THE OLYMPIAD CORNER

No. 283

R.E. Woodrow

Another year, and it is time to thank the many people who have contributed solutions, problems, comments, and advice during 2009.

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Thank you to both Jill Ainsworth, who produced the text for the *Corner* for the first four issues, and to Joanne Canape, who continued the rest of the issues, for their skilled and tireless efforts to translate my scribbles and notes into a nice presentation.

To start the New Year for the *Corner*, we give the problems proposed but not used at the 2007 IMO in Vietnam. My thanks go to Bill Sands, Canadian Team Leader, for collecting them for our use.

2007 IMO in VIETNAM
Problems Proposed But Not Used

Contributing Countries. Austria, Australia, Belgium, Bulgaria, Canada, Croatia, Czech Republic, Estonia, Finland, Greece, India, Indonesia, Iran, Japan, Korea (North), Korea (South), Lithuania, Luxembourg, Mexico, Moldova, Netherlands, New Zealand, Poland, Romania, Russia, Serbia, South Africa, Sweden, Thailand, Taiwan, Turkey, Ukraine, United Kingdom, and the United States of America

Problem Selection Committee. Ha Huy Khoai, Ilya Bogdanov, Tran Nam Dung, Le Tuan Hoa, Géza Kós.

Algebra

A1. Consider those functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy the condition

$$f(m+n) \;\geq\; f(m) + f\big(f(n)\big) - 1$$

for all m, $n \in \mathbb{N}$. Find all possible values of $f(2007)$. (Here $\mathbb{N}$ denotes the set of positive integers.)

A2. Let n be a positive integer, and let x and y be positive real numbers such that $x^n + y^n = 1$. Prove that

$$\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)(1-y)}.$$

A3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f\big(x + f(y)\big) = f(x+y) + f(y)$$

for all x, $y \in \mathbb{R}^+$. (Here $\mathbb{R}^+$ denotes the set of positive real numbers.)

A4. Let $c > 2$, and let $a(1)$, $a(2)$, $\ldots$, be a sequence of nonnegative real numbers such that

(a) $a(m+n) \leq 2a(m) + 2a(n)$ for all m, $n \geq 1$, and

(b) $a\big(2^k\big) \leq \dfrac{1}{(k+1)^c}$ for all $k \geq 0$.

Prove that the sequence $\{a(n)\}_{n=1}^{\infty}$ is bounded.

A5. Let a_1, a_2, $\ldots$, a_{100} be nonnegative real numbers satisfying the relation $a_1^2 + a_2^2 + \cdots + a_{100}^2 = 1$. Prove that $a_1^2 a_2 + a_2^2 a_3 + \cdots + a_{100}^2 a_1 < \frac{12}{25}$.

Combinatorics

C1. Let $n > 1$ be an integer. Find all sequences a_1, a_2, $\ldots$, a_{n^2+n} such that

(a) $a_i \in \{0, 1\}$ for all $1 \leq i \leq n^2 + n$, and

(b) $a_{i+1} + a_{i+2} + \cdots + a_{i+n} < a_{i+n+1} + a_{i+n+2} + \cdots + a_{i+2n}$ holds for all $0 \leq i \leq n^2 - n$.

C2. A unit square is dissected into $n > 1$ rectangles whose sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that one of the rectangles has no point on the boundary of the square.

C3. Determine all positive integers n for which the numbers in the set $S = \{1, 2, \ldots, n\}$ can be coloured red and blue so that $S \times S \times S$ contains exactly 2007 ordered triples (x, y, z) with these two properties :

(a) x, y, z are of the same colour ; and

(b) $x + y + z$ is divisible by n.

C4. Let $A_0 = (a_1, a_2, \ldots, a_n)$ be a sequence of real numbers. For each integer $k \geq 0$, form a new sequence A_{k+1} from $A_k = (x_1, \ldots, x_n)$ as follows :

(a) Choose a partition $\{1, 2, \ldots, n\} = I \cup J$ such that $\left| \sum_{i \in I} x_i - \sum_{j \in J} x_j \right|$ is minimized (if I or J is empty, then the corresponding sum is 0; if several such partitions exist, then choose one arbitrarily).

(b) Set $A_{k+1} = (y_1, y_2 \ldots, y_n)$, where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.

Prove that for some k, the sequence A_k has a term x with $|x| \geq \frac{n}{2}$.

C5. In the Cartesian coordinate plane let $S_n = \{(x, y) : n \leq x < n + 1\}$ for each integer n, and paint each region S_n either red or blue. Prove that any rectangle whose side lengths are distinct positive integers may be placed in the plane so that its vertices lie in regions of the same colour.

C6. Let $\alpha < \frac{3 - \sqrt{5}}{2}$ be a positive real number. Prove that there exist positive integers n and $p > \alpha 2^n$ for which one can select $2p$ pairwise distinct subsets $S_1, \ldots, S_p, T_1, \ldots, T_p$ of the set $\{1, 2, \ldots, n\}$ such that $S_i \cap T_j \neq \emptyset$ for all $1 \leq i, j \leq p$.

C7. A convex n-gon P in the plane is given. For every three vertices of P, the triangle determined by them is *good* if all its sides are of unit length. Prove that P has at most $\frac{2n}{3}$ good triangles.

Geometry

G1. An isosceles triangle ABC with $AB = AC$ is given. The midpoint of side BC is denoted by M. Let X be a variable point on the shorter arc MA of the circumcircle of triangle ABM. Let T be the point in the angle domain BMA, for which $\angle TMX = 90°$ and $TX = BX$. Prove that $\angle MTB - \angle CTM$ does not depend on X.

G2. The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

G3. Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle of ABC. Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C' respectively. Assume that the points A, B, C, A', B', C' are distinct, and that the lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

G4. Let $ABCD$ be a convex quadrilateral, and let points A_1, B_1, C_1, and D_1 lie on sides AB, BC, CD, and DA, respectively. Consider the areas of triangles AA_1D_1, BB_1A_1, CC_1B_1, and DD_1C_1; let S be the sum of the two smallest ones, and let S_1 be the area of quadrilateral $A_1B_1C_1D_1$.

Find the smallest positive real number k such that $kS_1 \geq S$ is always the case.

G5. Triangle ABC is acute with $\angle ABC > \angle ACB$, incentre I, and circumradius R. Point D is the foot of the altitude from vertex A, point K lies on line AD such that $AK = 2R$, and D separates A and K. Finally, lines DI and KI meet sides AC and BC at E and F, respectively.

Prove that if $IE = IF$, then $\angle ABC > 3\angle ACB$.

G6. Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incentre. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively, Let lines AC and BD meet at E, and let lines AD and BL meet at F. Prove that points E, I, and F are collinear.

Number Theory

N1. Find all pairs of positive integers (k, n) such that $\left(7^k - 3^n\right) \mid \left(k^4 + n^2\right)$.

N2. Let b, $n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $k \mid \left(b - a_k^n\right)$. Prove that $b = A^n$ for some integer A.

N3. Let X be a set of $10,000$ integers, none of them divisible by 47. Prove that there exists a 2007-element subset Y of X such that $a - b + c - d + e$ is not divisible by 47 for any a, b, c, d, $e \in Y$.

N4. For every integer $k \geq 2$, prove that 2^{3k} divides the number

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}}$$

but 2^{3k+1} does not.

N5. Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every m, $n \in \mathbb{N}$ and every prime p, the number $f(m + n)$ is divisible by p if and only if $f(m) + f(n)$ is divisible by p. ($\mathbb{N}$ is the set of all positive integers.)

