SKOLIAD No. 122

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **1 July, 2010**. A copy of ***CRUX*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest for this number of the Skoliad is the 27th New Brunswick Mathematics Competition, 2009, Grade 9, Part C. We thank Daryl Tingley, Department of Mathematics and Statistics, University of New Brunswick, and Paul Deguire, Département de mathématiques et de statistique, Faculté des sciences, Université de Moncton, for providing us with this contest and for permission to publish it.

Complete justification is required in order for a solver to obtain credit for his/her solution.

27th New Brunswick Mathematics Competition, 2009 Grade 9, Part C

Approximately 30 minutes allowed

1. If you write all integers from 1 to 100, how many even digits will be written? (When you write the number 42, two even digits are written.)

(A) 50 (B) 71 (C) 80 (D) 89 (E) 91

2. In a farm there are hens (no hump, two legs), camels (two humps, four legs) and dromedaries (one hump, four legs). If the number of legs is four times the number of humps, then the number of hens divided by the number of camels will be?

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) Not enough information

3. A cubic box of side 1 m is placed on the floor. A second cubic box of side $\frac{2}{3}$ m is placed on top of the first box so that the centre of the second box is directly above the centre of the first box. A painter paints all of the surface area of the two boxes that can be reached without moving the boxes. What is the total area of surface that is painted?

(A) $\frac{49}{9}$ m^2 (B) $\frac{57}{9}$ m^2 (C) $\frac{61}{9}$ m^2 (D) $\frac{72}{9}$ m^2 (E) None of these

4. What is the ones digit of 2^{2009}?

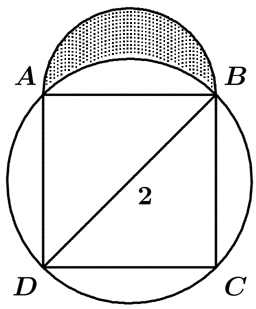
(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

5. The numbers 1, 2, 3, 4, 5, and 6 are to be arranged in a row. In how many ways can this be done if 2 is always to the left of 4, and 4 is always to the left of 6? (For example 2, 5, 3, 4, 6, 1 is an arrangement with 2 to the left of 4 and 4 to the left of 6.)

(A) 20 (B) 36 (C) 60 (D) 120 (E) 240

6. The square $ABCD$ is inscribed in a circle with diameter BD of length 2. If AB is the diameter of the semicircle on top of the square, what is the area of the shaded region?

(A) $\frac{4-\pi}{4}$ (B) $\frac{\pi-2}{4}$ (C) $\frac{1}{2}$
(D) 1 (E) Not enough information



27^{e} Concours de Mathématiques du Nouveau-Brunswick, 2009
9^{e} année, Partie C
Durée : environ 30 minutes

1. Si vous écrivez tous les entiers de 1 à 100, combien de chiffres pairs seront écrits ? (Quand vous écrivez le nombre 42, vous écrivez deux chiffres pairs.)

(A) 50 (B) 71 (C) 80 (D) 89 (E) 91

2. Dans une ferme il y a des poules (pas de bosse, deux pattes), des chameaux (deux bosses, quatre pattes) et des dromadaires (une bosse, quatre pattes). Si le nombre de pattes est quatre fois le nombre de bosses, alors le nombre de poules, divisé par le nombre de chameaux sera de ?

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) Information insuffisante

3. Une boîte cubique dont le côté mesure 1 m est placée sur le sol. Une seconde boîte cubique dont le côté mesure $\frac{2}{3}$ m est placée sur la première de manière à ce que son centre soit exactement au dessus du centre de la première boîte. Un peintre peint alors les surfaces des deux boîtes qu'il peut rejoindre sans bouger les boîtes. Quelle est la surface totale qui est peinte ?

(A) $\frac{49}{9}$ m^2 (B) $\frac{57}{9}$ m^2 (C) $\frac{61}{9}$ m^2 (D) $\frac{72}{9}$ m^2 (E) Aucune de ces réponses

4. Quel est le chiffre des unités de 2^{2009} ?

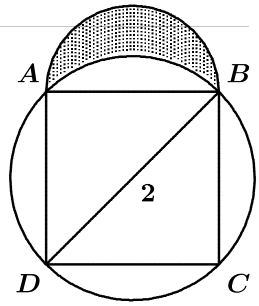
(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

5. Les nombres 1, 2, 3, 4, 5 et 6 sont placés en ligne. De combien de façons cela peut-il être fait si 2 est toujours à la gauche de 4 et 4 est toujours à la gauche de 6 ? (2, 5, 3, 4, 6, 1 est un tel placement avec 2 à la gauche de 4 et 4 à la gauche de 6).

(A) 20 (B) 36 (C) 60 (D) 120 (E) 240

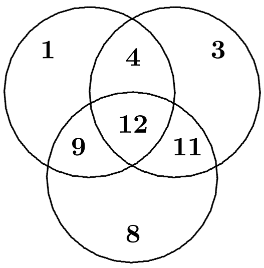
6. Le carré $ABCD$ est inscrit dans un cercle dont le diamètre BD est de longueur 2. Si AB est le diamètre du demi-cercle au-dessus du carré, quelle est l'aire de la région ombragée ?

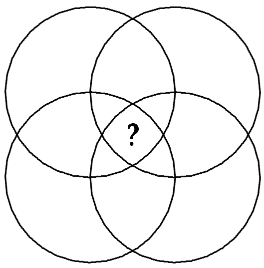
(A) $\frac{4-\pi}{4}$ (B) $\frac{\pi-2}{4}$ (C) $\frac{1}{2}$
(D) 1 (E) Information insuffisante



Next follow solutions to the Swedish Junior High School Mathematics Contest, Final Round, 2007/2008 [2009 : 129–131].

1. Values are assigned to a number of circles, and these values are written in the circles. When two or more circles overlap, the sum of the values of the overlapping circles is written in the common region. In the example on the left below, the three circles have been assigned the values 1, 3, and 8. Where the circle with value 1 overlaps the circle with value 3 we write 4 $(= 1 + 3)$. In the region in the middle, we add all three values and write 12.

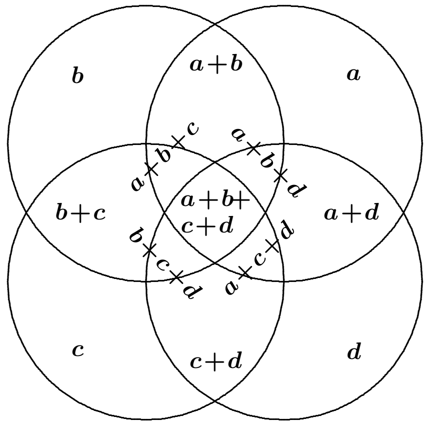




In the figure on the right above are four circles and, thus, thirteen regions. Find the number in the middle if the sum of all thirteen numbers is 294.

Solution by Alison Tam, student, Burnaby South Secondary School, Burnaby, BC.

Assign the values a, b, c, and d to the four circles, as shown in the diagram on the following page. Then the overlapping regions get the values shown. The sum of the thirteen regions is $7a + 7b + 7c + 7d$. Since this sum is required to be 294, it follows that the value in the middle region is $a+b+c+d=\frac{294}{7}=42$.



Also solved by CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; EMILY HUANG, student, Burnaby Central Secondary School, Burnaby, BC; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; LENA CHOI, student, École Banting Middle School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; and THOMAS HSU, student, Moscrop Secondary School, Burnaby, BC.

2. This is the 20th edition of the Swedish Junior High School Mathematics Contest. The first qualification round was held in the fall of 1988, and this year's final is held in2008. That is twenty-one calendar years, 1988–2008, but the table at right has room for only eighteen of them. Which three must be omitted if the digit sum in every row and every column must be divisible by 9? (Two solutions exist.)

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

The digits in the first columns of each of the two parts of the chart are 1's and 2's. Say one such column has k copies of 1 and $9-k$ copies of 2. Then the sum is $k+2(9-k)=18-k$. If this sum is a multiple of 9, then k is a multiple of 9, so either $k=0$ or $k=9$. Thus, one first column contains only 1's while the other first column contains only 2's. Since only nine years begin with 2, all of them must be used and they must all be in the same part of the chart. See the left-hand diagram below.

1	9				2	0	0	0
1	9				2	0	0	1
1	9				2	0	0	2
1	9				2	0	0	3
1	9				2	0	0	4
1	9				2	0	0	5
1	9				2	0	0	6
1	9				2	0	0	7
1	9				2	0	0	8

1	9	9	6		2	0	0	0
1	9	9	5		2	0	0	1
1	9	9	4		2	0	0	2
1	9	9	3		2	0	0	3
1	9	9	2		2	0	0	4
1	9	9	1		2	0	0	5
1	9	9	x		2	0	0	6
1	9	9	8		2	0	0	7
1	9	9	7		2	0	0	8

Here x is either 0 or 9

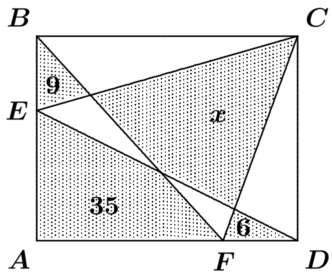
The third column in the 1900's part of the chart contains 8's and 9's.

Since there are at most two 8's and the column sum is divisible by 9, the column must consist of only 9's. Thus the years 1988 and 1989 must be excluded from the chart. Using that the row sums are divisible by 9, it is now easy to fill in the chart as in the right-hand diagram.

Hence the excluded years are either 1988, 1989, and 1990; or 1988, 1989 and 1999.

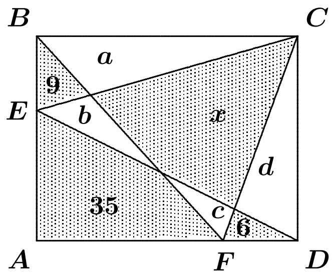
Also solved by LENA CHOI, student, École Banting Middle School, Coquitlam, BC.

3. The line segments DE, CE, BF, and CF divide the rectangle $ABCD$ into a number of smaller regions. Four of these, two triangles and two quadrilaterals, are shaded in the figure at right. The areas of the four shaded regions are 9, 35, 6, and x (see the figure). Determine the value of x.



Solution by Lena Choi, student, École Banting Middle School, Coquitlam, BC.

Since the base of $\triangle BCF$ is $|BC|$ and its height is $|CD|$, the area of $\triangle BCF$ is half the area of rectangle $ABCD$. But then $\triangle ABF$ and $\triangle CDF$ take up the other half of the rectangle. Similarly, the area of $\triangle CDE$ equals half the area of the rectangle as does the combined area of $\triangle BCE$ and $\triangle ADE$.



Assign letters to the areas of the four unshaded regions as in the figure. Then the paragraph above amounts to saying that $a+x+c$, $9+b+35+d+6$, $b+x+d$, and $9+a+35+c+6$ are all equal. In particular, $a+x+c = 9+a+35+c+6$, so $x = 9+35+6 = 50$.

Also solved by ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia.

4. A goody bag contains a two-digit number of goodies. Lisa adds the two digits and then removes as many goodies as the sum yields. Lisa repeats this procedure until the number of goodies left is a single digit number larger than zero. Find this single digit number.

Solution by the editors.

Say the number of goodies is $10a+b$. Then Lisa removes $a+b$ goodies and is left with $9a$ goodies. It follows that once Lisa has removed goodies at least once, then the number of goodies left is divisible by nine. Thus, when the number of goodies left reaches a single-digit number, that number must be nine.

