SKOLIAD No. 122

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Please send your solutions to problems in this Skoliad by **1 July, 2010**. A copy of ***CRUX*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest for this number of the Skoliad is the 27th New Brunswick Mathematics Competition, 2009, Grade 9, Part C. We thank Daryl Tingley, Department of Mathematics and Statistics, University of New Brunswick, and Paul Deguire, Département de mathématiques et de statistique, Faculté des sciences, Université de Moncton, for providing us with this contest and for permission to publish it.

Complete justification is required in order for a solver to obtain credit for his/her solution.

27th New Brunswick Mathematics Competition, 2009
Grade 9, Part C
Approximately 30 minutes allowed

1. If you write all integers from 1 to 100, how many even digits will be written? (When you write the number 42, two even digits are written.)

(A) 50 (B) 71 (C) 80 (D) 89 (E) 91

2. In a farm there are hens (no hump, two legs), camels (two humps, four legs) and dromedaries (one hump, four legs). If the number of legs is four times the number of humps, then the number of hens divided by the number of camels will be?

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) Not enough information

3. A cubic box of side 1 m is placed on the floor. A second cubic box of side $\frac{2}{3}$ m is placed on top of the first box so that the centre of the second box is directly above the centre of the first box. A painter paints all of the surface area of the two boxes that can be reached without moving the boxes. What is the total area of surface that is painted?

(A) $\frac{49}{9}$ m^2 (B) $\frac{57}{9}$ m^2 (C) $\frac{61}{9}$ m^2 (D) $\frac{72}{9}$ m^2 (E) None of these

4. What is the ones digit of 2^{2009}?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

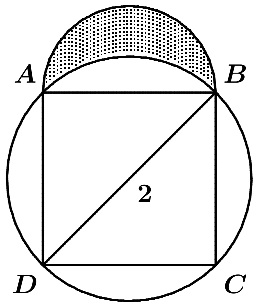
5. The numbers 1, 2, 3, 4, 5, and 6 are to be arranged in a row. In how many ways can this be done if 2 is always to the left of 4, and 4 is always to the left of 6? (For example $2, 5, 3, 4, 6, 1$ is an arrangement with 2 to the left of 4 and 4 to the left of 6.)

(A) 20 (B) 36 (C) 60 (D) 120 (E) 240

6. The square $ABCD$ is inscribed in a circle with diameter BD of length 2. If AB is the diameter of the semicircle on top of the square, what is the area of the shaded region?

(A) $\frac{4-\pi}{4}$ (B) $\frac{\pi-2}{4}$ (C) $\frac{1}{2}$

(D) 1 (E) Not enough information



27^e Concours de Mathématiques du Nouveau-Brunswick, 2009
9^e année, Partie C
Durée : environ 30 minutes

1. Si vous écrivez tous les entiers de 1 à 100, combien de chiffres pairs seront écrits ? (Quand vous écrivez le nombre 42, vous écrivez deux chiffres pairs.)

(A) 50 (B) 71 (C) 80 (D) 89 (E) 91

2. Dans une ferme il y a des poules (pas de bosse, deux pattes), des chameaux (deux bosses, quatre pattes) et des dromadaires (une bosse, quatre pattes). Si le nombre de pattes est quatre fois le nombre de bosses, alors le nombre de poules, divisé par le nombre de chameaux sera de ?

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) Information insuffisante

3. Une boîte cubique dont le côté mesure 1 m est placée sur le sol. Une seconde boîte cubique dont le côté mesure $\frac{2}{3}$ m est placée sur la première de manière à ce que son centre soit exactement au dessus du centre de la première boîte. Un peintre peint alors les surfaces des deux boîtes qu'il peut rejoindre sans bouger les boîtes. Quelle est la surface totale qui est peinte ?

(A) $\frac{49}{9}$ m^2 (B) $\frac{57}{9}$ m^2 (C) $\frac{61}{9}$ m^2 (D) $\frac{72}{9}$ m^2 (E) Aucune de ces réponses

4. Quel est le chiffre des unités de 2^{2009} ?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

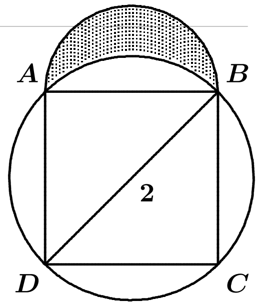
5. Les nombres 1, 2, 3, 4, 5 et 6 sont placés en ligne. De combien de façons cela peut-il être fait si 2 est toujours à la gauche de 4 et 4 est toujours à la gauche de 6 ? (2, 5, 3, 4, 6, 1 est un tel placement avec 2 à la gauche de 4 et 4 à la gauche de 6).