N6. For a prime p and a positive integer n, denote by $\nu_p(n)$ the exponent of p in the prime factorization of $n!$. Given a positive integer d and a finite set of primes $\{p_1, p_2, \ldots, p_k\}$, show that there are infinitely many positive integers n such that $d \mid \nu_{p_i}(n)$ for all $1 \leq i \leq k$.

As a final pair of contests for your puzzling pleasure, we give two rounds of the Bundeswettbewerb Mathematik. Thanks go to Bill Sands, Canadian Team Leader to the 2007 IMO in Vietnam, for collecting them for our use.

BUNDESWETTBEWERB MATHEMATIK 2006
Second Round

1. A circle is divided into $2n$ congruent sectors, n of them coloured black and n of them coloured white. Starting with an arbitrarily chosen sector, the white sectors are numbered clockwise from 1 to n. Subsequently, the black sectors are numbered counterclockwise from 1 to n, again starting at an arbitrary sector.

Prove that there exist n consecutive sectors containing all of the numbers from 1 to n.

2. Let $\mathbb{Q}^+$ (resp. $\mathbb{R}^+$) denote the set of positive rational (resp. real) numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{R}^+$ that satisfy

$$f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)} \quad \text{for all } x, y \in \mathbb{Q}^+ .$$

3. The point P lies inside the acute-angled triangle ABC and C', A', B' are the feet of the perpendiculars from P to AB, BC, CA. Find all positions of P such that $\angle BAC = \angle B'A'C'$ and $\angle CBA = \angle C'B'A'$.

4. A positive integer n is *deficient* if there are at most nine different digits in the decimal representation of n. (Leading zeroes are not counted.)

Prove that for any finite set S of deficient numbers, the sum of the reciprocals of its elements is less than 180.

BUNDESWETTBEWERB MATHEMATIK 2007
First Round

1. Show that one can distribute the integers from 1 to 4014 on the vertices and the midpoints of the sides of a regular 2007-gon so that the sum of the three numbers along any side is constant.

2. Each positive integer is coloured either red or green so that

(a) The sum of three (not necessarily distinct) red numbers is red.

(b) The sum of three (not necessarily distinct) green numbers is green.

(c) There is at least one green number and one red number.

Find all colourings satisfying these conditions.

3. In triangle ABC the points E and F lie in the interiors of sides AC and BC (respectively) so that $|AE| = |BF|$. Furthermore, the circle through A, C and F and the circle through B, C and E intersect in a point $D \neq C$.

Prove that the line CD is the bisector of $\angle ACB$.

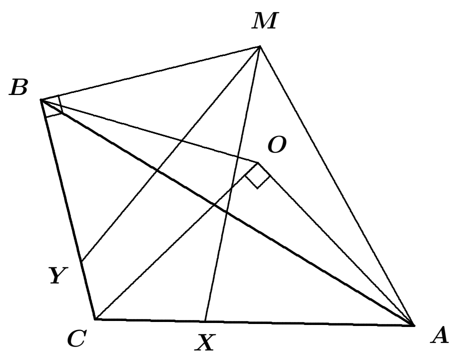
4. Let a be a positive integer. How many nonnegative integers x satisfy

$$\left\lfloor \frac{x}{a} \right\rfloor = \left\lfloor \frac{x}{a+1} \right\rfloor \ ?$$

Our solutions in the New Year begin with solutions from our readers to problems of the Bulgarian National Olympiad 2006 given in the *Corner* at [2008 : 409–410].

5. (Emil Kolev) Let $\triangle ABC$ be such that $\angle BAC = 30°$ and $\angle ABC = 45°$. Consider all pairs of points X and Y such that X is on the ray $\overrightarrow{AC}$, Y is on the ray $\overrightarrow{BC}$, and $OX = BY$, where O is the circumcenter of $\triangle ABC$. Prove that the perpendicular bisectors of the segments XY pass through a fixed point.

Solution by Titu Zvonaru, Comăneşti, Romania.



Without loss of generality, suppose that $OA = OB = OC = 1$. Since $\angle BAC = 30°$, we have that $\angle BOC = 60°$ and $BC = 1$, that is, $\triangle BOC$ is equilateral.

Since $\angle ABC = 45°$, it follows that $\triangle AOC$ has a right angle at O, hence $\angle OCA = \angle OAC = 45°$ and $AC = \sqrt{2}$.

If $X = A$, then $OX = 1$ and $BY = 1$, and hence $Y = C$; as a result, the fixed point we seek lies on the perpendicular bisector of AC.

Let M be the point on the same side of AC as O such that $\triangle AMC$ is equilateral. Then $\angle OCM = 60° - 45° = 15°$, $\angle MCB = 45°$, and by the Law of Cosines in $\triangle MCB$ we obtain

$$\begin{aligned} BM^2 &= MC^2 + BC^2 - 2MC \cdot BC \cos \angle MCB \\ &= 2 + 1 - 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1, \end{aligned}$$

hence $\triangle MCB$ has a right angle B.

Let $AX = \alpha$. By the Law of Cosines we have

$$\begin{aligned} OX^2 &= OA^2 + AX^2 - 2OA \cdot AX \cos \angle OAX = 1 + \alpha^2 - \alpha\sqrt{2}, \\ MX^2 &= MA^2 + AX^2 - 2MA \cdot AX \cos \angle MAC = 2 + \alpha^2 - \alpha\sqrt{2}, \quad (1) \end{aligned}$$

and by the Pythagorean Theorem we have

$$MY^2 = MB^2 + BY^2 = MB^2 + OX^2 = \alpha^2 - \alpha\sqrt{2} + 2. \quad (2)$$

By (1) and (2) it follows that the point M belongs to the perpendicular bisectors of the segments XY.

We continue with solutions from our readers to problems of the Indian Mathematical Olympiad 2006 (Team Selection Problems) given in the *Corner* at [2008 : 410–412].

1. Let n be a positive integer divisible by 4. Find the number of permutations σ of $(1,2,3,\ldots,n)$ which satisfy the condition $\sigma(j)+\sigma^{-1}(j)=n+1$ for all $j\in\{1,\,2,\,3,\,\ldots,\,n\}$.

Solution by Michel Bataille, Rouen, France.

Let $\mathcal{S}_n$ be the set of all permutations of $[n]=\{1,\,2,\,\ldots,\,n\}$ and let

$$\begin{aligned}\mathcal{Q}_n &= \{\sigma\in\mathcal{S}_n : \forall j\in[n]\,,\ \sigma(j)+\sigma^{-1}(j) \;=\; n+1\} \\ &= \{\sigma\in\mathcal{S}_n : \forall k\in[n]\,,\ \sigma\circ\sigma(k) \;=\; n+1-k\}\,.\end{aligned}$$

We show that if $n=4m$, then $|\mathcal{Q}_n|=\dfrac{(2m)!}{m!}=2^m N_m$ where N_m denotes the product $1\times 3\times\cdots\times(2m-1)$ of the first m odd natural numbers.