Several solvers found that Lisa is left with nine goodies with several choices of the initial number of goodies. However, that does not prove that Lisa ***always*** *ends up with nine goodies.*

5. In how many ways can the list $[1, 2, 3, 4, 5, 6]$ be permuted if the product of neighbouring numbers must always be even?

Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.

If the product of neighbouring numbers is even, then the parity of the numbers must follow one of the patterns

$$eoeoeo\,, \quad oeoeoe\,, \quad oeoeeo\,, \quad \text{or} \quad oeeoeo\,,$$

where e is an even number and o is an odd number. Once you have chosen one of the four patterns, you can arrange the three even numbers into the even slots in six ways, and you can arrange the odd numbers in six ways. Thus the total number of permutations is $4 \cdot 6 \cdot 6 = 144$.

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

6. The digits of a five-digit number are $abcde$. Prove that $abcde$ is divisible by 7 if and only if the number $abcd - 2 \cdot e$ is divisible by 7.

Solution by the editors.

Let A and B denote the two numbers $abcde$ and $abcd - 2e$, respectively. Let x denote the four-digit number $abcd$. Then $A = 10x + e$ and $B = x - 2e$. Therefore,

$$2A + B \;=\; 20x + 2e + (x - 2e) \;=\; 21x\,.$$

If A is divisible by 7, then $A = 7m$ for some integer m, so

$$B \;=\; 21x - 2A \;=\; 21x - 14m \;=\; 7(3x - 2m)\,,$$

which is divisible by 7.

If B is divisible by 7, then $B = 7n$ for some integer n, so

$$2A \;=\; 21x - B \;=\; 21x - 7n \;=\; 7(3x - n)\,,$$

which is divisible by 7. But if $2A$ is divisible by 7, then so is A.

Thus, A is divisible by 7 if and only if B is divisible by 7.

This issue's prize of one copy of ***CRUX with MAYHEM*** for the best solutions goes to Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON. We would very much appreciate receiving more solutions from our readers. Solutions to just some of the problems are also very welcome.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON) and Eric Robert (Leo Hayes High School, Fredericton, NB).

Mayhem Problems

*Please send your solutions to the problems in this edition by **1 May 2010**. Solutions received after this date will only be considered if there is time before publication of the solutions. The Mayhem Staff ask that each solution be submitted on a separate page and that the solver's name and contact information be included with each solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M420. *Proposed by the Mayhem Staff.*

Riley is a poor starving university student, but is mathematically astute. He notices that five suppers in residence cost the same as seven lunches. After one week of skipping supper most nights, he notices that five lunches and one supper cost $\$48$ in total. How much do 16 suppers cost?

M421. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Let $\lfloor x \rfloor$ be the greatest integer less than or equal to the real number x. Determine all real numbers x such that

$$\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{3}{x} \right\rfloor = 4 \,.$$

M422. *Proposed by Adnan Arapovic, student, University of Waterloo, Waterloo, ON.*

Prove that

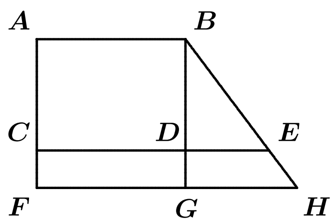
$$\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6} \,.$$

M423. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The tens digit of a perfect square S is three greater than the ones digit of S. Determine all possible remainders when S is divided by 100.

M424. *Proposed by Margo Kondratieva, Memorial University of Newfoundland, St. John's, NL.*

In the diagram, line segments AB, CDE, and FGH are parallel. Also, line segments ACF and BDG are perpendicular to AB. Suppose that the area of rectangle $ABDC$ is x, the area of rectangle $CDGF$ is y, and the area of $\triangle BDE$ is z. Determine the area of $DEHG$ in terms of x, y, and z.



M425. *Proposed by Titu Zvonaru, Comănești, Romania.*

In $\triangle ABC$, $\angle BAC = 90°$ and I is the incentre. The interior bisector of angle C meets AB at D. The line segment through D perpendicular to BI intersects BC at E. The line segment through D parallel to BI meets AC at F. Prove that E, I, and F are collinear.

. . . .

M420. *Proposé par l'Équipe de Mayhem.*

Richard est un étudiant pauvre et affamé, mais mathématiquement doué. Il a remarqué qu'à la résidence, cinq soupers coûtent le même prix que sept lunchs. Après avoir sauté les soupers presque tous les soirs pendant une semaine, il constate que cinq lunchs et un souper coûtent 48 au total. Combien coûtent 16 soupers ?

M421. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Soit $\lfloor x \rfloor$ le plus grand entier plus petit ou égal au nombre réel x. Trouver tous les nombres réels tels que

$$\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{3}{x} \right\rfloor = 4 \,.$$

M422. *Proposé par Adnan Arapovic, étudiant, Université de Waterloo, Waterloo, ON.*

Montrer que

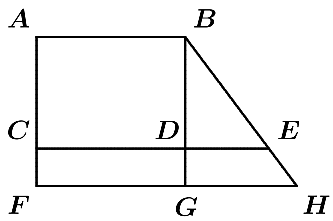
$$\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6} \,.$$

M423. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

La différence entre le chiffre des dizaines et celui des unités d'un carré parfait S est de trois. Trouver tous les restes de la division de S par 100.

M424. *Proposé par Margo Kondratieva, Université Memorial de Terre-Neuve, St. John's, NL.*

Dans la figure ci-contre, les segments de droite AB, CDE, et FGH sont parallèles. De plus, les segments ACF et BDG sont perpendiculaires à AB. Supposons que les aires respectives des rectangles $ABDC$, $CDGF$, et $\triangle BDE$ sont x, y et z. Trouver l'aire de $DEHG$ en fonction de x, y et z.



M425. *Proposé par Titu Zvonaru, Comănești, Roumanie.*

Dans le triangle ABC, $\angle BAC = 90°$ et soit I le centre du cercle inscrit. La bissectrice intérieure de l'angle C coupe AB en D. La droite passant par D et perpendiculaire à BI coupe BC en E. La droite passant par D et parallèle à BI coupe AC en F. Montrer que E, I et F sont colinéaires.

Mayhem Solutions

Last year we received some late solutions that did not appear in the December issue. We therefore acknowledge a correct solution to M383 by Mridul Singh, student, Kendriya Vidyalaya School, Shillong, India, and correct solutions to problems M383, M384, and M386 by Hugo Luyo Sánchez, Pontificia Universidad Católica del Peru, Lima, Peru.

M388. *Proposed by Kyle Sampson, St. John's, NL.*

A sequence is generated by listing (from smallest to largest) for each positive integer n the multiples of n up to and including n^2. Thus, the sequence begins 1, 2, 4, 3, 6, 9, 4, 8, 12, 16, 5, 10, 15, 20, 25, 6, 12, $\ldots$. Determine the 2009^{th} term in the sequence.

Solution by Kristóf Huszár, Valéria Koch Grammar School, Pécs, Hungary.

First, we notice that there are k positive integral multiples of k less than or equal to k^2. If we group the terms of the sequence as the multiples of 1, then the multiples of 2, then the multiples 3, and so on, we notice that the groups have 1 term, then 2 terms, then 3 terms, and so on.

If n is a positive integer, then the sum of the first n positive integers is equal to $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. Therefore, n^2 is the $\left(\frac{n(n+1)}{2}\right)^{\text{th}}$ term of the sequence.

Hence, $63^2 = 3969$ is the $\left(\frac{63 \cdot 64}{2}\right)^{\text{th}} = 2016^{\text{th}}$ term. Since 3969 occurs $2016 - 2009 = 7$ terms after the 2009^{th} term, we find that the 2009^{th} term is $63^2 - 7(63) = 3528$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; JACLYN CHANG, student, Western Canada High School, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. One incorrect solution was submitted.

M389. *Proposed by Lino Demasi, student, Simon Fraser University, Burnaby, BC.*

There are 2009 students and each has a card with a different positive integer on it. If the sum of the numbers on these cards is 2020049, what are the possible values for the median of the numbers on the cards?

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Let a be the smallest of the numbers on the cards. The smallest possible sum of the 2009 numbers on the cards is then

$$\begin{aligned} S &= a + (a+1) + (a+2) + \cdots + (a+2008) \\ &= 2009a + \frac{1}{2}(2008)(2009) \;=\; 2009a + 2017036\,. \end{aligned}$$

If $a \geq 2$ then $S \geq 2021054 > 2020049$. Therefore, $a = 1$.

Next, consider the sequence of numbers 1, 2, 3, $\ldots$, 2009. The sum of these numbers is 2019045 (which is 1004 less than the desired sum of 2020049). Also, their median is 1005.

In order to obtain the desired sum of 2020049, some of the numbers in this sequence need to be increased. When a certain term in the sequence is increased, then every greater term must be increased as well in order for the terms of the sequence to remain distinct. If a term that is less than 1006 is increased, then every larger term will also have to increase, resulting in an increase of the initial sum 2019045 by at least 1005 (since at least 1005 terms are increased), yielding a new sum of at least $2019045 + 1005 = 2020050$, which is too large. Therefore, only terms that are 1006 or greater may be increased.

When only terms greater than or equal to 1006 are increased, then the median 1005 remains unchanged. Thus, 1005 is the only possible value for the median.

Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. There was one incorrect solution submitted.

M390. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

A Pythagorean triangle is a right-angled triangle with all three sides of integer length. Let a and b be the legs of a Pythagorean triangle and let h be the altitude to the hypotenuse. Determine all such triangles for which

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{h} = 1\,.$$

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

Let A be the area of the triangle and c the length of its hypotenuse. Then A equals both $\frac{1}{2}ab$ and $\frac{1}{2}ch$, and so $ab = ch$.

Also,

$$\begin{aligned} 1 &= \frac{1}{a}+\frac{1}{b}+\frac{1}{h} = \frac{bh+ah+ab}{abh} = \frac{ah+bh+ch}{abh} \\ &= \frac{(a+b+c)h}{abh} = \frac{a+b+c}{ab}\,, \end{aligned}$$

hence $ab = a+b+c$.

By the Pythagorean Theorem, $c = \sqrt{a^2+b^2}$. Since $a+b+c = ab$, we then obtain the equivalent equations

$$\begin{aligned} ab &= a+b+\sqrt{a^2+b^2}\,, \\ ab-a-b &= \sqrt{a^2+b^2}\,, \\ (ab-a-b)^2 &= a^2+b^2\,, \\ a^2b^2+a^2+b^2-2a^2b-2ab^2+2ab &= a^2+b^2\,, \\ a^2b^2-2a^2b-2ab^2+2ab &= 0\,, \\ ab(ab-2a-2b+2) &= 0\,. \end{aligned}$$

Since $ab > 0$, then $ab-2a-2b+2 = 0$, or $b(a-2) = 2a-2$, which implies that $b = \dfrac{2a-2}{a-2} = 2 + \left(\dfrac{2}{a-2}\right)$.

Since a and b are positive integers, then $a-2$ is a positive divisor of 2; that is, $a-2 = 2$ or $a-2 = 1$. If $a-2 = 2$, then $a = 4$, $b = 3$, and $c = 5$. If $a-2 = 1$, then $a = 3$, $b = 4$, and $c = 5$.

