three terms on the left side of the claimed inequality. Hence,

$$\frac{ab+bc+ca}{a^3+b^3+c^3}+\frac{ab+bd+da}{a^3+b^3+d^3}+\frac{ac+cd+da}{a^3+c^3+d^3}+\frac{bc+cd+db}{b^3+c^3+d^3}$$
$$\leq\; \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\,. \qquad (1)$$

The AM–GM and AM–QM inequalities imply that $(ab)^{1/2} \leq \sqrt{\dfrac{a^2+b^2}{2}}$ and $a+b \leq 2\sqrt{\dfrac{a^2+b^2}{2}}$, respectively. Multiplying across these inequalities yields $(ab)^{1/2}(a+b) \leq a^2+b^2$. Hence,

$$\frac{1}{a}+\frac{1}{b} \;=\; \frac{a+b}{ab} \;\leq\; \frac{a^2+b^2}{(ab)^{3/2}}\,.$$

Analogous inequalities again hold for the other pairs of variables, thus

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$
$$\leq\;\min\left\{\frac{a^2+b^2}{(ab)^{3/2}}+\frac{c^2+d^2}{(cd)^{3/2}},\,\frac{a^2+c^2}{(ac)^{3/2}}+\frac{b^2+d^2}{(bd)^{3/2}},\,\frac{a^2+d^2}{(ad)^{3/2}}+\frac{b^2+c^2}{(bc)^{3/2}}\right\}\,.$$

The desired inequality now follows from the above inequality and (1).

Also solved by MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer. One incorrect solution was submitted.

A computer generated solution totaling 228 pages was submitted, which due to its length and complexity could not be verified in the available time.

3410. [2009 : 43, 46] *Proposed by Joe Howard, Portales, NM, USA.*

Let a, b, and c be the sides of triangle ABC, let R be its circumradius, and let F be its area. Prove that

$$\sum_{\text{cyclic}} \frac{bc\sin^2 A/2}{b+c} \;\geq\; \frac{F}{2R}\,.$$

Similar solutions by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let r and s denote the inradius and semiperimeter of ΔABC, respectively. By the Law of Cosines, $\dfrac{b^2+c^2-a^2}{2bc} = \cos A = 2\cos^2\dfrac{A}{2} - 1$, hence

$$bc\sin^2\frac{A}{2} \;=\; s(s-a)\tan^2\frac{A}{2}\,.$$

Since $F = rs$, it follows that the proposed inequality is equivalent to

$$\sum_{\text{cyclic}} \frac{b+c-a}{b+c}\tan^2\frac{A}{2} \;\geq\; \frac{r}{R}\,.$$

Using the Law of Sines, we have

$$\begin{aligned} \frac{b+c-a}{b+c} \;&=\; \frac{\sin B + \sin C - \sin A}{\sin B + \sin C} \;=\; \frac{4\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}}{2\cos\frac{A}{2}\cos\frac{B-C}{2}} \\ &\geq\; 2\sin\frac{B}{2}\sin\frac{C}{2}\,. \end{aligned}$$

Now, using the last inequality and the well-known and easy to prove identity $r = 4R\sin\dfrac{B}{2}\sin\dfrac{C}{2}\sin\dfrac{A}{2}$, we have

$$\frac{b+c-a}{b+c}\tan^2\frac{A}{2} \;\geq\; \left(2\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}\right)\cdot\frac{\sin\frac{A}{2}}{\cos^2\frac{A}{2}} \;=\; \frac{r}{2R}\cdot\frac{\sin\frac{A}{2}}{\cos^2\frac{A}{2}}\,.$$

Thus, it suffices to prove that

$$\sum_{\text{cyclic}} \frac{\sin\frac{A}{2}}{\cos^2\frac{A}{2}} \;\geq\; 2\,.$$

To this end, consider $f(x) = \dfrac{\sin x}{\cos^2 x}$ for $x \in \left(0, \frac{\pi}{2}\right)$. An easy calculation yields $f''(x) = (\cos x)^{-4}(\sin x)\left(5+\sin^2 x\right) > 0$, so that f is a convex function on the interval $\left(0,\frac{\pi}{2}\right)$. From Jensen's inequality,

$$f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right) \;\geq\; 3f\left(\frac{A/2+B/2+C/2}{3}\right)\,,$$

that is,

$$\sum_{\text{cyclic}} \frac{\sin\frac{A}{2}}{\cos^2\frac{A}{2}} \;\geq\; 3\cdot\frac{\sin\left(\frac{\pi}{6}\right)}{\cos^2\left(\frac{\pi}{6}\right)} \;=\; 2\,,$$

which completes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvarez Cubero, Priego de Córdoba, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; TITU ZVONARU, Comănești, Romania; and the proposer.