Let $\sigma\in\mathcal{Q}_n$, $k\in[n]$ and let $j=\sigma(k)$. Then, $j\neq k$ (otherwise $k=n+1-k$, contradicting the fact that $n+1$ is odd) and

$$\begin{aligned}\sigma(j) &= \sigma\circ\sigma(k) \;=\; n+1-k\,,\\ \sigma(n+1-k) &= \sigma\circ\sigma(j) \;=\; n+1-j\,,\\ \sigma(n+1-j) &= \sigma\circ\sigma(n+1-k) \;=\; n+1-(n+1-k) \;=\; k\,.\end{aligned}$$

Thus, the cycle containing k is $(k,j,n+1-k,n+1-j)$. Since k is arbitrary, the standard expression of σ as a product of disjoint cycles (the cycle decomposition of σ) consists of m 4-cycles of the form $(k,j,n+1-k,n+1-j)$. Conversely, if $\sigma\in\mathcal{S}_n$ has such a cycle decomposition, then clearly $\sigma\in\mathcal{Q}_n$.

Consider now the set $S=\{p_k : k\in[2m]\}$ where $p_k=\{k,\,n+1-k\}$ and let $\sigma\in\mathcal{Q}_n$. Using its cycle decomposition, we associate with σ in a natural way a partition $\{P_1,\,P_2,\,\ldots,\,P_m\}$ where each P_k is a 2-subset of S ($P_k\cap P_l=\emptyset$ if $k\neq l$; $\bigcup_{i=1}^{m}P_i=S$). There are N_m such partitions (see the lemma below) and each of them is obtained from exactly 2^m elements of $\mathcal{Q}_n$, since any 2-subset $P=\{p_k,\,p_j\}$ of S is obtained from two distinct 4-cycles, namely $(j,k,n+1-j,n+1-k)$ and $(k,j,n+1-k,n+1-j)$. It follows that $|\mathcal{Q}_n|=2^m N_m$.

Lemma If S is a set with $|S|=2m$, then the number of partitions of S formed by 2-subsets of S is $N_m=(2m-1)(2m-3)\cdots(3)(1)=\dfrac{(2m)!}{2^m m!}$.

Proof. Fix $s\in S$. We partition S into 2-subsets by first choosing one of the $2m-1$ 2-subsets $\{s,\,t\}$, where $t\in S$ and $t\neq s$, and then augmenting this subset with a partition into 2-subsets of $S-\{s,\,t\}$ (and there are N_{m-1} such partitions). Thus, $N_m=(2m-1)N_{m-1}$ when $m>1$, and the result follows since $N_1=1$.

3. (Short list, IMO 2005) There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. In each step (if possible) we pick a marker with the white side up that is not an outermost marker, remove it, and turn over the closest marker to the left and the closest marker to the right of it. Prove that one can reach a terminal state of exactly two markers if and only if $(n-1)$ is not divisible by 3.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We prove that if $3 \nmid (n-1)$, then two markers can be left behind. This is clear if $n = 2, 3$. Now assume that by starting with n markers we can leave behind two markers, and that $n+3$ markers are given. By successively choosing the second, third, then second marker from the left, we obtain n markers all white side up, and then we can finish with two markers. This completes the proof by induction.

It only remains to prove that if two markers can be left behind, then $3 \nmid (n-1)$. In a fixed state S, we let $b(m)$ be the number of black markers to the left of marker m. Further, we assign the number $T(S) = \sum\limits_{m \text{ white}} (-1)^{b(m)}$ to the state S. Let B or W denote a marker with the black or white side up, respectively; then the admissible reduction steps are

$$\text{(i)}\ \ BWB \to WW\,, \qquad \text{(iii)}\ \ WWB \to BW\,,$$
$$\text{(ii)}\ \ BWW \to WB\,, \qquad \text{(iv)}\ \ WWW \to BB\,.$$

Note that the parity of the number of B's is invariant, hence it is always even, and also we have $T(BB) = 0$, $T(WW) = 2$, and for the initial state I with n white markers $T(I) = n$. It therefore suffices to prove that $T(S)$ modulo 3 is invariant under the transitions (i)-(iv), because this implies that if $F \in \{BB, WW\}$ is reachable from I, then

$$n \;=\; T(I) \;\equiv\; T(F) \;\not\equiv\; 1 \pmod 3\ .$$

In the case of transition (i), if the white marker m that is picked has value $(-1)^{b(m)} = v \in \{-1, 1\}$ in state S, then the two markers m_1' and m_2' which are turned over have values $(-1)^{b(m_1')} = (-1)^{b(m_2')} = -v$ in the new state S', whereas the values of all other markers remain unchanged, thus

$$T(S') \;=\; T(S) - v - 2v \;=\; T(S) - 3v \;\equiv\; T(S) \pmod 3\ .$$

The transitions (ii) to (iv) are analyzed similarly.

This completes the proof.

7. Let ABC be a triangle with inradius r, circumradius R, and with sides $a = BC$, $b = AC$, and $c = AB$. Prove that

$$\frac{R}{2r} \;\ge\; \left(\frac{64a^2b^2c^2}{\big(4a^2-(b-c)^2\big)\big(4b^2-(c-a)^2\big)\big(4c^2-(a-b)^2\big)}\right)^2 .$$

Commentary by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Solved by George Apostolopoulos, Messolonghi, Greece. We give the comment of Amengual Covas.

This problem appears as Problem 11195 of the *American Mathematical Monthly*, Vol. 113 (January 2006), p. 79.

The solution appears on p. 648 of the August–September 2007 issue of the *American Mathematical Monthly*, Vol. 114, and includes two generalizations of the given inequality.

Next we give the write-up of Apostolopoulos.

By Heron's formula, the area of $\triangle ABC$ is $\sqrt{s(s-a)(s-b)(s-c)}$, and also the same area is given by $\dfrac{abc}{4R}$. Hence, $R = \dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$. Furthermore, the area of $\triangle ABC$ is rs, and a comparison with Heron's formula yields $r = \dfrac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$, where $s = \dfrac{a+b+c}{2}$. Therefore,

$$\frac{R}{2r} = \frac{abc}{8(s-a)(s-b)(s-c)}\,. \tag{1}$$

By using the AM–GM Inequality, we obtain

$$\begin{aligned} 4a^2-(b-c)^2 &= a^2+a^2+a^2+4(s-b)(s-c) \\ &\geq 4\sqrt[4]{a^2a^2a^24(s-b)(s-c)} \\ &= 4a\sqrt[4]{4a^2(s-b)(s-c)}\,. \end{aligned}$$

Similarly, we have the inequality $4b^2-(c-a)^2 \geq 4b\sqrt[4]{4b^2(s-c)(s-a)}$ and $4c^2-(a-b)^2 \geq 4c\sqrt[4]{4c^2(s-a)(s-b)}$, so that

$$\begin{aligned} &\frac{4a^2}{4a^2-(b-c)^2}\cdot\frac{4b^2}{4b^2-(c-a)^2}\cdot\frac{4c^2}{4c^2-(a-b)^2} \\ &\quad\leq \frac{a}{\sqrt[4]{4a^2(s-b)(s-c)}}\cdot\frac{b}{\sqrt[4]{4b^2(s-c)(s-a)}}\cdot\frac{c}{\sqrt[4]{4c^2(s-a)(s-b)}}\,. \end{aligned}$$

Finally, we have

$$\begin{aligned} &\left(\frac{64a^2b^2c^2}{\left(4a^2-(b-c)^2\right)\left(4b^2-(c-a)^2\right)\left(4c^2-(a-b)^2\right)}\right)^2 \\ &\qquad\leq \frac{a^2b^2c^2}{\sqrt{64a^2b^2c^2(s-a)^2(s-b)^2(s-c)^2}} \\ &\qquad\leq \frac{abc}{8(s-a)(s-b)(s-c)} = \frac{R}{2r}\,. \end{aligned}$$

as desired, where the last equality is from (1).