There is therefore just one Pythagorean triangle for which $\frac{1}{a}+\frac{1}{b}+\frac{1}{h} = 1$, namely the triangle with legs 3 and 4, and hypotenuse 5.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KRISTÓF HUSZÁR, Valéria Koch Grammar School, Pécs, Hungary; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier

University, Waterloo, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

M391. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Determine all pairs (a, b) of positive integers for which both $\dfrac{a+1}{b}$ and $\dfrac{b+2}{a}$ are positive integers.

Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina, modified by the editor.

Let $x = \dfrac{a+1}{b}$ and $y = \dfrac{b+2}{a}$, where x and y are positive integers. Rearranging, we obtain $a = bx - 1$ and $b = ay - 2$.

Substituting for a in the second equation, we obtain $b = (bx-1)y - 2$ and so $y + 2 = bxy - b$ or $b = \dfrac{y+2}{xy-1}$.

If $x = 1$, then $b = \dfrac{y+2}{y-1} = \dfrac{y-1+3}{y-1} = 1 + \dfrac{3}{y-1}$. Since b and y are positive integers, then $y = 2$ or $y = 4$ (giving $b = 4$ and $b = 2$, respectively).

If $x = 2$, then $b = \dfrac{y+2}{2y-1}$. Since b is a positive integer, then we must have $y + 2 \geq 2y - 1$ and so $y \leq 3$. Checking $y = 1$, $y = 2$, and $y = 3$ shows that $y = 1$ and $y = 3$ give positive integer values for b (namely $b = 3$ and $b = 1$, respectively).

If $x = 3$, then $b = \dfrac{y+2}{3y-1}$. Since b is a positive integer, then we must have $y + 2 \geq 3y - 1$ and so $y \leq \dfrac{3}{2}$. The only possible value of y is $y = 1$, which does not give an integer value for b.

If $x = 4$, then $b = \dfrac{y+2}{4y-1}$. Since b is a positive integer, then we must have $y + 2 \geq 4y - 1$ and so $y \leq 1$. If $y = 1$, then $b = 1$.

If $x \geq 5$, then $xy - 1 \geq 5y - 1 > y + 2$ for all positive integers y, so $b = \dfrac{y+2}{xy-1}$ cannot be a positive integer.

We finish by calculating the values of a that go with the values of b to obtain the pairs $(a, b) = (3, 4)$, $(1, 2)$, $(5, 3)$, $(1, 1)$, $(3, 1)$.

Also solved by ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KRISTÓF HUSZÁR, Valéria Koch Grammar School, Pécs, Hungary; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; JOSÉ JAIME SAN JUAN CASTELLANOS, student, Universidad technológica de la Mixteca, Oaxaca, Mexico; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

M392. *Proposed by the Mayhem Staff.*

Determine, with justification, the fraction $\dfrac{p}{q}$, where p and q are positive integers and $q < 1000$, that is closest to, but not equal to, $\dfrac{19}{72}$.

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON, modified by the editor.

In order to find the desired fraction, we need to minimize the value of

$$\left|\frac{p}{q} - \frac{19}{72}\right| = \frac{|72p - 19q|}{72q},$$

where p and q are positive integers and $q < 1000$.

To do this, we attempt to make the numerator of this difference as small as possible, while at the same time keeping the denominator as large as possible, hence by making q as large as possible.

To minimize the numerator, we try to make $72p-19q$ equal to 1 or -1. Consider $72p-19q$ modulo 19. Since $72 \equiv -4 \pmod{19}$, then $72p-19q \equiv -4p \pmod{19}$, so we try to find p such that $-4p \equiv \pm 1 \pmod{19}$. Solving this congruence, we obtain $p \equiv 14 \pmod{19}$, or $p \equiv 5 \pmod{19}$, and so $p = 14+19k$ for some integer $k \geq 0$ or $p = 5+19k$ for some integer $k \geq 0$.

In the first case, $72p - 19q = 1$, so $q = \dfrac{72p-1}{19} = 53 + 72k$; since $q < 1000$, then $k \leq 13$; when $k = 13$, $q = 989$. In the second case, $72p-19q = -1$, so $q = \dfrac{72p+1}{19} = 19+72k$; since $q < 1000$, then $k \leq 13$; when $k = 13$, $q = 955$.

In either of these cases, the difference is equal to $\dfrac{1}{72q}$, and so is minimized when q is maximized. Thus, in these cases, the minimum possible difference occurs when q is as large as possible, or $q = 989$ (and so $k = 13$ and $p = 261$). This difference is $\dfrac{1}{72 \cdot 989}$.

It is not possible to achieve a smaller difference when $|72p - 19q| \geq 2$ and $q < 1000$, since this difference would always be at least $\dfrac{2}{72 \cdot 1000}$ which is larger than the difference that we have already found.

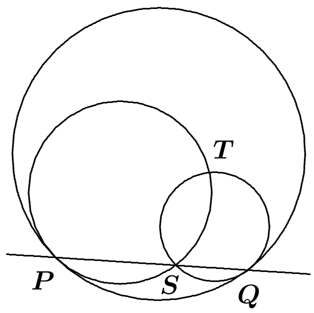
Therefore, the fraction closest to, but not equal to $\dfrac{19}{72}$ under the given conditions is $\dfrac{p}{q} = \dfrac{261}{989}$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and KRISTÓF HUSZÁR, Valéria Koch Grammar School, Pécs, Hungary; One incorrect solution was also submitted.

A very similar problem appeared in the 2006 Canadian Open Mathematics Challenge (problem 4(a), Part B).

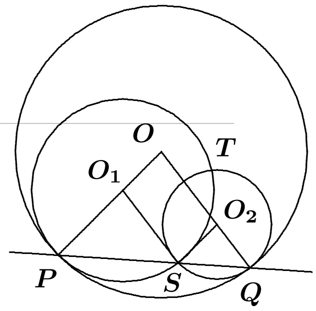
M393. *Proposed by the Mayhem Staff.*

Inside a large circle of radius r two smaller circles of radii a and b are drawn, as shown, so that the smaller circles are tangent to the larger circle at P and Q. The smaller circles intersect at S and T. If P, S, and Q are collinear (that is, they lie on the same straight line), prove that $r = a + b$.



Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain and Geoffrey A. Kandall, Hamden, CT, USA (independently).

Let O be the centre of the large circle of radius r. Let O_1 be the centre of the smaller circle of radius a tangent to the large circle at point P, and let O_2 be the centre of the smaller circle of radius b tangent to the large circle at point Q.



Since the circles centred at O_1 and O_2 are tangent to the large circle, then O, O_1, P are collinear, as are O, O_2, Q.

Triangle OPQ is isosceles with $OP = OQ$, triangle O_1PS is isosceles with $O_1P = O_1S$, and triangle O_2QS is isosceles with $O_2Q = O_2S$ (since each of these triangles has two radii of one of the circles as sides). Therefore, $\angle OPQ = \angle OQP$, $\angle O_1PS = \angle O_1SP$, and $\angle O_2QS = \angle O_2SQ$.

Since P, S, and Q are collinear, then

$$\angle PSO_1 = \angle O_1PS = \angle OPQ = \angle PQO\,,$$

which tells us that O_1S and OQ are parallel. Similarly,

$$\angle QSO_2 = \angle O_2QS = \angle OQP = \angle QPO\,,$$

which tells us that O_2S and OP are parallel. Therefore, quadrilateral OO_1SO_2 is a parallelogram.

Thus, $OO_1 = SO_2$. But $SO_2 = b$ and $OO_1 = OP - O_1P = r - a$, and so $r - a = b$, or $r = a + b$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; ANTONIO GODOY TOHARIA, Madrid, Spain; KRISTÓF HUSZÁR, Valéria Koch Grammar School, Pécs, Hungary; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

Problem of the Month

Ian VanderBurgh

To start the new year of Problems of the Month, we'll look at a problem that relies on a concept that we learn early on – addition – but requires us to think in some fairly deep ways to come up with a complete solution and total understanding of what is going on.

Since we've had a short break since the last issue, we should start with a warm-up problem. Your task is to compute the following sum :

$$2 + 22 + 222 + 2222 + 22222 + 222222 + 2222222\,.$$

But before you start, there are two rules : no calculators are allowed, and you have to compute the sum aloud.

If you're as out of practice on this sort of thing as some of us are, this isn't all that easy. We can rewrite the sum first in a form that makes it easier to compute :

$$
\begin{array}{cccccccc}
 & & & & & & & 2\\
 & & & & & & 2 & 2\\
 & & & & & 2 & 2 & 2\\
 & & & & 2 & 2 & 2 & 2\\
 & & & 2 & 2 & 2 & 2 & 2\\
 & & 2 & 2 & 2 & 2 & 2 & 2\\
+ & 2 & 2 & 2 & 2 & 2 & 2 & 2\\ \hline
\end{array}
$$

Here's an attempt to do this in words :

> The sum in the units' column is 14. Put down the 4; carry the 1. The sum in the tens' column is 12 plus the carry of 1 gives 13. Put down the 3; carry the 1. The sum in the hundreds' column is 10 plus the carry of 1 gives 11. Put down the 1; carry the 1. The sum in the thousands' column is 8 plus the carry of 1 gives 9. Put down the 9; no carry. The sum in the ten thousands' column is 6. Put down the 6. The sum in the hundred thousands' column is 4. Put down the 4. The sum in the millions' column is 2. Put down the 2. The final sum is thus 2469134.

That's a bit of a workout, isn't it? We should clarify the role of the digit and carry. If the sum in a column is 14, we write this as $14 = 10(1) + 4$; the units digit (the 4) is the digit that we write down, while the quotient when dividing by 10 (the 1) is the carry. (The units digit is also the remainder when we divide the sum by 10.) Let's have a look at our Problem of the Month, then.

Problem (2009 Fryer Contest) The addition shown below, representing $2{+}22{+}222{+}2222{+}\cdots$, has 101 rows and the last term consists of 101 2's :

$$
\begin{array}{ccccccccc}
 & & & & & & & & 2\\
 & & & & & & & 2 & 2\\
 & & & & & & 2 & 2 & 2\\
 & & & & & 2 & 2 & 2 & 2\\
 & & & & & \vdots & & & \vdots\\
 & & 2 & 2 & \cdots & 2 & 2 & 2 & 2\\
+ & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & 2\\ \hline
 & & & & & \cdots & C & B & A
\end{array}
$$

(a) Determine the value of the ones digit A .

(b) Determine the value of the tens digit B
and the value of the hundreds digit C.

(c) Determine the middle digit of the sum.

This problem looks pretty scary at first glance. Despite this, at least (a) and (b) can be answered exactly as in our warm-up problem. Let's do these parts and then think a bit about part (c).

Solution to (a) and (b) We proceed exactly as we did above. The units' column consists of 101 copies of the digit 2. Therefore, the sum in the units' column is $101 \times 2 = 202$. We put down the 2 and carry 20.

The tens' column consists of 100 copies of the digit 2 plus the carry of 20. Therefore, the sum in the tens' column is $100 \times 2 + 20 = 220$. We put down the 0 and carry 22.

The hundreds' column consists of 99 copies of the digit 2 plus the carry of 22. Therefore, the sum in the hundreds' column is $99 \times 2 + 22 = 220$. We put down the 0 and carry 22.

We can stop at this point, since we have determined the hundreds, tens, and units digits of the sum. Therefore, $A = 2$, $B = 0$, and $C = 0$. ■

Great – that was much less scary than it looked like it could be. Now we need to try to tackle (c), which actually *is* quite scary.