3411. [2009 :44, 46] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers such that

$$a^6 + b^6 + c^6 \ < \ \frac{32}{33}\left(a^3 + b^3 + c^3\right)^2 \ .$$

Prove that at least one of the quadratics $ax^2 + bx + c$, $bx^2 + cx + a$, or $cx^2 + ax + b$ has no real roots.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA, modified by the editor.

We prove that in fact the conclusion holds for all positive real numbers a, b, and c. Suppose that each quadratic has at least one real root. Then we have $a^2 \geq 4bc$, $b^2 \geq 4ac$, and $c^2 \geq 4ab$. Multiplying these inequalities we obtain $a^2b^2c^2 \geq 64a^2b^2c^2$, which is a contradiction.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; D.J. SMEENK, Zaltbommel, the Netherlands; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3412. [2009 :44, 46] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers such that $abc = 1$. Prove that

$$\sum_{\text{cyclic}} \frac{1}{\sqrt{a^3 + 2b^3 + 6}} \ \leq \ 1 \, .$$

Solution by Dung Nguyen Manh, High School of HUS, Hanoi, Vietnam.

Let x, y, z be positive real numbers such that $\frac{x}{y} = a$, $\frac{y}{z} = b$, $\frac{z}{x} = c$. By using the AM–GM Inequality we obtain

$$\begin{aligned} a^3 + 2b^3 + 6 &= \left(a^3 + b^3 + 1\right) + \left(b^3 + 1 + 1\right) + 3 \\ &\geq 3(ab + b + 1) \ = \ 3\left(\frac{x}{z} + \frac{y}{z} + 1\right) \ = \ \frac{3(x+y+z)}{z} \, . \end{aligned}$$

Hence, $\dfrac{1}{\sqrt{a^3+2b^3+6}} \le \dfrac{\sqrt{z}}{\sqrt{3(x+y+z)}}$. Adding up the cyclic variants of this inequality, we obtain

$$\sum_{\text{cyclic}} \frac{1}{\sqrt{a^3+2b^3+6}} \;\le\; \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{3(x+y+z)}}\,. \tag{1}$$

On the other hand, by the Cauchy–Schwarz Inequality we have

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \;\le\; \sqrt{3(x+y+z)}\,. \tag{2}$$

The result now follows from (2) and (1).

Equality holds if and only if $a=b=c=1$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3413. [2009 : 44, 46] *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Let a, b, c, and d be real numbers in the interval $[1,2]$. Prove that

$$\frac{a+b}{c+d} + \frac{c+d}{a+b} - \frac{a+c}{b+d} \le \frac{3}{2}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany, modified by the editor.

Since the inequality is invariant under the permutation

$$\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix},$$

then without loss of generality we can also assume that $b \le d$.

Let $f(a,b,c,d) = \dfrac{a+b}{c+d} + \dfrac{c+d}{a+b} - \dfrac{a+c}{b+d}$. We need to prove that $f(a,b,c,d) \le \dfrac{3}{2}$ for all a, b, c, d in the interval $[1,2]$ with $b \le d$. We will prove more generally that $f(a,b,c,d) \le \dfrac{3}{2}$ in the region

$$D = \{(a,b,c,d) \in \mathbb{R}^4 \;:\; 1 \le b \le d \le 2 \text{ and } \frac{d}{2} \le a,c \le 2b\}\,.$$

Since the function f is continuous and the region D is compact, f attains a maximum in D.