8. The positive divisors $d_1, d_2, \ldots, d_l$ of a positive integer n are ordered

$$1 = d_1 < d_2 < \cdots < d_l = m\,.$$

Suppose it is known that $d_7^2 + d_{15}^2 = d_{16}^2$. Find all possible values of d_{17}.

Solution by Titu Zvonaru, Comănești, Romania.

It is known (see the comment at the end of this solution) that if a, b, c are positive integers such that $a^2 = b^2 + c^2$, then abc is divisible by 60 and one of a, b, c is divisible by 4.

Let $D = \{d_1, \ldots, d_l\}$ be the set of divisors of n. Since $d_7^2 + d_{15}^2 = d_{16}^2$, it follows that n is divisible by 60, hence $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $d_4 = 4$, $d_5 = 5$, $d_6 = 6$ and $10 \in D$. We deduce that $d_7 \in \{7, 8, 9, 10\}$.

Case 1. Suppose that $d_7 = 10$. Then we have $d_{16}^2 - d_{15}^2 = 100$, and factoring yields $(d_{16} - d_{15})(d_{16} + d_{15}) = 100$. Since $d_{16} + d_{15}$ and $d_{16} - d_{15}$ have the same parity, we must have $d_{16} - d_{15} = 2$ and $d_{16} + d_{15} = 50$, and hence $d_{15} = 24$, $d_{16} = 26$. This means that $8 \in D$, contradicting $d_7 = 10$.

Case 2. Suppose that $d_7 = 9$. Then $(d_{16} - d_{15})(d_{16} + d_{15}) = 81$. Since $d_{15} + d_{16} \geq 15 + 16 = 31$, we must have $d_{16} - d_{15} = 1$ and $d_{16} + d_{15} = 81$, and hence $d_{15} = 40$, $d_{16} = 41$. This means that $8 \in D$, contradicting $d_7 = 9$.

Case 3. Suppose that $d_7 = 8$. Since $7 \notin D$, we deduce $d_{15} > 15$, $d_{16} > 16$, hence $d_{15} + d_{16} \geq 33$. Then $(d_{16} - d_{15})(d_{16} + d_{15}) = 64$ has no solution, because $d_{15} + d_{16} \geq 33$ and $d_{16} + d_{15}$ and $d_{16} - d_{15}$ have the same parity.

Case 4. Suppose $d_7 = 7$. Then necessarily $d_{16} - d_{15} = 1$ and $d_{16} + d_{15} = 49$, and hence $d_{15} = 24$, $d_{16} = 25$. If n is divisible by 9, 11, or 13, then $d_{15} < 24$, a contradiction. As a result, the positive divisors of n are

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
d_i	1	2	3	4	5	6	7	8	10	12	14	15	20	21	24	25

and we deduce that $d_{17} = 28$.

Comment. The fact that $60 \mid abc$ whenever $a^2 = b^2 + c^2$ can be seen as follows. There are positive integers m, n with $m > n$ such that $a = m^2 + n^2$, $b = m^2 - n^2$, $c = 2mn$. Let $R_i = \{x \in \mathbb{Z} : x \equiv \pm i \pmod 5\}$.

- If $m \in R_0$ or $n \in R_0$, then $c \equiv 0 \pmod 5$.
- If $m, n \in R_1$ or $m, n \in R_2$, then $b \equiv 0 \pmod 5$.
- If $m \in R_1$ and $n \in R_2$ (or vice-versa), then $a \equiv 0 \pmod 5$.

Hence, $5 \mid abc$. The divisibility by 3 and 4 is proved similarly.

Next we present solutions from our readers to problems given in the November 2008 number of the *Corner*, and the Third Round, Senior Division, of the 2004 South African Mathematical Olympiad given at [2008 : 412–413].

1. Let $a = 1111\cdots1111$ and $b = 1111\cdots1111$, where a has forty ones and b has twelve ones. Determine the greatest common divisor of a and b.

Solved by George Apostolopoulos, Messolonghi, Greece ; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's version.

Let $d = \gcd(a,b)$. Observe that $a = \frac{1}{9}\big(10^{40}-1\big)$ and $b = \frac{1}{9}\big(10^{12}-1\big)$.

Since d divides both $\frac{1}{9}\big(10^{40}-1\big)$ and $\frac{1}{9}\big(10^{12}-1\big)$, and we also have $10^4 - 1 = \big(10^{40}-1\big) - \big(10^{28}+10^{16}+10^4\big)\big(10^{12}-1\big)$, then

$$d \mid \frac{1}{9}\big(10^4 - 1\big)\,. \tag{1}$$

Conversely, since $10^{40}-1 = \big(10^4\big)^{10} - 1$ and $10^{12}-1 = \big(10^4\big)^3 - 1$ are both divisible by 10^4-1, we see that both a and b are divisible by $\frac{1}{9}\big(10^4-1\big)$. Hence,

$$\frac{1}{9}\big(10^4 - 1\big) \mid d\,. \tag{2}$$

From (1) and (2), it follows that $d = \frac{1}{9}\big(10^4 - 1\big) = 1111$.

5. Let $n \geq 2$ be an integer. Find the number of integers x with $0 \leq x < n$ and such that x^2 leaves a remainder of 1 when divided by n.

Solution by Michel Bataille, Rouen, France.

For each positive integer n let $S(n)$ be the set of all integers x such that $0 \leq x < n$ and $x^2 \equiv 1 \pmod{n}$ and let $s(n)$ denote its cardinality. We will prove that if $\lambda(n)$ is the number of odd prime divisors of n and $\mu(n)$ is the greatest nonnegative integer such that 2^m divides n, then

$$s(n) = \begin{cases} 2^{\lambda(n)} & \text{if } \mu(n) = 0, 1\,; \\ 2^{\lambda(n)+1} & \text{if } \mu(n) = 2\,; \\ 2^{\lambda(n)+2} & \text{if } \mu(n) > 2\,. \end{cases}$$

We first show that s is a multiplicative function. Clearly, $s(1) = 1$. Suppose that $n = ab$ where a, b are coprime positive integers. If $x \in S(n)$, then $x^2 \equiv 1 \pmod{ab}$, hence $x^2 \equiv 1 \pmod{a}$ and $x^2 \equiv 1 \pmod{b}$ and therefore $x \equiv y \pmod{a}$ and $x \equiv z \pmod{b}$ for some unique pair of integers $(y,z) \in S(a) \times S(b)$. Conversely, if $(y,z) \in S(a) \times S(b)$, then the Chinese Remainder Theorem ensures that the system $x \equiv y \pmod{a}$, $x \equiv z \pmod{b}$ has a unique solution x with $0 \leq x < ab = n$. Then both a, b divide $x^2 - 1$, hence $x^2 - 1 \equiv 0 \pmod{ab}$ (since a, b are coprime) and so $x \in S(n)$. Thus, $S(n)$ and $S(a) \times S(b)$ correspond bijectively and $s(n) = s(a)s(b)$.

It just remains to determine $s(p^r)$ and $s(2^r)$ where p is an odd prime and r is a positive integer. If $x^2 - 1 \equiv 0 \pmod{p^r}$, then p divides $x - 1$ or $x + 1$ but not both, or else it would divide $(x+1) - (x-1) = 2$. It follows that either $x - 1 \equiv 0 \pmod{p^r}$ or $x + 1 \equiv 0 \pmod{p^r}$. As a result, we have $S(p^r) = \{1, p^r - 1\}$ and $s(p^r) = 2$.