One approach would be to proceed by "brute force" and work our way systematically column by column from the units' column towards the left. Of course, we don't need to go all of the way to the leftmost column, since we can stop when we get to the middle digit of the sum. Which digit will this be? In order to answer this, we need to know how many digits the final sum has. How many digits do you think that it has? My best guess is 101 digits, since it seems pretty unlikely that the single 2 in the leftmost column is going to have enough of a carry from the column to its right to create two-digit sum in this leftmost column. How do we know for sure that this is correct?

If we knew this for sure, then the middle digit would be the 51st digit, since there would be 50 digits to its left and 50 digits to its right. Now, this 51^{st} column consists of 51 copies of the digit 2, so its sum is 102 plus whatever carry comes from the column to the right. The column to the right consists of 52 copies of 2, so its sum is 104 plus the carry from the column to its right, whose sum is at least 106 (that is, 106 from the 2's plus the carry). This is getting complicated!

Let's try this again with a bit of agreement on our terminology. We'll denote the leftmost column $C1$ and the rightmost column $C101$; we label the columns in between in the logical way. We also use s_n to represent the sum in the n^{th} column, including the carry.

We saw above that the sum of the digits in $C51$ is 102, in $C52$ is 104, and in $C53$ is 106. Therefore, $s_{53} \geq 106$. (We haven't included any carry here from $C54$.) Therefore, the carry from $C53$ to $C52$ is at least 10, so

$s_{52} \geq 104 + 10 = 114$. Therefore, the carry from $C52$ to $C51$ is at least 11, so $s_{51} \geq 102 + 11 = 113$.

If $s_{51} = 113$, then the middle digit is 3, and we're done. But is it actually the case that $s_{51} = 113$? Could it be bigger?

If s_{51} was at least 114, then the carry from $C52$ to $C51$ would be at least $114 - 102 = 12$, which would mean that $s_{52} \geq 120$. If $s_{52} \geq 120$, then the carry from $C53$ to $C52$ would be at least $120 - 104 = 16$, so $s_{53} \geq 160$. If $s_{53} \geq 160$, then carry from $C54$ to $C53$ would be at least $160 - 106 = 54$, so $s_{54} \geq 540$. If $s_{54} \geq 540$, then the carry from $C55$ to $C54$ would be at least $540 - 108 = 432$, which is getting just plain silly, given that in parts (a) and (b) the carries that we got from the "largest columns" were 22 only.

So it seems pretty clear that s_{51} should be 113, so the middle digit should be 3.

Now, I don't know about you, but I'm just about convinced. However, I'm not sure if "the middle digit should be 3" is all that rigorous and "just plain silly" counts as a solid mathematical proof. So we should prove some restriction on the carries. Let's do this, and also write out a cohesive solution to part (c). We'll use a bit of algebraic notation to simplify things.

Solution to (c) We label the columns as above and let s_n be the sum in the n^{th} column, including the carry from the $(n+1)^{\text{th}}$ column; we denote this carry by c_{n+1}. Column n consists of n copies of the digit 2, so $s_n = 2n + c_{n+1}$.

From our solution to (a) and (b), we know that $c_{101} = 20$ and that $c_{100} = c_{99} = 22$. Let's argue that $c_n \leq 22$ for all n with $1 \leq n \leq 101$.

We use an informal backwards induction. Suppose that $c_{n+1} \leq 22$. (We know that this is true for $n = 98$, 99, 100.) Then $s_n = 2n + c_{n+1}$ is at most $2(101) + 22 = 224$ and so $c_n \leq 22$. Thus, if $c_{n+1} \leq 22$, then $c_n \leq 22$. Since $c_{101} \leq 22$, then we can carry this chain along to show that $c_n \leq 22$ for all n with $1 \leq n \leq 101$.

We can use this to show that the sum has exactly 101 digits. For the sum to have more than 101 digits, we would need to have $s_1 \geq 10$. But $s_1 = 2 + c_2$, so this would mean that $c_2 \geq 8$ and so $s_2 \geq 80$. But $s_2 = 4 + c_3$, so this would mean that $c_3 \geq 76$, which is not possible. Therefore, the sum has exactly 101 digits.

Finally, we can determine the 51^{st} digit. We know that $s_{51} = 102 + c_{52}$ and $s_{52} = 104 + c_{53}$ and $s_{53} = 106 + c_{54}$. Since $0 \leq c_{54} \leq 22$, then $106 \leq s_{53} \leq 128$. Thus, $10 \leq c_{53} \leq 12$.

Since $10 \leq c_{53} \leq 12$, then $114 \leq s_{52} \leq 116$. Thus, $c_{52} = 11$, which means that $s_{51} = 113$, and so the 51^{st} (that is, the middle) digit of the sum is 3. ∎

Let's make a couple of observations to finish this off. First, a little bit of algebra and notation helped to save us a large number of words and convoluted explanations. Second, a relatively simple topic like addition gave us a problem that requires some pretty high-level thinking. To me, one of the great beauties of mathematics is that simplicity and complexity can be so completely interwoven.

THE OLYMPIAD CORNER

No. 283

R.E. Woodrow

Another year, and it is time to thank the many people who have contributed solutions, problems, comments, and advice during 2009.

Arkady Alt	Pavlos Maragoudakis
Miguel Amengual Covas	Robert Morewood
George Apostolopoulos	Bill Sands
Šefket Arslanagić	D.J. Smeenk
Ricardo Barroso Campos	Daniel Tsai
Michel Bataille	George Tsapakidis
José Luis Díaz-Barrero	Edward T.H. Wang
Oliver Geupel	Konstantine Zelator
Jean-David Houle	Titu Zvonaru
John Grant McLoughlin	

Thank you to both Jill Ainsworth, who produced the text for the *Corner* for the first four issues, and to Joanne Canape, who continued the rest of the issues, for their skilled and tireless efforts to translate my scribbles and notes into a nice presentation.

To start the New Year for the *Corner*, we give the problems proposed but not used at the 2007 IMO in Vietnam. My thanks go to Bill Sands, Canadian Team Leader, for collecting them for our use.

2007 IMO in VIETNAM
Problems Proposed But Not Used

Contributing Countries. Austria, Australia, Belgium, Bulgaria, Canada, Croatia, Czech Republic, Estonia, Finland, Greece, India, Indonesia, Iran, Japan, Korea (North), Korea (South), Lithuania, Luxembourg, Mexico, Moldova, Netherlands, New Zealand, Poland, Romania, Russia, Serbia, South Africa, Sweden, Thailand, Taiwan, Turkey, Ukraine, United Kingdom, and the United States of America

Problem Selection Committee. Ha Huy Khoai, Ilya Bogdanov, Tran Nam Dung, Le Tuan Hoa, Géza Kós.

Algebra

A1. Consider those functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy the condition

$$f(m+n) \;\geq\; f(m) + f\big(f(n)\big) - 1$$

for all m, $n \in \mathbb{N}$. Find all possible values of $f(2007)$. (Here $\mathbb{N}$ denotes the set of positive integers.)

A2. Let n be a positive integer, and let x and y be positive real numbers such that $x^n + y^n = 1$. Prove that

$$\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)(1-y)}.$$

A3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f\big(x + f(y)\big) = f(x+y) + f(y)$$

for all x, $y \in \mathbb{R}^+$. (Here $\mathbb{R}^+$ denotes the set of positive real numbers.)

A4. Let $c > 2$, and let $a(1)$, $a(2)$, $\dots$, be a sequence of nonnegative real numbers such that

(a) $a(m+n) \le 2a(m) + 2a(n)$ for all m, $n \ge 1$, and

(b) $a\big(2^k\big) \le \frac{1}{(k+1)^c}$ for all $k \ge 0$.

Prove that the sequence $\{a(n)\}_{n=1}^{\infty}$ is bounded.

A5. Let a_1, a_2, $\dots$, a_{100} be nonnegative real numbers satisfying the relation $a_1^2 + a_2^2 + \cdots + a_{100}^2 = 1$. Prove that $a_1^2 a_2 + a_2^2 a_3 + \cdots + a_{100}^2 a_1 < \frac{12}{25}$.

Combinatorics

C1. Let $n > 1$ be an integer. Find all sequences a_1, a_2, $\dots$, a_{n^2+n} such that

(a) $a_i \in \{0, 1\}$ for all $1 \le i \le n^2 + n$, and

(b) $a_{i+1} + a_{i+2} + \cdots + a_{i+n} < a_{i+n+1} + a_{i+n+2} + \cdots + a_{i+2n}$ holds for all $0 \le i \le n^2 - n$.

C2. A unit square is dissected into $n > 1$ rectangles whose sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that one of the rectangles has no point on the boundary of the square.

C3. Determine all positive integers n for which the numbers in the set $S = \{1, 2, \dots, n\}$ can be coloured red and blue so that $S \times S \times S$ contains exactly 2007 ordered triples (x, y, z) with these two properties :

(a) x, y, z are of the same colour ; and

(b) $x + y + z$ is divisible by n.

C4. Let $A_0 = (a_1, a_2, \dots, a_n)$ be a sequence of real numbers. For each integer $k \ge 0$, form a new sequence A_{k+1} from $A_k = (x_1, \dots, x_n)$ as follows :

(a) Choose a partition $\{1, 2, \ldots, n\} = I \cup J$ such that $\left|\sum_{i\in I} x_i - \sum_{j\in J} x_j\right|$ is minimized (if I or J is empty, then the corresponding sum is 0; if several such partitions exist, then choose one arbitrarily).

(b) Set $A_{k+1} = (y_1, y_2 \ldots, y_n)$, where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.

Prove that for some k, the sequence A_k has a term x with $|x| \geq \frac{n}{2}$.

C5. In the Cartesian coordinate plane let $S_n = \{(x, y) : n \leq x < n + 1\}$ for each integer n, and paint each region S_n either red or blue. Prove that any rectangle whose side lengths are distinct positive integers may be placed in the plane so that its vertices lie in regions of the same colour.

C6. Let $\alpha < \frac{3-\sqrt{5}}{2}$ be a positive real number. Prove that there exist positive integers n and $p > \alpha 2^n$ for which one can select $2p$ pairwise distinct subsets $S_1, \ldots, S_p, T_1, \ldots, T_p$ of the set $\{1, 2, \ldots, n\}$ such that $S_i \cap T_j \neq \emptyset$ for all $1 \leq i, j \leq p$.

C7. A convex n-gon P in the plane is given. For every three vertices of P, the triangle determined by them is *good* if all its sides are of unit length. Prove that P has at most $\frac{2n}{3}$ good triangles.

Geometry

G1. An isosceles triangle ABC with $AB = AC$ is given. The midpoint of side BC is denoted by M. Let X be a variable point on the shorter arc MA of the circumcircle of triangle ABM. Let T be the point in the angle domain BMA, for which $\angle TMX = 90^\circ$ and $TX = BX$. Prove that $\angle MTB - \angle CTM$ does not depend on X.

G2. The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

G3. Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle of ABC. Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C' respectively. Assume that the points A, B, C, A', B', C' are distinct, and that the lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

G4. Let $ABCD$ be a convex quadrilateral, and let points A_1, B_1, C_1, and D_1 lie on sides AB, BC, CD, and DA, respectively. Consider the areas of triangles AA_1D_1, BB_1A_1, CC_1B_1, and DD_1C_1; let S be the sum of the two smallest ones, and let S_1 be the area of quadrilateral $A_1B_1C_1D_1$.