(A) 20 (B) 36 (C) 60 (D) 120 (E) 240

6. Le carré $ABCD$ est inscrit dans un cercle dont le diamètre BD est de longueur 2. Si AB est le diamètre du demi-cercle au-dessus du carré, quelle est l'aire de la région ombragée ?

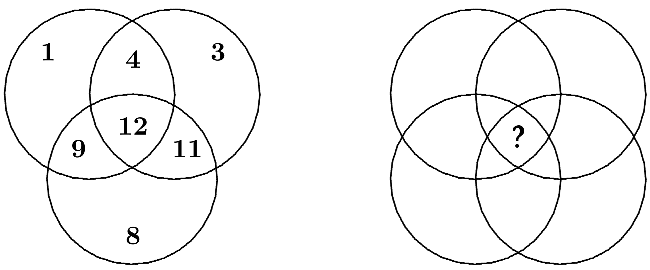
(A) $\frac{4-\pi}{4}$ (B) $\frac{\pi-2}{4}$ (C) $\frac{1}{2}$

(D) 1 (E) Information insuffisante



Next follow solutions to the Swedish Junior High School Mathematics Contest, Final Round, 2007/2008 [2009 : 129–131].

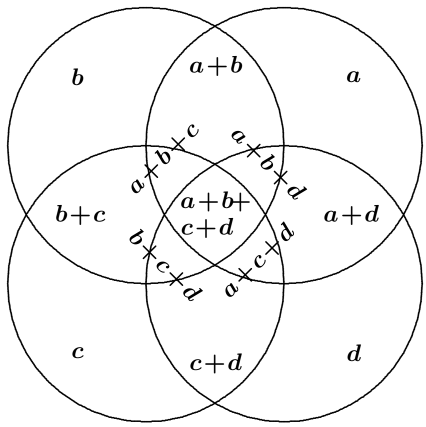
1. Values are assigned to a number of circles, and these values are written in the circles. When two or more circles overlap, the sum of the values of the overlapping circles is written in the common region. In the example on the left below, the three circles have been assigned the values 1, 3, and 8. Where the circle with value 1 overlaps the circle with value 3 we write 4 ($= 1 + 3$). In the region in the middle, we add all three values and write 12.



In the figure on the right above are four circles and, thus, thirteen regions. Find the number in the middle if the sum of all thirteen numbers is 294.

Solution by Alison Tam, student, Burnaby South Secondary School, Burnaby, BC.

Assign the values a, b, c, and d to the four circles, as shown in the diagram on the following page. Then the overlapping regions get the values shown. The sum of the thirteen regions is $7a + 7b + 7c + 7d$. Since this sum is required to be 294, it follows that the value in the middle region is $a + b + c + d = \frac{294}{7} = 42$.



Also solved by CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; EMILY HUANG, student, Burnaby Central Secondary School, Burnaby, BC; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; LENA CHOI, student, École Banting Middle School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; and THOMAS HSU, student, Moscrop Secondary School, Burnaby, BC.

2. This is the 20th edition of the Swedish Junior High School Mathematics Contest. The first qualification round was held in the fall of 1988, and this year's final is held in2008. That is twenty-one calendar years, 1988–2008, but the table at right has room for only eighteen of them. Which three must be omitted if the digit sum in every row and every column must be divisible by 9? (Two solutions exist.)

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

The digits in the first columns of each of the two parts of the chart are 1's and 2's. Say one such column has k copies of 1 and $9 - k$ copies of 2. Then the sum is $k + 2(9 - k) = 18 - k$. If this sum is a multiple of 9, then k is a multiple of 9, so either $k = 0$ or $k = 9$. Thus, one first column contains only 1's while the other first column contains only 2's. Since only nine years begin with 2, all of them must be used and they must all be in the same part of the chart. See the left-hand diagram below.

1	9			2	0	0	0
1	9			2	0	0	1
1	9			2	0	0	2
1	9			2	0	0	3
1	9			2	0	0	4
1	9			2	0	0	5
1	9			2	0	0	6
1	9			2	0	0	7
1	9			2	0	0	8

1	9	9	6	2	0	0	0
1	9	9	5	2	0	0	1
1	9	9	4	2	0	0	2
1	9	9	3	2	0	0	3
1	9	9	2	2	0	0	4
1	9	9	1	2	0	0	5
1	9	9	x	2	0	0	6
1	9	9	8	2	0	0	7
1	9	9	7	2	0	0	8

Here x is either 0 or 9

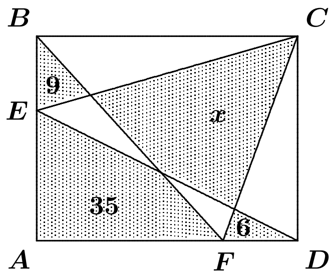
The third column in the 1900's part of the chart contains 8's and 9's.