For fixed b and d, the function f has positive partial derivatives

$$\frac{\partial^2 f}{\partial a^2} = \frac{2(c+d)}{(a+b)^3}\,; \qquad \frac{\partial^2 f}{\partial c^2} = \frac{2(a+b)}{(c+d)^3}\,,$$

and therefore it is convex for a, $c > 0$. Thus, f attains its maximum in D at a point with $(a,c) \in \left\{ \left(\frac{d}{2},\frac{d}{2}\right), \left(\frac{d}{2},2b\right), \left(2b,\frac{d}{2}\right), (2b,2b)\right\}$.

By multiplying each side of the inequality by $8(c+d)(a+b)(b+d)$, we see that proving the inequality amounts to proving that

$$\begin{aligned} g(a,c) \;=\; & 8b^3 + 8d^3 + 8a^2b - 8a^2c + 16ab^2 - 8ac^2 \\ & - 12ad^2 - 12b^2c - 4b^2d - 4bd^2 + 8c^2d + 16cd^2 \\ & - 20abc - 4abd - 20acd - 4bcd \;\le\; 0 \end{aligned}$$

for $(a,c) \in \left\{ \left(\frac{d}{2},\frac{d}{2}\right), \left(\frac{d}{2},2b\right), \left(2b,\frac{d}{2}\right), (2b,2b) \right\}$ and $1 \le b \le d \le 2$.

We have

$$\begin{aligned} g\left(\frac{d}{2},\frac{d}{2}\right) &= 8b^3 - 2b^2d - 11bd^2 + 5d^3 \;=\; (b-d)(2b-d)(4b+5d) \;\le\; 0\,, \\ g\left(\frac{d}{2},2b\right) &= -16b^3 - 8b^2d + 4bd^2 + 2d^3 \;=\; 2(d-2b)(2b+d)^2 \;\le\; 0\,, \\ g\left(2b,\frac{d}{2}\right) &= 72b^3 - 54b^2d - 54bd^2 + 18d^3 \\ &= 18\big[(d^2(d-2b) + b(b-d)(4b+d)\big] \;\le\; 0\,, \\ g(2b,2b) &= -160b^3 - 68b^2d + 4bd^2 + 8d^3 \\ &= 8\big[d^3 - (2b)^3\big] + 4bd(d-2b) - 60b^2d - 96b^3 \;\le\; 0\,, \end{aligned}$$

and the inequality follows.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece, ALBERT STADLER, Herrliberg, Switzerland, and the proposer. There were three incomplete solutions submitted.

3414. [2009 : 108, 111] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

As triangle ABC varies, its circumcircle $\gamma_1(O,R)$ and its incircle $\gamma_2(I,r)$ are fixed, where O and I are the respective centres and R and r are the respective radii. Find the locus of the orthocentre H of triangle ABC.

Solution by Michel Bataille, Rouen, France.

We show that the locus of H is the circle $\gamma_3(J, R-2r)$, where J is the reflection of O in the point I; note that O, I and, consequently, J are fixed while the triangle ABC varies.

(a) Let N be the centre of the nine-point (or Euler) circle, and F be the Feuerbach point (that is, the point where the nine-point circle is tangent to the incircle γ_2). Since the radii NF and IF have respective lengths $\frac{R}{2}$ and r, N must lie on the circle with centre I and radius $\frac{R}{2} - r$. Since N is the midpoint of OH while I is the midpoint of OJ, we must have JH parallel to and twice the length of IN; in other words, H must lie on the circle with centre J and radius $2\left(\frac{R}{2} - r\right) = R - 2r$.

(b) Conversely, let H be an arbitrary point of the circle $\gamma_3(J, R-2r)$. We shall use complex numbers to show that there exist points A, B, C on the circumcircle $\gamma_1(O,R)$ for which the triangle ABC has incircle $\gamma_2(I,r)$ and orthocentre H. Without loss of generality we assume that γ_1 is the unit circle (that is, $R = 1$ and O is represented by the complex number 0); moreover, we will take I on the real axis. Because Euler's formula gives $OI^2 = R^2 - 2Rr$, I will be represented by the real number $u := \sqrt{1-2r}$. Because we assume that H is on γ_3, it is represented by the complex number $h := 2u + u^2 e^{i\theta}$ for some real number θ. Now, let z_1, z_2, z_3 denote the complex roots of the polynomial

$$P(z) \;=\; z^3 - \left(2u + u^2 e^{i\theta}\right) z^2 + (u^2 + 2ue^{i\theta})z - e^{i\theta}\,. \tag{1}$$