Find the smallest positive real number k such that $kS_1 \geq S$ is always the case.

G5. Triangle ABC is acute with $\angle ABC > \angle ACB$, incentre I, and circumradius R. Point D is the foot of the altitude from vertex A, point K lies on line AD such that $AK = 2R$, and D separates A and K. Finally, lines DI and KI meet sides AC and BC at E and F, respectively.

Prove that if $IE = IF$, then $\angle ABC > 3\angle ACB$.

G6. Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incentre. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively, Let lines AC and BD meet at E, and let lines AD and BL meet at F. Prove that points E, I, and F are collinear.

Number Theory

N1. Find all pairs of positive integers (k, n) such that $\left(7^k - 3^n\right) \mid \left(k^4 + n^2\right)$.

N2. Let b, $n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $k \mid \left(b - a_k^n\right)$. Prove that $b = A^n$ for some integer A.

N3. Let X be a set of $10,000$ integers, none of them divisible by 47. Prove that there exists a 2007-element subset Y of X such that $a - b + c - d + e$ is not divisible by 47 for any a, b, c, d, $e \in Y$.

N4. For every integer $k \geq 2$, prove that 2^{3k} divides the number

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}}$$

but 2^{3k+1} does not.

N5. Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every m, $n \in \mathbb{N}$ and every prime p, the number $f(m + n)$ is divisible by p if and only if $f(m) + f(n)$ is divisible by p. ($\mathbb{N}$ is the set of all positive integers.)

N6. For a prime p and a positive integer n, denote by $\nu_p(n)$ the exponent of p in the prime factorization of $n!$. Given a positive integer d and a finite set of primes $\{p_1, p_2, \ldots, p_k\}$, show that there are infinitely many positive integers n such that $d \mid \nu_{p_i}(n)$ for all $1 \leq i \leq k$.

As a final pair of contests for your puzzling pleasure, we give two rounds of the Bundeswettbewerb Mathematik. Thanks go to Bill Sands, Canadian Team Leader to the 2007 IMO in Vietnam, for collecting them for our use.

BUNDESWETTBEWERB MATHEMATIK 2006
Second Round

1. A circle is divided into $2n$ congruent sectors, n of them coloured black and n of them coloured white. Starting with an arbitrarily chosen sector, the white sectors are numbered clockwise from 1 to n. Subsequently, the black sectors are numbered counterclockwise from 1 to n, again starting at an arbitrary sector.

Prove that there exist n consecutive sectors containing all of the numbers from 1 to n.

2. Let $\mathbb{Q}^+$ (resp. $\mathbb{R}^+$) denote the set of positive rational (resp. real) numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{R}^+$ that satisfy

$$f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)} \quad \text{for all } x, y \in \mathbb{Q}^+ .$$

3. The point P lies inside the acute-angled triangle ABC and C', A', B' are the feet of the perpendiculars from P to AB, BC, CA. Find all positions of P such that $\angle BAC = \angle B'A'C'$ and $\angle CBA = \angle C'B'A'$.

4. A positive integer n is *deficient* if there are at most nine different digits in the decimal representation of n. (Leading zeroes are not counted.)

Prove that for any finite set S of deficient numbers, the sum of the reciprocals of its elements is less than 180.

BUNDESWETTBEWERB MATHEMATIK 2007
First Round

1. Show that one can distribute the integers from 1 to 4014 on the vertices and the midpoints of the sides of a regular 2007-gon so that the sum of the three numbers along any side is constant.

2. Each positive integer is coloured either red or green so that

(a) The sum of three (not necessarily distinct) red numbers is red.

(b) The sum of three (not necessarily distinct) green numbers is green.

(c) There is at least one green number and one red number.

Find all colourings satisfying these conditions.

3. In triangle ABC the points E and F lie in the interiors of sides AC and BC (respectively) so that $|AE| = |BF|$. Furthermore, the circle through A, C and F and the circle through B, C and E intersect in a point $D \neq C$.

Prove that the line CD is the bisector of $\angle ACB$.

4. Let a be a positive integer. How many nonnegative integers x satisfy

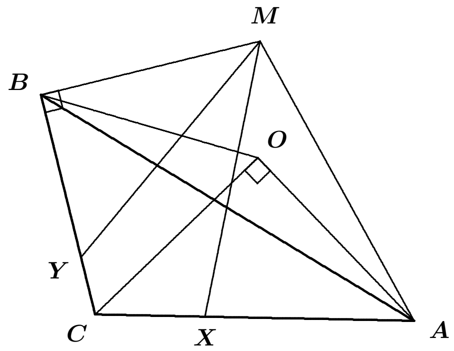
$$\left\lfloor \frac{x}{a} \right\rfloor = \left\lfloor \frac{x}{a+1} \right\rfloor ?$$

Our solutions in the New Year begin with solutions from our readers to problems of the Bulgarian National Olympiad 2006 given in the *Corner* at [2008 : 409–410].

5. (Emil Kolev) Let $\triangle ABC$ be such that $\angle BAC = 30^\circ$ and $\angle ABC = 45^\circ$. Consider all pairs of points X and Y such that X is on the ray $\overrightarrow{AC}$, Y is on the ray $\overrightarrow{BC}$, and $OX = BY$, where O is the circumcenter of $\triangle ABC$. Prove that the perpendicular bisectors of the segments XY pass through a fixed point.

Solution by Titu Zvonaru, Comăneşti, Romania.

Without loss of generality, suppose that $OA = OB = OC = 1$. Since $\angle BAC = 30^\circ$, we have that $\angle BOC = 60^\circ$ and $BC = 1$, that is, $\triangle BOC$ is equilateral.



Since $\angle ABC = 45^\circ$, it follows that $\triangle AOC$ has a right angle at O, hence $\angle OCA = \angle OAC = 45^\circ$ and $AC = \sqrt{2}$.

If $X = A$, then $OX = 1$ and $BY = 1$, and hence $Y = C$; as a result, the fixed point we seek lies on the perpendicular bisector of AC.

Let M be the point on the same side of AC as O such that $\triangle AMC$ is equilateral. Then $\angle OCM = 60^\circ - 45^\circ = 15^\circ$, $\angle MCB = 45^\circ$, and by the Law of Cosines in $\triangle MCB$ we obtain

$$\begin{aligned} BM^2 &= MC^2 + BC^2 - 2MC \cdot BC \cos\angle MCB \\ &= 2 + 1 - 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1 , \end{aligned}$$

hence $\triangle MCB$ has a right angle B.

Let $AX = \alpha$. By the Law of Cosines we have

$$\begin{aligned} OX^2 &= OA^2 + AX^2 - 2OA \cdot AX \cos\angle OAX = 1 + \alpha^2 - \alpha\sqrt{2} , \\ MX^2 &= MA^2 + AX^2 - 2MA \cdot AX \cos\angle MAC = 2 + \alpha^2 - \alpha\sqrt{2} , \end{aligned} \qquad (1)$$

and by the Pythagorean Theorem we have

$$MY^2 = MB^2 + BY^2 = MB^2 + OX^2 = \alpha^2 - \alpha\sqrt{2} + 2 . \qquad (2)$$

By (1) and (2) it follows that the point M belongs to the perpendicular bisectors of the segments XY.

We continue with solutions from our readers to problems of the Indian Mathematical Olympiad 2006 (Team Selection Problems) given in the *Corner* at [2008 : 410–412].

1. Let n be a positive integer divisible by 4. Find the number of permutations σ of $(1,2,3,\ldots,n)$ which satisfy the condition $\sigma(j)+\sigma^{-1}(j)=n+1$ for all $j\in\{1,\,2,\,3,\,\ldots,\,n\}$.

Solution by Michel Bataille, Rouen, France.

Let $\mathcal{S}_n$ be the set of all permutations of $[n]=\{1,\,2,\,\ldots,\,n\}$ and let

$$\begin{aligned}\mathcal{Q}_n &= \{\sigma\in\mathcal{S}_n : \forall j\in[n]\,,\ \sigma(j)+\sigma^{-1}(j) \;=\; n+1\}\\ &= \{\sigma\in\mathcal{S}_n : \forall k\in[n]\,,\ \sigma\circ\sigma(k) \;=\; n+1-k\}\,.\end{aligned}$$

We show that if $n=4m$, then $|\mathcal{Q}_n|=\dfrac{(2m)!}{m!}=2^mN_m$ where N_m denotes the product $1\times3\times\cdots\times(2m-1)$ of the first m odd natural numbers.

Let $\sigma\in\mathcal{Q}_n$, $k\in[n]$ and let $j=\sigma(k)$. Then, $j\neq k$ (otherwise $k=n+1-k$, contradicting the fact that $n+1$ is odd) and

$$\begin{aligned}\sigma(j) &= \sigma\circ\sigma(k) \;=\; n+1-k\,,\\ \sigma(n+1-k) &= \sigma\circ\sigma(j) \;=\; n+1-j\,,\\ \sigma(n+1-j) &= \sigma\circ\sigma(n+1-k) \;=\; n+1-(n+1-k) \;=\; k\,.\end{aligned}$$

Thus, the cycle containing k is $(k,j,n+1-k,n+1-j)$. Since k is arbitrary, the standard expression of σ as a product of disjoint cycles (the cycle decomposition of σ) consists of m 4-cycles of the form $(k,j,n+1-k,n+1-j)$. Conversely, if $\sigma\in\mathcal{S}_n$ has such a cycle decomposition, then clearly $\sigma\in\mathcal{Q}_n$.

Consider now the set $S=\{p_k : k\in[2m]\}$ where $p_k=\{k,\,n+1-k\}$ and let $\sigma\in\mathcal{Q}_n$. Using its cycle decomposition, we associate with σ in a natural way a partition $\{P_1,\,P_2,\,\ldots,\,P_m\}$ where each P_k is a 2-subset of S ($P_k\cap P_l=\emptyset$ if $k\neq l$; $\bigcup_{i=1}^m P_i=S$). There are N_m such partitions (see the lemma below) and each of them is obtained from exactly 2^m elements of $\mathcal{Q}_n$, since any 2-subset $P=\{p_k,\,p_j\}$ of S is obtained from two distinct 4-cycles, namely $(j,k,n+1-j,n+1-k)$ and $(k,j,n+1-k,n+1-j)$. It follows that $|\mathcal{Q}_n|=2^mN_m$.

Lemma If S is a set with $|S|=2m$, then the number of partitions of S formed by 2-subsets of S is $N_m=(2m-1)(2m-3)\cdots(3)(1)=\dfrac{(2m)!}{2^m m!}$.

Proof. Fix $s\in S$. We partition S into 2-subsets by first choosing one of the $2m-1$ 2-subsets $\{s,\,t\}$, where $t\in S$ and $t\neq s$, and then augmenting this subset with a partition into 2-subsets of $S-\{s,\,t\}$ (and there are N_{m-1} such partitions). Thus, $N_m=(2m-1)N_{m-1}$ when $m>1$, and the result follows since $N_1=1$.

3. (Short list, IMO 2005) There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. In each step (if possible) we pick a marker with the white side up that is not an outermost marker, remove it, and turn over the closest marker to the left and the closest marker to the right of it. Prove that one can reach a terminal state of exactly two markers if and only if $(n-1)$ is not divisible by 3.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We prove that if $3 \nmid (n-1)$, then two markers can be left behind. This is clear if $n = 2, 3$. Now assume that by starting with n markers we can leave behind two markers, and that $n+3$ markers are given. By successively choosing the second, third, then second marker from the left, we obtain n markers all white side up, and then we can finish with two markers. This completes the proof by induction.