Since there are at most two 8's and the column sum is divisible by 9, the column must consist of only 9's. Thus the years 1988 and 1989 must be excluded from the chart. Using that the row sums are divisible by 9, it is now easy to fill in the chart as in the right-hand diagram.

Hence the excluded years are either 1988, 1989, and 1990; or 1988, 1989 and 1999.

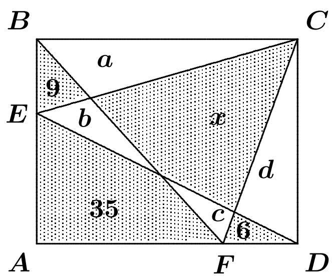
Also solved by LENA CHOI, student, École Banting Middle School, Coquitlam, BC.

3. The line segments DE, CE, BF, and CF divide the rectangle $ABCD$ into a number of smaller regions. Four of these, two triangles and two quadrilaterals, are shaded in the figure at right. The areas of the four shaded regions are 9, 35, 6, and x (see the figure). Determine the value of x.



Solution by Lena Choi, student, École Banting Middle School, Coquitlam, BC.

Since the base of $\triangle BCF$ is $|BC|$ and its height is $|CD|$, the area of $\triangle BCF$ is half the area of rectangle $ABCD$. But then $\triangle ABF$ and $\triangle CDF$ take up the other half of the rectangle. Similarly, the area of $\triangle CDE$ equals half the area of the rectangle as does the combined area of $\triangle BCE$ and $\triangle ADE$.



Assign letters to the areas of the four unshaded regions as in the figure. Then the paragraph above amounts to saying that $a+x+c$, $9+b+35+d+6$, $b+x+d$, and $9+a+35+c+6$ are all equal. In particular, $a+x+c = 9+a+35+c+6$, so $x = 9+35+6 = 50$.

Also solved by ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia.

4. A goody bag contains a two-digit number of goodies. Lisa adds the two digits and then removes as many goodies as the sum yields. Lisa repeats this procedure until the number of goodies left is a single digit number larger than zero. Find this single digit number.

Solution by the editors.

Say the number of goodies is $10a+b$. Then Lisa removes $a+b$ goodies and is left with $9a$ goodies. It follows that once Lisa has removed goodies at least once, then the number of goodies left is divisible by nine. Thus, when the number of goodies left reaches a single-digit number, that number must be nine.

Several solvers found that Lisa is left with nine goodies with several choices of the initial number of goodies. However, that does not prove that Lisa ***always*** *ends up with nine goodies.*

5. In how many ways can the list $[1, 2, 3, 4, 5, 6]$ be permuted if the product of neighbouring numbers must always be even?

Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.

If the product of neighbouring numbers is even, then the parity of the numbers must follow one of the patterns

$$eoeoeo\,, \quad oeoeoe\,, \quad oeoeeo\,, \quad \text{or} \quad oeeoeo\,,$$

where e is an even number and o is an odd number. Once you have chosen one of the four patterns, you can arrange the three even numbers into the even slots in six ways, and you can arrange the odd numbers in six ways. Thus the total number of permutations is $4 \cdot 6 \cdot 6 = 144$.

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

6. The digits of a five-digit number are $abcde$. Prove that $abcde$ is divisible by 7 if and only if the number $abcd - 2 \cdot e$ is divisible by 7.

Solution by the editors.

Let A and B denote the two numbers $abcde$ and $abcd - 2e$, respectively. Let x denote the four-digit number $abcd$. Then $A = 10x + e$ and $B = x - 2e$. Therefore,

$$2A + B \;=\; 20x + 2e + (x - 2e) \;=\; 21x\,.$$

If A is divisible by 7, then $A = 7m$ for some integer m, so

$$B \;=\; 21x - 2A \;=\; 21x - 14m \;=\; 7(3x - 2m)\,,$$

which is divisible by 7.

If B is divisible by 7, then $B = 7n$ for some integer n, so

$$2A \;=\; 21x - B \;=\; 21x - 7n \;=\; 7(3x - n)\,,$$

which is divisible by 7. But if $2A$ is divisible by 7, then so is A.

Thus, A is divisible by 7 if and only if B is divisible by 7.

This issue's prize of one copy of ***CRUX with MAYHEM*** for the best solutions goes to Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON. We would very much appreciate receiving more solutions from our readers. Solutions to just some of the problems are also very welcome.