We readily verify that $s(2) = 1$, $s(2^2) = 2$. If $r \geq 3$, then 2 is the highest power of 2 that can possibly divide both $x - 1$ and $x + 1$ whenever $x^2 - 1 \equiv 0 \pmod{2^r}$. Hence, in addition to the obvious solutions 1, $2^r - 1$ to the latter, we also have $2^{r-1} + 1$ and $2^{r-1} - 1$, corresponding to the cases $2^{r-1} | (x - 1)$, $2 | (x + 1)$ and $2 | (x - 1)$, $2^{r-1} | (x + 1)$, respectively. Thus, $s(2^r) = 4$ whenever $r \geq 3$. The result follows.

Now we turn to the 2006 Vietnamese Mathematical Olympiad given at [2008 : 413–414].

1. Find all real solutions of the system of equations

$$\begin{aligned} \sqrt{x^2 - 2x + 6} \cdot \log_3(6 - y) &= x\,, \\ \sqrt{y^2 - 2y + 6} \cdot \log_3(6 - z) &= y\,, \\ \sqrt{z^2 - 2z + 6} \cdot \log_3(6 - x) &= z\,. \end{aligned}$$

Solved by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain ; and Titu Zvonaru, Comănești, Romania. We give the solution of Díaz-Barrero, modified by the editor.

First, we observe that x, y, $z \in (-\infty, 6)$. To find all real solutions we first rewrite the system in the form

$$\begin{aligned} y &= 6 - 3^{\frac{x}{\sqrt{x^2-2x+6}}}\,, \\ z &= 6 - 3^{\frac{y}{\sqrt{y^2-26+6}}}\,, \\ x &= 6 - 3^{\frac{z}{\sqrt{z^2-2z+6}}}\,. \end{aligned}$$

Let $f(t) = 6 - 3^{\frac{t}{\sqrt{t^2-2t+6}}}$ for $t \in (-\infty, 6)$. Since on the domain of f we have

$$\frac{d}{dt} \frac{t}{\sqrt{t^2 - 2t + 6}} = \frac{(6 - t)}{\left(t^2 - 2t + 6\right)^{3/2}} > 0\,,$$

then f is a strictly decreasing function, whence $f(f(f(t)))$ is also a decreasing function for $t \in (-\infty, 6)$. However, $f(x) = y$, $f(y) = z$, $f(z) = x$; and thus $f(f(f(x))) = x$. We claim that if $f(f(f(x))) = x$, then $f(x) = x$. Indeed, let f^n denote the n-fold composition of f, and note that both $f^3(x) = x$ and $f(x) < x$ yield $f(x) = f(f^3(x)) = f^3(f(x)) > f^3(x) = x$, a contradiction. The case $f(x) > x$ similarly leads to a contradiction, and our claim is established.

Now, $6 - x$ is a decreasing function and $3^{\frac{x}{\sqrt{x^2-2x+6}}}$ is increasing for $x \in (-\infty, 6)$. Hence, $f(x) = x$ has at most one root. Since $f(3) = 3$, it follows that $(x, y, z) = (3, 3, 3)$ is the unique real solution of the system.

2. Let $ABCD$ be a given convex quadrilateral. A point M moves on the line AB but does not coincide with A or B. Let N be the second point of intersection (distinct from M) of the circles (MAC) and (MBD), where (XYZ) denotes the circle passing through the points X, Y, Z. Prove that

(a) N moves on a fixed circle,

(b) the line MN passes through a fixed point.

Solution to part (a) by Michel Bataille, Rouen, France. Solution to part (b) by J. Chris Fisher, University of Regina, Regina, SK.

(a) We denote by $\angle(TU, VW)$ the directed angle of the lines TU and VW. Let the diagonals AC and BD intersect at O. The point N moves on the circle (CDO) as it immediately follows from the characterization of concyclicity in terms of angles and the following calculation :

$$\begin{aligned}
& \angle(NC, ND) \\
= \; & \angle(NC, NM) + \angle(NM, ND) \\
= \; & \angle(AC, AM) + \angle(BM, BD) \\
= \; & \angle(AC, BD) \\
= \; & \angle(OC, OD)\,.
\end{aligned}$$

(b) Because of the use of directed angles, convexity was not required in part (a). Likewise in part (b), A, B, C, and D can be any four points in the plane, no three collinear. Let Γ be the circle (CDO) from part (a). For any two positions of M on AB, say M_1 and M_2, we know that the circles (AMC) meet Γ at the corresponding points N, say N_1 and N_2. It suffices to prove that the lines M_1N_1 and M_2N_2 intersect at a point of Γ. To this end, we define P to be the point where these two lines intersect, and we apply Miquel's theorem to triangle M_1M_2P and the points A on M_1M_2, N_1 on M_1P, and N_2 on M_2P. By construction the circles (AM_1N_1) and (AM_2N_2) meet at C. By Miquel's theorem the circle (PN_1N_2) also passes through C. Since $\Gamma = (N_1N_2C)$, we conclude that P lies on Γ, as claimed.

4. Consider the function

$$f(x) \;=\; -x + \sqrt{(x+a)(x+b)}$$

where a and b are distinct positive real numbers. Prove that for every real number s in the interval $(0,1)$, there exists a unique positive real number α such that

$$f(\alpha) \;=\; \left(\frac{a^s + b^s}{2}\right)^{1/s}\,.$$

Solution by Michel Bataille, Rouen, France.

Let v, w lie in $(0,\infty)$ with $w > v$. Then

$$\begin{aligned} f(w) - f(v) &= \sqrt{(w+a)(w+b)} - \sqrt{(v+a)(v+b)} - (w-v) \\ &= \frac{(w+a)(w+b) - (v+a)(v+b)}{\sqrt{(w+a)(w+b)} + \sqrt{(v+a)(v+b)}} - (w-v) \\ &= (w-v)\left(\frac{N}{D} - 1\right) . \end{aligned}$$

where $N = v + w + a + b$ and $D = \sqrt{(w+a)(w+b)} + \sqrt{(v+a)(v+b)}$. By the AM–GM Inequality,

$$D < \frac{2w+a+b}{2} + \frac{2v+a+b}{2} = N ,$$

hence $f(w) > f(v)$ and f is strictly increasing on $(0,\infty)$. Also, f is continuous on $(0,\infty)$ and

$$\lim_{x\to 0+} f(x) = \sqrt{ab} , \qquad \lim_{x\to\infty} f(x) = \frac{a+b}{2} ,$$

the latter holding because

$$f(x) = \frac{(x+a)(x+b) - x^2}{\sqrt{(x+a)(x+b)} + x} = \frac{a + b + (ab/x)}{1 + \sqrt{1 + (a+b)/x + (ab)/x^2}} .$$

Thus, f is a bijection from $(0,\infty)$ onto (m_0, m_1), where $m_0 = \sqrt{ab}$ and $m_1 = \frac{a+b}{2}$. The s-mean of a and b is $m_s = \left(\frac{a^s + b^s}{2}\right)^{1/s}$, and we know from the Power Mean Inequality that $m_0 < m_s < m_1$ whenever $0 < s < 1$. Since f is bijective, $m_s = f(\alpha)$ for some unique $\alpha \in (0,\infty)$.

5. Find all polynomials $P(x)$ with real coefficients satisfying

$$P(x^2) + x\big(3P(x) + P(-x)\big) = P(x)^2 + 2x^2 ,$$

for all real numbers x.