Since $P(z) = z(z-u)^2 - e^{i\theta}(1-uz)^2$, we have for $j = 1, 2, 3$,

$$\left(\frac{z_j - u}{1 - uz_j}\right)^2 \;=\; \frac{e^{i\theta}}{z_j}\,. \tag{2}$$

Note that $\left|\frac{z-u}{1-uz}\right|$ is less than or greater than 1 according as $|z| < 1$ or $|z| > 1$, while $\left|\frac{e^{i\theta}}{z}\right| = \frac{1}{|z|}$; consequently, equation (2) implies that $|z_j| = 1$. Thus, the points A, B, C that correspond respectively to z_1, z_2, z_3 are on the circle $\gamma_1 = \gamma_1(O,1)$. In addition, since $z_1 + z_2 + z_3 = h$ (from (1)), we have $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$. We recognize this last equation to be the vector formulation of the theorem that the segment AH is parallel to and twice as long as the segment joining O to the midpoint of BC, which implies that H is the orthocentre of $\triangle ABC$. It remains to prove that I is its incentre. We denote by L the second point where the line AI intersects γ_1, and represent it by the complex number ℓ; the complex equation of AI is then, $z + z_1\ell\bar{z} = z_1 + \ell$. Because u (representing the point I) must satisfy this equation, we have

$$\ell \;=\; \frac{u - z_1}{1 - uz_1}\,,$$

so that from (2), $\ell^2 = \frac{e^{i\theta}}{z_1} = \frac{z_1 z_2 z_3}{z_1} = z_2 z_3$. Thus, $2\arg\ell = \arg z_2 + \arg z_3$, which tells us that the perpendicular bisector of BC meets γ_1 in L; it follows that $AI \equiv AL$ is one of the bisectors of $\angle BAC$. Similarly, BI and CI

are bisectors of $\angle CBA$ and $\angle ACB$, respectively. Since I is interior to the circumcircle γ_1 of $\triangle ABC$, I must be the incentre of $\triangle ABC$.

Comment. I came across the problem in the references listed below; however, the converse, treated above in part (b), was either absent from these sources or very incomplete.

References

[1] William Gallatly, *The Modern Geometry of the Triangle*, 2nd ed. Hodgson (1913).

[2] Jos.E. Hofmann, Zur elementaren Dreiecksgeometrie in der komplexen Ebene. *L'Enseignement mathématique* 4 (1958) pp. 197-199. This article has been translated into French by Lisiane Nivelle, *L'Ouvert*, **98** (2000) pp. 1-22; it is available at `http://irem.u-strasbg.fr/php/articles/98_Nivelle.pdf`.

[3] T. Lalesco, *La géometrie du triangle*. J. Gabay (2003), p. 21.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Observe that as a consequence of the argument in part (b), each point H on its circle γ_3 determines three positions for the vertex A: at z_1, z_2, and z_3; in other words, as the vertex A of the moving triangle travels once around γ_1, H will travel three times around γ_3. Woo observed that because the centroid G of $\triangle ABC$ is on the Euler line two-thirds of the way from O to N, the argument of part (a) also shows that the locus of G is a circle whose radius is two-thirds that of the locus of N, namely $\frac{2}{3}\left(\frac{R}{2} - r\right)$; its centre is the point two-thirds of the way from O to I.

Almost all solvers showed that the locus of orthocentre H was a subset of the circle $\gamma_3(J, R-2r)$ using an argument similar to part (a) of our featured solution. Geupel, however, used complex numbers; he provided the only solution other than Bataille's to address satisfactorily the converse problem of showing the locus to be all of γ_3. It would be nice if somebody could treat this converse using elementary geometry in the spirit of part (a). The proposer found the problem in the December 1912 issue of the Journal de mathématiques élémentaires *with a "rather long" proof; he does not mention if the converse problem was addressed there.*