It only remains to prove that if two markers can be left behind, then $3 \nmid (n-1)$. In a fixed state S, we let $b(m)$ be the number of black markers to the left of marker m. Further, we assign the number $T(S) = \sum\limits_{m \text{ white}} (-1)^{b(m)}$ to the state S. Let B or W denote a marker with the black or white side up, respectively; then the admissible reduction steps are

$$\text{(i)}\ \ BWB \to WW\,, \qquad \text{(iii)}\ \ WWB \to BW\,,$$
$$\text{(ii)}\ \ BWW \to WB\,, \qquad \text{(iv)}\ \ WWW \to BB\,.$$

Note that the parity of the number of B's is invariant, hence it is always even, and also we have $T(BB) = 0$, $T(WW) = 2$, and for the initial state I with n white markers $T(I) = n$. It therefore suffices to prove that $T(S)$ modulo 3 is invariant under the transitions (i)-(iv), because this implies that if $F \in \{BB, WW\}$ is reachable from I, then

$$n \;=\; T(I) \;\equiv\; T(F) \;\not\equiv\; 1 \pmod 3\,.$$

In the case of transition (i), if the white marker m that is picked has value $(-1)^{b(m)} = v \in \{-1, 1\}$ in state S, then the two markers m_1' and m_2' which are turned over have values $(-1)^{b(m_1')} = (-1)^{b(m_2')} = -v$ in the new state S', whereas the values of all other markers remain unchanged, thus

$$T(S') \;=\; T(S) - v - 2v \;=\; T(S) - 3v \;\equiv\; T(S) \pmod 3\,.$$

The transitions (ii) to (iv) are analyzed similarly.

This completes the proof.

7. Let ABC be a triangle with inradius r, circumradius R, and with sides $a = BC$, $b = AC$, and $c = AB$. Prove that

$$\frac{R}{2r} \;\geq\; \left(\frac{64a^2b^2c^2}{\big(4a^2-(b-c)^2\big)\big(4b^2-(c-a)^2\big)\big(4c^2-(a-b)^2\big)}\right)^2 .$$

Commentary by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Solved by George Apostolopoulos, Messolonghi, Greece. We give the comment of Amengual Covas.

This problem appears as Problem 11195 of the *American Mathematical Monthly*, Vol. 113 (January 2006), p. 79.

The solution appears on p. 648 of the August–September 2007 issue of the *American Mathematical Monthly*, Vol. 114, and includes two generalizations of the given inequality.

Next we give the write-up of Apostolopoulos.

By Heron's formula, the area of $\triangle ABC$ is $\sqrt{s(s-a)(s-b)(s-c)}$, and also the same area is given by $\dfrac{abc}{4R}$. Hence, $R = \dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$. Furthermore, the area of $\triangle ABC$ is rs, and a comparison with Heron's formula yields $r = \dfrac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$, where $s = \dfrac{a+b+c}{2}$. Therefore,

$$\frac{R}{2r} = \frac{abc}{8(s-a)(s-b)(s-c)}\,. \qquad (1)$$

By using the AM–GM Inequality, we obtain

$$\begin{aligned} 4a^2-(b-c)^2 &= a^2+a^2+a^2+4(s-b)(s-c) \\ &\geq 4\sqrt[4]{a^2a^2a^2 4(s-b)(s-c)} \\ &= 4a\sqrt[4]{4a^2(s-b)(s-c)}\,. \end{aligned}$$

Similarly, we have the inequality $4b^2-(c-a)^2 \geq 4b\sqrt[4]{4b^2(s-c)(s-a)}$ and $4c^2-(a-b)^2 \geq 4c\sqrt[4]{4c^2(s-a)(s-b)}$, so that

$$\begin{aligned} &\frac{4a^2}{4a^2-(b-c)^2}\cdot\frac{4b^2}{4b^2-(c-a)^2}\cdot\frac{4c^2}{4c^2-(a-b)^2} \\ &\leq \frac{a}{\sqrt[4]{4a^2(s-b)(s-c)}}\cdot\frac{b}{\sqrt[4]{4b^2(s-c)(s-a)}}\cdot\frac{c}{\sqrt[4]{4c^2(s-a)(s-b)}}\,. \end{aligned}$$

Finally, we have

$$\begin{aligned} &\left(\frac{64a^2b^2c^2}{\left(4a^2-(b-c)^2\right)\left(4b^2-(c-a)^2\right)\left(4c^2-(a-b)^2\right)}\right)^2 \\ &\leq \frac{a^2b^2c^2}{\sqrt{64a^2b^2c^2(s-a)^2(s-b)^2(s-c)^2}} \\ &\leq \frac{abc}{8(s-a)(s-b)(s-c)} = \frac{R}{2r}\,. \end{aligned}$$

as desired, where the last equality is from (1).

8. The positive divisors $d_1, d_2, \ldots, d_l$ of a positive integer n are ordered

$$1 = d_1 < d_2 < \cdots < d_l = m.$$

Suppose it is known that $d_7^2 + d_{15}^2 = d_{16}^2$. Find all possible values of d_{17}.

Solution by Titu Zvonaru, Comăneşti, Romania.

It is known (see the comment at the end of this solution) that if a, b, c are positive integers such that $a^2 = b^2 + c^2$, then abc is divisible by 60 and one of a, b, c is divisible by 4.

Let $D = \{d_1, \ldots, d_l\}$ be the set of divisors of n. Since $d_7^2 + d_{15}^2 = d_{16}^2$, it follows that n is divisible by 60, hence $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $d_4 = 4$, $d_5 = 5$, $d_6 = 6$ and $10 \in D$. We deduce that $d_7 \in \{7, 8, 9, 10\}$.

Case 1. Suppose that $d_7 = 10$. Then we have $d_{16}^2 - d_{15}^2 = 100$, and factoring yields $(d_{16} - d_{15})(d_{16} + d_{15}) = 100$. Since $d_{16} + d_{15}$ and $d_{16} - d_{15}$ have the same parity, we must have $d_{16} - d_{15} = 2$ and $d_{16} + d_{15} = 50$, and hence $d_{15} = 24$, $d_{16} = 26$. This means that $8 \in D$, contradicting $d_7 = 10$.

Case 2. Suppose that $d_7 = 9$. Then $(d_{16} - d_{15})(d_{16} + d_{15}) = 81$. Since $d_{15} + d_{16} \geq 15 + 16 = 31$, we must have $d_{16} - d_{15} = 1$ and $d_{16} + d_{15} = 81$, and hence $d_{15} = 40$, $d_{16} = 41$. This means that $8 \in D$, contradicting $d_7 = 9$.

Case 3. Suppose that $d_7 = 8$. Since $7 \notin D$, we deduce $d_{15} > 15$, $d_{16} > 16$, hence $d_{15} + d_{16} \geq 33$. Then $(d_{16} - d_{15})(d_{16} + d_{15}) = 64$ has no solution, because $d_{15} + d_{16} \geq 33$ and $d_{16} + d_{15}$ and $d_{16} - d_{15}$ have the same parity.

Case 4. Suppose $d_7 = 7$. Then necessarily $d_{16} - d_{15} = 1$ and $d_{16} + d_{15} = 49$, and hence $d_{15} = 24$, $d_{16} = 25$. If n is divisible by 9, 11, or 13, then $d_{15} < 24$, a contradiction. As a result, the positive divisors of n are

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
d_i	1	2	3	4	5	6	7	8	10	12	14	15	20	21	24	25

and we deduce that $d_{17} = 28$.

Comment. The fact that $60 \mid abc$ whenever $a^2 = b^2 + c^2$ can be seen as follows. There are positive integers m, n with $m > n$ such that $a = m^2 + n^2$, $b = m^2 - n^2$, $c = 2mn$. Let $R_i = \{x \in \mathbb{Z} : x \equiv \pm i \pmod 5\}$.

- If $m \in R_0$ or $n \in R_0$, then $c \equiv 0 \pmod 5$.
- If $m, n \in R_1$ or $m, n \in R_2$, then $b \equiv 0 \pmod 5$.
- If $m \in R_1$ and $n \in R_2$ (or vice-versa), then $a \equiv 0 \pmod 5$.

Hence, $5 \mid abc$. The divisibility by 3 and 4 is proved similarly.

Next we present solutions from our readers to problems given in the November 2008 number of the *Corner*, and the Third Round, Senior Division, of the 2004 South African Mathematical Olympiad given at [2008 : 412–413].

1. Let $a = 1111 \cdots 1111$ and $b = 1111 \cdots 1111$, where a has forty ones and b has twelve ones. Determine the greatest common divisor of a and b.

Solved by George Apostolopoulos, Messolonghi, Greece ; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's version.

Let $d = \gcd(a, b)$. Observe that $a = \frac{1}{9}\big(10^{40} - 1\big)$ and $b = \frac{1}{9}\big(10^{12} - 1\big)$.

Since d divides both $\frac{1}{9}\big(10^{40} - 1\big)$ and $\frac{1}{9}\big(10^{12} - 1\big)$, and we also have $10^4 - 1 = \big(10^{40} - 1\big) - \big(10^{28} + 10^{16} + 10^4\big)\big(10^{12} - 1\big)$, then

$$d \mid \frac{1}{9}\big(10^4 - 1\big)\,. \tag{1}$$

Conversely, since $10^{40} - 1 = \big(10^4\big)^{10} - 1$ and $10^{12} - 1 = \big(10^4\big)^3 - 1$ are both divisible by $10^4 - 1$, we see that both a and b are divisible by $\frac{1}{9}\big(10^4 - 1\big)$. Hence,

$$\frac{1}{9}\big(10^4 - 1\big) \mid d\,. \tag{2}$$

From (1) and (2), it follows that $d = \frac{1}{9}\big(10^4 - 1\big) = 1111$.

5. Let $n \geq 2$ be an integer. Find the number of integers x with $0 \leq x < n$ and such that x^2 leaves a remainder of 1 when divided by n.

Solution by Michel Bataille, Rouen, France.

For each positive integer n let $S(n)$ be the set of all integers x such that $0 \leq x < n$ and $x^2 \equiv 1 \pmod{n}$ and let $s(n)$ denote its cardinality. We will prove that if $\lambda(n)$ is the number of odd prime divisors of n and $\mu(n)$ is the greatest nonnegative integer such that 2^m divides n, then

$$s(n) \;=\; \begin{cases} 2^{\lambda(n)} & \text{if } \mu(n) = 0, 1\,; \\ 2^{\lambda(n)+1} & \text{if } \mu(n) = 2\,; \\ 2^{\lambda(n)+2} & \text{if } \mu(n) > 2\,. \end{cases}$$

We first show that s is a multiplicative function. Clearly, $s(1) = 1$. Suppose that $n = ab$ where a, b are coprime positive integers. If $x \in S(n)$, then $x^2 \equiv 1 \pmod{ab}$, hence $x^2 \equiv 1 \pmod{a}$ and $x^2 \equiv 1 \pmod{b}$ and therefore $x \equiv y \pmod{a}$ and $x \equiv z \pmod{b}$ for some unique pair of integers $(y, z) \in S(a) \times S(b)$. Conversely, if $(y, z) \in S(a) \times S(b)$, then the Chinese Remainder Theorem ensures that the system $x \equiv y \pmod{a}$, $x \equiv z \pmod{b}$ has a unique solution x with $0 \leq x < ab = n$. Then both a, b divide $x^2 - 1$, hence $x^2 - 1 \equiv 0 \pmod{ab}$ (since a, b are coprime) and so $x \in S(n)$. Thus, $S(n)$ and $S(a) \times S(b)$ correspond bijectively and $s(n) = s(a)s(b)$.