Solution by Titu Zvonaru, Comănești, Romania.

First we will prove a lemma.

Lemma Let f be a polynomial that satisfies $f(x^2) = f(x)^2$ for all x. Then either $f = 0$ or $f(x) = x^m$.

Proof : We will prove by induction that $f\big(x^{2^p}\big) = f(x)^{2^p}$. For $p = 1$ the identity holds by hypothesis. Assume the identity holds for $p - 1 \geq 1$, then

$$f\big(x^{2^p}\big) = f\Big(\big(x^{2^{p-1}}\big)^2\Big) = f\big(x^{2^{p-1}}\big)^2 = \big(f(x)^{2^{p-1}}\big)^2 = f(x)^{2^p} ,$$

which completes the induction.

Now we choose a natural number p such that $g = 2^p > \deg f$. Let x_0 be any complex root of f and let $z_1, z_2, \ldots, z_g$ be the (distinct) complex roots of $z^g = x_0$. We then have

$$\big(f(z_i)\big)^g = f(z_i^g) = f(x_0) = 0 ,$$

so that $f(z_k)$ vanishes for each z_k. We deduce that the polynomial f has $g > \deg f$ distinct roots, or $x_0 = 0$. Therefore, $f = 0$ or $f(x) = ax^m$ for some number a. In the latter case $f(x^2) = f(x)^2$ becomes $ax^{2m} = a^2x^{2m}$, hence $f \equiv 0$ or $f(x) = x^m$. ■

The given equation, for x and $-x$, is

$$\begin{aligned} P(x^2) + x\big(3P(x) + P(-x)\big) &= P(x)^2 + 2x^2 , \\ P(x^2) - x\big(3P(-x) + P(x)\big) &= P(-x)^2 + 2x^2 . \end{aligned}$$

Subtracting, we obtain

$$4x\big(P(x) + P(-x)\big) = \big(P(x) - P(-x)\big)\big(P(x) + P(-x)\big) ,$$

which leads to two cases.

Case 1. $P(x) + P(-x) = 0$. Here the following are equivalent :

$$\begin{aligned} P(x^2) + 2xP(x) &= P(x)^2 + 2x^2 , \\ P(x^2) - x^2 &= P(x)^2 - 2xP(x) + x^2 , \\ P(x^2) - x^2 &= \big(P(x) - x\big)^2 . \end{aligned}$$

Thus, $f(x) = P(x) - x$ satisfies $f(x^2) = f(x)^2$, and by the Lemma we have either $f \equiv 0$ and $P(x) = x$, or $f(x) = x^m$ and $P(x) = x^m + x$.

In the latter case, since $P(x) + P(-x) = 0$, we have

$$x^m + x + (-x)^m - x = x^m + (-x)^m = 0 ,$$

that is, m is odd.

We conclude that $P(x) = x$ or $P(x) = x^{2n+1} + x$, with n a nonnegative integer.

Case 2. $P(x) - P(-x) = 4x$. Here the following are equivalent :

$$\begin{aligned} P(x^2) + x\big(3P(x) + P(x) - 4x\big) &= P(x)^2 + 2x^2 , \\ P(x^2) - 2x^2 &= P(x)^2 - 4xP(x) + 4x^2 , \\ P(x^2) - 2x^2 &= \big(P(x) - 2x\big)^2 . \end{aligned}$$

Thus, $f(x) = P(x) - 2x$ satisfies $f(x^2) = f(x)^2$, and by similar reasoning as in Case 1, we deduce that either $P(x) = 2x$ or $P(x) = x^m + 2x$.

In the latter case, since $P(x) - P(-x) = 4x$, we have

$$x^m + 2x - (-x)^m + 2x \;=\; 4x\,,$$

or $x^m - (-x)^m = 0$, that is, m is even.

So, in this case, we obtain the polynomials $P(x) = 2x$ and $P(x) = x^{2n} + 2x$ with n a nonnegative integer.

For $n = 0$ we have $x^{2n+1} + x = 2x$, so a complete list of polynomials $P(x)$ is x, $x^{2n+1} + x$, and $x^{2n} + 2x$, where n is a nonnegative integer.

6. A set of integers T is called *sum-free* if for every two (not necessarily distinct) elements u and v in T, their sum $u + v$ does not belong to T. Prove that

(a) a sum-free subset of $S = \{1, 2, \ldots, 2006\}$ has at most 1003 elements,

(b) any set S consisting of 2006 positive integers has a sum-free subset consisting of 669 elements.

Solution by Oliver Geupel, Brühl, NRW, Germany.

For part (a), we claim that a sum free subset T of $S = \{1, 2, \ldots, n\}$ has at most $\left\lceil\frac{n}{2}\right\rceil$ elements. To prove this fact, let $m = \max S$. If m is even then each of the sets $\{1, m-1\}$, $\{2, m-2\}$, $\ldots$, $\left\{\frac{m-2}{2}, \frac{m+2}{2}\right\}$ contains at most one element of T; hence $|T| \le \frac{m}{2} \le \frac{n}{2}$. Now suppose that m is odd. Then each of the sets $\{1, m-1\}$, $\{2, m-2\}$, $\ldots$, $\left\{\frac{m-1}{2}, \frac{m+1}{2}\right\}$ contains at most one element of S; thus $|S| \le \frac{m+1}{2}$. If n is even, then $\frac{m+1}{2} \le \frac{n}{2}$, while if n is odd, then $\frac{m+1}{2} \le \frac{n+1}{2} = \left\lceil\frac{n}{2}\right\rceil$. This completes the proof of part (a).

For part (b), we claim that any set S consisting of n positive integers has a sum free subset consisting of $\left\lceil\frac{n}{3}\right\rceil$ elements.

To prove this, choose a prime number $p > \max S$ such that $3 \mid (p+1)$, which is possible by Dirichlet's theorem. Let

$$I \;=\; \left\{k \in \mathbb{N} :\; \frac{p+1}{3} \;\le\; k \;\le\; \frac{2p-1}{3}\right\}.$$

Consider the bipartite graph with sets of nodes $P = \{1, 2, \ldots, p-1\}$ and I, where a node $k \in P$ is adjacent to a node $l \in I$ if and only if there exists a number $a \in S$ such that $ka \equiv l \pmod{p}$. Note that for each edge, the number $a \in S$ is unique, and label the respective edge with the number a. For each $a \in S$ there are exactly $|I| = \frac{p+1}{3}$ numbers $k \in P$ such that ka has a residue modulo p which belongs to I, hence our graph has $\frac{n(p+1)}{3}$ edges. By the Pigeonhole Principle, there is a node $k \in P$ with degree not less than $\frac{n(p+1)}{3(p-1)}$, hence not less than $\left\lceil \frac{n}{3} \right\rceil$. Let $T \subseteq S$ be the set of labels of its respective edges.

We claim that T has the required properties. Clearly, $|T| \geq \left\lceil \frac{n}{3} \right\rceil$. Assume that $a+b=c$ for some a, b, $c \in T$. Then ka, kb, kc are each congruent modulo p to numbers in I. Moreover, $ka + kb = kc$, so that $ka + kb$ is congruent modulo p to a number in I. On the other hand, it is easy to check that for all l, $l' \in I$, the residue modulo p of the number $l + l'$ does not belong to I. This contradicts the hyothesis $a + b = c$, and completes our proof.