It just remains to determine $s(p^r)$ and $s(2^r)$ where p is an odd prime and r is a positive integer. If $x^2 - 1 \equiv 0 \pmod{p^r}$, then p divides $x - 1$ or $x + 1$ but not both, or else it would divide $(x + 1) - (x - 1) = 2$. It follows that either $x - 1 \equiv 0 \pmod{p^r}$ or $x + 1 \equiv 0 \pmod{p^r}$. As a result, we have $S(p^r) = \{1, p^r - 1\}$ and $s(p^r) = 2$.

We readily verify that $s(2) = 1$, $s(2^2) = 2$. If $r \geq 3$, then 2 is the highest power of 2 that can possibly divide both $x - 1$ and $x + 1$ whenever $x^2 - 1 \equiv 0 \pmod{2^r}$. Hence, in addition to the obvious solutions 1, $2^r - 1$ to the latter, we also have $2^{r-1} + 1$ and $2^{r-1} - 1$, corresponding to the cases $2^{r-1}|(x - 1)$, $2|(x + 1)$ and $2|(x - 1)$, $2^{r-1}|(x + 1)$, respectively. Thus, $s(2^r) = 4$ whenever $r \geq 3$. The result follows.

Now we turn to the 2006 Vietnamese Mathematical Olympiad given at [2008 : 413–414].

1. Find all real solutions of the system of equations

$$\begin{aligned}
\sqrt{x^2 - 2x + 6} \cdot \log_3(6 - y) &= x\,,\\
\sqrt{y^2 - 2y + 6} \cdot \log_3(6 - z) &= y\,,\\
\sqrt{z^2 - 2z + 6} \cdot \log_3(6 - x) &= z\,.
\end{aligned}$$

Solved by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain ; and Titu Zvonaru, Comănești, Romania. We give the solution of Díaz-Barrero, modified by the editor.

First, we observe that x, y, $z \in (-\infty, 6)$. To find all real solutions we first rewrite the system in the form

$$\begin{aligned}
y &= 6 - 3^{\frac{x}{\sqrt{x^2-2x+6}}}\,,\\
z &= 6 - 3^{\frac{y}{\sqrt{y^2-26+6}}}\,,\\
x &= 6 - 3^{\frac{z}{\sqrt{z^2-2z+6}}}\,.
\end{aligned}$$

Let $f(t) = 6 - 3^{\frac{t}{\sqrt{t^2-2t+6}}}$ for $t \in (-\infty, 6)$. Since on the domain of f we have

$$\frac{d}{dt}\frac{t}{\sqrt{t^2 - 2t + 6}} = \frac{(6 - t)}{\left(t^2 - 2t + 6\right)^{3/2}} > 0\,,$$

then f is a strictly decreasing function, whence $f(f(f(t)))$ is also a decreasing function for $t \in (-\infty, 6)$. However, $f(x) = y$, $f(y) = z$, $f(z) = x$; and thus $f(f(f(x))) = x$. We claim that if $f(f(f(x))) = x$, then $f(x) = x$. Indeed, let f^n denote the n-fold composition of f, and note that both $f^3(x) = x$ and $f(x) < x$ yield $f(x) = f(f^3(x)) = f^3(f(x)) > f^3(x) = x$, a contradiction. The case $f(x) > x$ similarly leads to a contradiction, and our claim is established.

Now, $6 - x$ is a decreasing function and $3^{\frac{x}{\sqrt{x^2-2x+6}}}$ is increasing for $x \in (-\infty, 6)$. Hence, $f(x) = x$ has at most one root. Since $f(3) = 3$, it follows that $(x, y, z) = (3, 3, 3)$ is the unique real solution of the system.

2. Let $ABCD$ be a given convex quadrilateral. A point M moves on the line AB but does not coincide with A or B. Let N be the second point of intersection (distinct from M) of the circles (MAC) and (MBD), where (XYZ) denotes the circle passing through the points X, Y, Z. Prove that

(a) N moves on a fixed circle,

(b) the line MN passes through a fixed point.

Solution to part (a) by Michel Bataille, Rouen, France. Solution to part (b) by J. Chris Fisher, University of Regina, Regina, SK.

(a) We denote by $\angle(TU, VW)$ the directed angle of the lines TU and VW. Let the diagonals AC and BD intersect at O. The point N moves on the circle (CDO) as it immediately follows from the characterization of concyclicity in terms of angles and the following calculation :

$$\begin{aligned} & \angle(NC, ND) \\ = \; & \angle(NC, NM) + \angle(NM, ND) \\ = \; & \angle(AC, AM) + \angle(BM, BD) \\ = \; & \angle(AC, BD) \\ = \; & \angle(OC, OD)\,. \end{aligned}$$

(b) Because of the use of directed angles, convexity was not required in part (a). Likewise in part (b), A, B, C, and D can be any four points in the plane, no three collinear. Let Γ be the circle (CDO) from part (a). For any two positions of M on AB, say M_1 and M_2, we know that the circles (AMC) meet Γ at the corresponding points N, say N_1 and N_2. It suffices to prove that the lines M_1N_1 and M_2N_2 intersect at a point of Γ. To this end, we define P to be the point where these two lines intersect, and we apply Miquel's theorem to triangle M_1M_2P and the points A on M_1M_2, N_1 on M_1P, and N_2 on M_2P. By construction the circles (AM_1N_1) and (AM_2N_2) meet at C. By Miquel's theorem the circle (PN_1N_2) also passes through C. Since $\Gamma = (N_1N_2C)$, we conclude that P lies on Γ, as claimed.

4. Consider the function

$$f(x) \;=\; -x + \sqrt{(x+a)(x+b)}$$

where a and b are distinct positive real numbers. Prove that for every real number s in the interval $(0, 1)$, there exists a unique positive real number α such that

$$f(\alpha) \;=\; \left(\frac{a^s + b^s}{2}\right)^{1/s}\,.$$

Solution by Michel Bataille, Rouen, France.

Let v, w lie in $(0,\infty)$ with $w > v$. Then

$$\begin{aligned} f(w) - f(v) &= \sqrt{(w+a)(w+b)} - \sqrt{(v+a)(v+b)} - (w-v) \\ &= \frac{(w+a)(w+b) - (v+a)(v+b)}{\sqrt{(w+a)(w+b)} + \sqrt{(v+a)(v+b)}} - (w-v) \\ &= (w-v)\left(\frac{N}{D} - 1\right) . \end{aligned}$$

where $N = v + w + a + b$ and $D = \sqrt{(w+a)(w+b)} + \sqrt{(v+a)(v+b)}$. By the AM–GM Inequality,

$$D < \frac{2w+a+b}{2} + \frac{2v+a+b}{2} = N ,$$

hence $f(w) > f(v)$ and f is strictly increasing on $(0,\infty)$. Also, f is continuous on $(0,\infty)$ and

$$\lim_{x\to 0+} f(x) = \sqrt{ab} , \qquad \lim_{x\to\infty} f(x) = \frac{a+b}{2} ,$$

the latter holding because

$$f(x) = \frac{(x+a)(x+b) - x^2}{\sqrt{(x+a)(x+b)} + x} = \frac{a+b+(ab/x)}{1 + \sqrt{1 + (a+b)/x + (ab)/x^2}} .$$

Thus, f is a bijection from $(0,\infty)$ onto (m_0, m_1), where $m_0 = \sqrt{ab}$ and $m_1 = \frac{a+b}{2}$. The s-mean of a and b is $m_s = \left(\frac{a^s + b^s}{2}\right)^{1/s}$, and we know from the Power Mean Inequality that $m_0 < m_s < m_1$ whenever $0 < s < 1$. Since f is bijective, $m_s = f(\alpha)$ for some unique $\alpha \in (0,\infty)$.

5. Find all polynomials $P(x)$ with real coefficients satisfying

$$P(x^2) + x\big(3P(x) + P(-x)\big) = P(x)^2 + 2x^2 ,$$

for all real numbers x.

Solution by Titu Zvonaru, Comăneşti, Romania.

First we will prove a lemma.

Lemma Let f be a polynomial that satisfies $f(x^2) = f(x)^2$ for all x. Then either $f = 0$ or $f(x) = x^m$.

Proof : We will prove by induction that $f\left(x^{2^p}\right) = f(x)^{2^p}$. For $p = 1$ the identity holds by hypothesis. Assume the identity holds for $p - 1 \geq 1$, then

$$f\left(x^{2^p}\right) = f\left(\left(x^{2^{p-1}}\right)^2\right) = f\left(x^{2^{p-1}}\right)^2 = \left(f(x)^{2^{p-1}}\right)^2 = f(x)^{2^p} ,$$

which completes the induction.

Now we choose a natural number p such that $g = 2^p > \deg f$. Let x_0 be any complex root of f and let $z_1, z_2, \ldots, z_g$ be the (distinct) complex roots of $z^g = x_0$. We then have

$$\big(f(z_i)\big)^g = f(z_i^g) = f(x_0) = 0,$$

so that $f(z_k)$ vanishes for each z_k. We deduce that the polynomial f has $g > \deg f$ distinct roots, or $x_0 = 0$. Therefore, $f = 0$ or $f(x) = ax^m$ for some number a. In the latter case $f(x^2) = f(x)^2$ becomes $ax^{2m} = a^2x^{2m}$, hence $f \equiv 0$ or $f(x) = x^m$. ■

The given equation, for x and $-x$, is

$$\begin{aligned} P(x^2) + x\big(3P(x) + P(-x)\big) &= P(x)^2 + 2x^2, \\ P(x^2) - x\big(3P(-x) + P(x)\big) &= P(-x)^2 + 2x^2. \end{aligned}$$

Subtracting, we obtain

$$4x\big(P(x) + P(-x)\big) = \big(P(x) - P(-x)\big)\big(P(x) + P(-x)\big),$$

which leads to two cases.

Case 1. $P(x) + P(-x) = 0$. Here the following are equivalent :

$$\begin{aligned} P(x^2) + 2xP(x) &= P(x)^2 + 2x^2, \\ P(x^2) - x^2 &= P(x)^2 - 2xP(x) + x^2, \\ P(x^2) - x^2 &= \big(P(x) - x\big)^2. \end{aligned}$$

Thus, $f(x) = P(x) - x$ satisfies $f(x^2) = f(x)^2$, and by the Lemma we have either $f \equiv 0$ and $P(x) = x$, or $f(x) = x^m$ and $P(x) = x^m + x$.

In the latter case, since $P(x) + P(-x) = 0$, we have

$$x^m + x + (-x)^m - x = x^m + (-x)^m = 0,$$

that is, m is odd.

We conclude that $P(x) = x$ or $P(x) = x^{2n+1} + x$, with n a nonnegative integer.

Case 2. $P(x) - P(-x) = 4x$. Here the following are equivalent :

$$\begin{aligned} P(x^2) + x\big(3P(x) + P(x) - 4x\big) &= P(x)^2 + 2x^2, \\ P(x^2) - 2x^2 &= P(x)^2 - 4xP(x) + 4x^2, \\ P(x^2) - 2x^2 &= \big(P(x) - 2x\big)^2. \end{aligned}$$

Thus, $f(x) = P(x) - 2x$ satisfies $f(x^2) = f(x)^2$, and by similar reasoning as in Case 1, we deduce that either $P(x) = 2x$ or $P(x) = x^m + 2x$.