Next we move to solutions from our readers to problems proposed but not used for the 47$^{\text{th}}$ International Mathematical Olympiad 2006 in Slovenia given at [2008 : 459–464].

A1. Given an arbitrary real number a_0, define a sequence of real numbers a_0, a_1, a_2, ... by the recursion

$$a_{i+1} = \lfloor a_i \rfloor \cdot \{a_i\}\,, \qquad i \geq 0\,,$$

where $\lfloor a_i \rfloor$ is the greatest integer not exceeding a_i, and $\{a_i\} = a_i - \lfloor a_i \rfloor$. Prove that $a_i = a_{i+2}$ for sufficiently large i.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We consider the three cases $a_0 = 0$, $a_0 > 0$, and $a_0 < 0$ separately.

If $a_0 = 0$, then the sequence (a_i) is constant.

Next, suppose $a_0 > 0$. Then all a_i are nonnegative. Let $\lfloor a_i \rfloor = n$ and $\{a_i\} = r$. If $n = 0$, then $a_{i+1} = a_{i+2} = a_{i+3} = \cdots = 0$. Otherwise we have $n \geq 1$ and $a_i - a_{i+1} = (n + r) - nr = (n - 1)(1 - r) + 1 \geq 1$; hence we obtain $0 \leq a_{i+n} < 1$, which reduces to the previous case.

Lastly, suppose $a_0 < 0$. Then $a_0 \leq a_1 \leq a_2 \leq \cdots \leq 0$. Hence, the integer sequence $(\lfloor a_i \rfloor)$ is non-decreasing and bounded above by 0. If just one term of this sequence is 0, then (a_i) terminates in zeros and we are done. The other possibility is that there exist positive integers i_0 and n such that for all $i \geq i_0$ we have $\lfloor a_i \rfloor = -n$. Let $\{a_{i_0}\} = r$. We will prove by induction that for each nonnegative integer k,

$$a_{i_0+k} = -\left(\frac{n^2 + (n - nr - r)(-n)^k}{n+1}\right)\,. \tag{1}$$

The equation (1) is immediate if $k = 0$. For the inductive step, assuming (1), we obtain

$$\begin{aligned} a_{i_0+k+1} &= -n(a_{i_0+k)}+n) \\ &= -n^2+n\left(\frac{n^2+(n-nr-r)(-n)^k}{n+1}\right) \\ &= -\left(\frac{n^2+(n-nr-r)(-n)^{k+1}}{n+1}\right), \end{aligned}$$

thus completing the proof of (1) by induction.

If $n > 1$, then we obtain from (1) that $\lim\limits_{k\to\infty} |a_{i_0+k}| = \infty$. But this contradicts our hypothesis that $\lfloor a_i \rfloor = -n$ for $i \geq i_0$. Consequently, $n = 1$. Then $a_{i_0} = -1+r$, $a_{i_0+1} = -r = -1+(1-r)$, and $a_{i_0+2} = -(1-r) = a_{i_0}$, which completes the proof.

A4. Let a_1, a_2, $\dots$, a_n be positive real numbers. Prove that

$$\sum_{i<j} \frac{a_i a_j}{a_i+a_j} \leq \frac{n}{2(a_1+a_2+\cdots+a_n)} \sum_{i<j} a_i a_j .$$

Solution by Michel Bataille, Rouen, France.

Equality holds for $n = 2$, so we will suppose that $n \geq 3$. We claim that if a, b, c are positive real numbers, then

$$\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \leq \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) . \qquad (1)$$

Proof : Observe that $\frac{2}{x+y}$ is the harmonic mean of $\frac{1}{x}$ and $\frac{1}{y}$ (for positive x, y). Using the AM–HM Inequality, it follows that

$$\frac{2}{b+c} + \frac{2}{c+a} + \frac{2}{a+b} \leq \frac{\frac{1}{b}+\frac{1}{c}}{2} + \frac{\frac{1}{c}+\frac{1}{a}}{2} + \frac{\frac{1}{a}+\frac{1}{b}}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} .$$

Back to the problem. The proposed inequality is equivalent to

$$(a_1+a_2+\cdots+a_n) \sum_{i<j} \frac{a_i a_j}{a_i+a_j} \leq \frac{n}{2} \sum_{i<j} a_i a_j , \qquad (2)$$

whose left-hand side is $\sum\limits_{i<j} a_i a_j + \sum\limits_{i<j} \frac{a_i a_j}{a_i+a_j} \left(\sum\limits_{k\neq i,j} a_k\right) = \sum\limits_{i<j} a_i a_j + L$.

Thus, (2) is equivalent to

$$L \leq \frac{n-2}{2} \sum_{i<j} a_i a_j . \qquad (3)$$

Note that $L = \sum\limits_{i,j,k \le n} a_i a_j a_k \left(\frac{1}{a_i + a_j} + \frac{1}{a_j + a_k} + \frac{1}{a_k + a_i} \right)$, the sum taken over distinct positive integers i, j, k. By (1) above,

$$\begin{aligned} L &\le \sum_{1 \le i < j < k \le n} \frac{a_i a_j a_k}{2} \left(\frac{1}{a_i} + \frac{1}{a_j} + \frac{1}{a_k} \right) \\ &= \sum_{1 \le i < j < k \le n} \left(\frac{a_i a_j}{2} + \frac{a_j a_k}{2} + \frac{a_k a_i}{2} \right) = \sum_{1 \le r < s \le n} (n-2) \frac{a_r a_s}{2}, \end{aligned}$$

and (3) is obtained, completing the proof.

A5. Let a, b, and c be the lengths of the sides of a triangle. Prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \le 3.$$

Solution by Titu Zvonaru, Comăneşti, Romania.

Let $x = \sqrt{a}$, $y = \sqrt{b}$, $z = \sqrt{c}$.

By the well-known inequality $(\alpha + \beta + \gamma)^2 \le 3\big(\alpha^2 + \beta^2 + \gamma^2\big)$ we obtain :

$$\left(\sum_{\text{cyclic}} \frac{\sqrt{-x^2 + y^2 + z^2}}{-x + y + z} \right)^2 \le 3 \sum_{\text{cyclic}} \frac{-x^2 + y^2 + z^2}{(-x+y+z)^2}.$$

The following inequalities are then equivalent

$$\sum_{\text{cyclic}} \frac{-x^2 + y^2 + z^2}{(-x+y+z)^2} \le 3 \iff \sum_{\text{cyclic}} \left(1 - \frac{-x^2 + y^2 + z^2}{(-x+y+z)^2} \right) \ge 0$$

$$\iff \sum_{\text{cyclic}} \frac{2x^2 - 2xy - 2xz + 2yz}{(-x+y+z)^2} \ge 0 \iff \sum_{\text{cyclic}} \frac{(x-y)(x-z)}{(-x+y+z)^2} \ge 0.$$

To prove the last inequality above, we may assume that $x \ge z$, $y \ge z$. After some algebra, we have :

$$\begin{aligned} &\frac{(x-y)(x-z)}{(-x+y+z)^2} + \frac{(y-z)(y-x)}{(x-y+z)^2} \\ &= (x-y) \cdot \frac{(x-z)(x-y+z)^2 - (y-z)(-x+y+z)^2}{(-x+y+z)^2(x-y+z)^2} \\ &= (x-y)^2 \cdot \frac{(x-y)^2 + 2z(x+y-2z) + z^2}{(-x+y+z)^2(x-y+z)^2} \ge 0, \end{aligned}$$

because $x + y \geq 2z$. Also, $\frac{(z-x)(z-y)}{(-z+x+y)^2} \geq 0$, since $x \geq z$ and $y \geq z$. Therefore, the last of the preceding equivalent inequalities is true, hence the original inequality is true.

Equality holds if and only if $x = y = z$, that is $a = b = c$.

C1. There are $n \geq 2$ lamps L_1, L_2, ..., L_n arranged in a row. Each of them is either *on* or *off*. Initially the lamp L_1 is on and all of the other lamps are off. Each second the state of each lamp changes as follows : if the lamp L_i and its neighbours (L_1 and L_n each have one neighbor, any other lamp has two neighbours) are in the same state, then L_i is switched off ; otherwise, L_i is switched on. Prove that there are

(a) infinitely many n for which all of the lamps will eventually be off,

(b) infinitely many n for which the lamps will never be all off.

Solved by Oliver Geupel, Brühl, NRW, Germany.

First, we prove by mathematical induction that, if $n = 2^k$ where k is a positive integer, all lamps are on after $n - 1$ steps, while not all lamps have equal states after each of the steps 1, 2, ..., $n - 2$.

This is clear for $k = 1$. Let $n = 2^k$ and assume the leftmost $m = \frac{n}{2}$ lamps are on after $m - 1$ steps. This is valid, as the m rightmost lamps do not change during the first $m - 1$ steps. After step m the two middle lamps are on and the other lamps are off. Afterwards, the on/off-pattern is symmetric about the centre, and from then on the two middle lamps will have the same state. Hence, the state of L_m is only affected by L_{m-1}, so the state of $(L_m, L_{m-1}, \ldots, L_1)$ after step $m+j$ coincides with that of $(L_1, L_2, \ldots, L_m)$ after step j. Hence, after $n-1$ steps the m leftmost lamps are again on, and by symmetry so are the m rightmost lamps. Moreover, not all the lamps are in the same state before then. The induction is complete.

Thus, for $n = 2^k$, all lamps are off after n steps, completing part (a).

For part (b), we claim that the lamps will never all be in the same state if $n = 2^k + 1$. To see this, note that after $n - 2$ steps the lamps L_1, L_2, ..., L_{n-1} are on and L_n is off. Moreover, by the analysis for part (a), the lamps are never all in the same state before then. After step $n - 1$, the lamps L_{n-1} and L_n are on and the rest are off. At this moment, the state of $(L_n, L_{n-1}, \ldots, L_1)$ coincides with that of $(L_1, L_2, \ldots, L_n)$ after the first step. Therefore, the state of $(L_1, L_2, \ldots, L_n)$ after step $j \geq 1$ is the same as that of $(L_n, L_{n-1}, \ldots, L_1)$ after step $n - 2 + j$. Hence, the sequence of states is eventually periodic with minimal period $2(n-2)$, and the lamps are never all in the same state.

C6. Let $\mathcal{P}$ be a convex polyhedron with no parallel edges and no edge parallel to a face other than the two faces it borders. A pair of points on $\mathcal{P}$ are *antipodal* if there exist two parallel planes each containing one of the points and such that $\mathcal{P}$ lies between them. Let A be the number of antipodal pairs of vertices and let B be the number of antipodal pairs of midpoints of edges. Express $A - B$ in terms of the numbers of vertices, edges, and faces of $\mathcal{P}$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let F_1, F_2, ..., F_f be the faces of $\mathcal{P}$ and n_1, n_2, ... n_f the respective outward normal unit vectors. Consider a graph $\mathcal{G}$ on the unit sphere $\mathcal{S}$ centred at the origin O. The vertices of $\mathcal{G}$ are the end points V_1, V_2, ..., V_f of the position vectors n_1, n_2, ..., n_f. An edge V_iV_j is drawn as an arc on a great circle of $\mathcal{S}$ if and only if the faces F_i and F_j of $\mathcal{P}$ have a common edge. The graph $\mathcal{G}$ is dual to $\mathcal{P}$, in that each vertex V, each edge E, and each face F of $\mathcal{P}$ corresponds to a unique face $d(V)$, edge $d(E)$, and vertex $d(F)$, respectively, of $\mathcal{G}$. Let $\mathcal{P}$ have f faces, e edges, and v vertices, then $\mathcal{G}$ has v faces, e edges, and f vertices.

Let σ be the reflection of $\mathcal{S}$ with respect to the point O. Then $\mathcal{G}$ is mapped to another graph $\sigma(\mathcal{G}) = \mathcal{G}'$. Finally we merge $\mathcal{G}$ and $\mathcal{G}'$ into a new graph, $\tilde{\mathcal{G}}$. The vertices of $\tilde{\mathcal{G}}$ are the vertices of $\mathcal{G} \cup \mathcal{G}'$ and the points of intersection of edges of $\mathcal{G}$ with edges of $\mathcal{G}'$. The edges of $\tilde{\mathcal{G}}$ are all segments of edges of $\mathcal{G} \cup \mathcal{G}'$.

Consider the planes π that contain an edge E of $\mathcal{P}$ bordering faces F_i and F_j. The outward normal unit vectors of these planes π all lie on the great circle of $\mathcal{S}$ containing $d(E)$ in $\mathcal{G}$. The plane π does not intersect the interior of $\mathcal{P}$ if and only if its outward normal unit vector is on the edge (that is, arc) $d(E)$. Parallel edges of $\mathcal{P}$ correspond to arcs on the same great circle of $\mathcal{S}$. An edge E is parallel to a face F in $\mathcal{P}$ if and only if $d(E)$ and the vertex $d(F)$ are on the same great circle of $\mathcal{S}$. By hypothesis, all edges of $\tilde{\mathcal{G}}$ are on distinct great circles, and no vertex of $\tilde{\mathcal{G}}$ lies on the same great circle as a non-adjacent edge.

The edges E_i and E_j of $\mathcal{P}$ have antipodal midpoints if and only if there are planes π_i and π_j containing E_i and E_j, respectively, and with opposite normal vectors, that is, if $d(E_i)$ and $\sigma\big(d(E_j)\big)$ intersect on $\mathcal{S}$. Hence, $\tilde{\mathcal{G}}$ has a total of $\tilde{v} = 2f + 2B$ vertices. Each of the $2B$ vertices splits 2 edges ; hence $\tilde{\mathcal{G}}$ has $\tilde{e} = 2e + 4B$ vertices.

Consider a plane π containing a vertex V of $\mathcal{P}$ if and only if its outward normal unit vector is in the face of $\mathcal{G}$ on $\mathcal{S}$ bordered by the edges $d(E_1)$, $d(E_2)$, ..., $d(E_k)$. Thus, vertices V_i and V_j of $\mathcal{P}$ are antipodal if and only if the faces $d(V_i)$ and $\sigma\big(d(V_j)\big)$ are non-disjoint on $\tilde{\mathcal{G}}$. Therefore, the number of faces of $\tilde{\mathcal{G}}$ is $\tilde{f} = 2A$.

By Euler's polyhedral formula, we obtain

$$\begin{aligned} 0 &= \tilde{v} + \tilde{f} - \tilde{e} - 2 \;=\; 2f + 2B + 2A - 2e - 4B - 2 \\ &= 2(A - B) + 2(f - e - 1) \;=\; 2(A - B) + 2(1 - v)\,. \end{aligned}$$

Consequently, $A - B = v - 1$ and $A - B$ depends only on the number of vertices of $\mathcal{P}$.

That completes the *Corner* for this issue. Send me your nice solutions and generalizations.