In the latter case, since $P(x) - P(-x) = 4x$, we have

$$x^m + 2x - (-x)^m + 2x \;=\; 4x\,,$$

or $x^m - (-x)^m = 0$, that is, m is even.

So, in this case, we obtain the polynomials $P(x) = 2x$ and $P(x) = x^{2n} + 2x$ with n a nonnegative integer.

For $n = 0$ we have $x^{2n+1} + x = 2x$, so a complete list of polynomials $P(x)$ is x, $x^{2n+1} + x$, and $x^{2n} + 2x$, where n is a nonnegative integer.

6. A set of integers T is called *sum-free* if for every two (not necessarily distinct) elements u and v in T, their sum $u + v$ does not belong to T. Prove that

(a) a sum-free subset of $S = \{1, 2, \ldots, 2006\}$ has at most 1003 elements,

(b) any set S consisting of 2006 positive integers has a sum-free subset consisting of 669 elements.

Solution by Oliver Geupel, Brühl, NRW, Germany.

For part (a), we claim that a sum free subset T of $S = \{1, 2, \ldots, n\}$ has at most $\left\lceil \frac{n}{2} \right\rceil$ elements. To prove this fact, let $m = \max S$. If m is even then each of the sets $\{1, m-1\}$, $\{2, m-2\}$, $\ldots$, $\left\{\frac{m-2}{2}, \frac{m+2}{2}\right\}$ contains at most one element of T; hence $|T| \le \frac{m}{2} \le \frac{n}{2}$. Now suppose that m is odd. Then each of the sets $\{1, m-1\}$, $\{2, m-2\}$, $\ldots$, $\left\{\frac{m-1}{2}, \frac{m+1}{2}\right\}$ contains at most one element of S; thus $|S| \le \frac{m+1}{2}$. If n is even, then $\frac{m+1}{2} \le \frac{n}{2}$, while if n is odd, then $\frac{m+1}{2} \le \frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil$. This completes the proof of part (a).

For part (b), we claim that any set S consisting of n positive integers has a sum free subset consisting of $\left\lceil \frac{n}{3} \right\rceil$ elements.

To prove this, choose a prime number $p > \max S$ such that $3 \mid (p+1)$, which is possible by Dirichlet's theorem. Let

$$I \;=\; \left\{k \in \mathbb{N} : \;\frac{p+1}{3} \;\le\; k \;\le\; \frac{2p-1}{3}\right\}.$$

Consider the bipartite graph with sets of nodes $P = \{1, 2, \ldots, p-1\}$ and I, where a node $k \in P$ is adjacent to a node $l \in I$ if and only if there exists a number $a \in S$ such that $ka \equiv l \pmod{p}$. Note that for each edge, the number $a \in S$ is unique, and label the respective edge with the number a. For each $a \in S$ there are exactly $|I| = \frac{p+1}{3}$ numbers $k \in P$ such that ka has a residue modulo p which belongs to I, hence our graph has $\frac{n(p+1)}{3}$ edges. By the Pigeonhole Principle, there is a node $k \in P$ with degree not less than $\frac{n(p+1)}{3(p-1)}$, hence not less than $\left\lceil\frac{n}{3}\right\rceil$. Let $T \subseteq S$ be the set of labels of its respective edges.

We claim that T has the required properties. Clearly, $|T| \geq \left\lceil\frac{n}{3}\right\rceil$. Assume that $a+b=c$ for some a, b, $c \in T$. Then ka, kb, kc are each congruent modulo p to numbers in I. Moreover, $ka + kb = kc$, so that $ka + kb$ is congruent modulo p to a number in I. On the other hand, it is easy to check that for all l, $l' \in I$, the residue modulo p of the number $l + l'$ does not belong to I. This contradicts the hyothesis $a + b = c$, and completes our proof.

Next we move to solutions from our readers to problems proposed but not used for the 47th International Mathematical Olympiad 2006 in Slovenia given at [2008 : 459–464].

A1. Given an arbitrary real number a_0, define a sequence of real numbers a_0, a_1, a_2, $\ldots$ by the recursion

$$a_{i+1} \;=\; \lfloor a_i \rfloor \cdot \{a_i\}\,, \qquad i \geq 0\,,$$

where $\lfloor a_i \rfloor$ is the greatest integer not exceeding a_i, and $\{a_i\} = a_i - \lfloor a_i \rfloor$. Prove that $a_i = a_{i+2}$ for sufficiently large i.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We consider the three cases $a_0 = 0$, $a_0 > 0$, and $a_0 < 0$ separately.

If $a_0 = 0$, then the sequence (a_i) is constant.

Next, suppose $a_0 > 0$. Then all a_i are nonnegative. Let $\lfloor a_i \rfloor = n$ and $\{a_i\} = r$. If $n = 0$, then $a_{i+1} = a_{i+2} = a_{i+3} = \cdots = 0$. Otherwise we have $n \geq 1$ and $a_i - a_{i+1} = (n+r) - nr = (n-1)(1-r) + 1 \geq 1$; hence we obtain $0 \leq a_{i+n} < 1$, which reduces to the previous case.

Lastly, suppose $a_0 < 0$. Then $a_0 \leq a_1 \leq a_2 \leq \cdots \leq 0$. Hence, the integer sequence $(\lfloor a_i \rfloor)$ is non-decreasing and bounded above by 0. If just one term of this sequence is 0, then (a_i) terminates in zeros and we are done. The other possibility is that there exist positive integers i_0 and n such that for all $i \geq i_0$ we have $\lfloor a_i \rfloor = -n$. Let $\{a_{i_0}\} = r$. We will prove by induction that for each nonnegative integer k,

$$a_{i_0+k} \;=\; -\left(\frac{n^2 + (n - nr - r)(-n)^k}{n+1}\right)\,. \tag{1}$$

The equation (1) is immediate if $k = 0$. For the inductive step, assuming (1), we obtain

$$\begin{aligned} a_{i_0+k+1} &= -n(a_{i_0+k}) + n) \\ &= -n^2 + n\left(\frac{n^2 + (n - nr - r)(-n)^k}{n+1}\right) \\ &= -\left(\frac{n^2 + (n - nr - r)(-n)^{k+1}}{n+1}\right), \end{aligned}$$

thus completing the proof of (1) by induction.

If $n > 1$, then we obtain from (1) that $\lim\limits_{k\to\infty} |a_{i_0+k}| = \infty$. But this contradicts our hypothesis that $\lfloor a_i \rfloor = -n$ for $i \geq i_0$. Consequently, $n = 1$. Then $a_{i_0} = -1+r$, $a_{i_0+1} = -r = -1+(1-r)$, and $a_{i_0+2} = -(1-r) = a_{i_0}$, which completes the proof.

A4. Let a_1, a_2, $\ldots$, a_n be positive real numbers. Prove that

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \sum_{i<j} a_i a_j .$$

Solution by Michel Bataille, Rouen, France.

Equality holds for $n = 2$, so we will suppose that $n \geq 3$. We claim that if a, b, c are positive real numbers, then

$$\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \leq \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) . \qquad (1)$$

Proof: Observe that $\frac{2}{x+y}$ is the harmonic mean of $\frac{1}{x}$ and $\frac{1}{y}$ (for positive x, y). Using the AM–HM Inequality, it follows that

$$\frac{2}{b+c} + \frac{2}{c+a} + \frac{2}{a+b} \leq \frac{\frac{1}{b}+\frac{1}{c}}{2} + \frac{\frac{1}{c}+\frac{1}{a}}{2} + \frac{\frac{1}{a}+\frac{1}{b}}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} .$$

Back to the problem. The proposed inequality is equivalent to

$$(a_1 + a_2 + \cdots + a_n) \sum_{i<j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2} \sum_{i<j} a_i a_j , \qquad (2)$$

whose left-hand side is $\sum\limits_{i<j} a_i a_j + \sum\limits_{i<j} \frac{a_i a_j}{a_i + a_j}\left(\sum\limits_{k\neq i,j} a_k\right) = \sum\limits_{i<j} a_i a_j + L$.

Thus, (2) is equivalent to

$$L \leq \frac{n-2}{2} \sum_{i<j} a_i a_j . \qquad (3)$$

Note that $L = \sum\limits_{i,j,k\le n} a_i a_j a_k \left(\frac{1}{a_i+a_j} + \frac{1}{a_j+a_k} + \frac{1}{a_k+a_i}\right)$, the sum taken over distinct positive integers i, j, k. By (1) above,

$$\begin{aligned} L &\le \sum_{1\le i<j<k\le n} \frac{a_i a_j a_k}{2}\left(\frac{1}{a_i} + \frac{1}{a_j} + \frac{1}{a_k}\right) \\ &= \sum_{1\le i<j<k\le n} \left(\frac{a_i a_j}{2} + \frac{a_j a_k}{2} + \frac{a_k a_i}{2}\right) = \sum_{1\le r<s\le n} (n-2)\frac{a_r a_s}{2}\,, \end{aligned}$$

and (3) is obtained, completing the proof.

A5. Let a, b, and c be the lengths of the sides of a triangle. Prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \le 3\,.$$

Solution by Titu Zvonaru, Comăneşti, Romania.

Let $x=\sqrt{a}$, $y=\sqrt{b}$, $z=\sqrt{c}$.

By the well-known inequality $(\alpha+\beta+\gamma)^2 \le 3\big(\alpha^2+\beta^2+\gamma^2\big)$ we obtain :

$$\left(\sum_{\text{cyclic}} \frac{\sqrt{-x^2+y^2+z^2}}{-x+y+z}\right)^2 \le 3\sum_{\text{cyclic}} \frac{-x^2+y^2+z^2}{(-x+y+z)^2}\,.$$

The following inequalities are then equivalent

$$\sum_{\text{cyclic}} \frac{-x^2+y^2+z^2}{(-x+y+z)^2} \le 3 \iff \sum_{\text{cyclic}}\left(1 - \frac{-x^2+y^2+z^2}{(-x+y+z)^2}\right) \ge 0$$

$$\iff \sum_{\text{cyclic}} \frac{2x^2-2xy-2xz+2yz}{(-x+y+z)^2} \ge 0 \iff \sum_{\text{cyclic}} \frac{(x-y)(x-z)}{(-x+y+z)^2} \ge 0\,.$$

To prove the last inequality above, we may assume that $x\ge z$, $y\ge z$. After some algebra, we have :

$$\begin{aligned} &\frac{(x-y)(x-z)}{(-x+y+z)^2} + \frac{(y-z)(y-x)}{(x-y+z)^2} \\ &= (x-y)\cdot\frac{(x-z)(x-y+z)^2-(y-z)(-x+y+z)^2}{(-x+y+z)^2(x-y+z)^2} \\ &= (x-y)^2\cdot\frac{(x-y)^2+2z(x+y-2z)+z^2}{(-x+y+z)^2(x-y+z)^2} \ge 0\,, \end{aligned}$$

because $x + y \geq 2z$. Also, $\frac{(z-x)(z-y)}{(-z+x+y)^2} \geq 0$, since $x \geq z$ and $y \geq z$. Therefore, the last of the preceding equivalent inequalities is true, hence the original inequality is true.

Equality holds if and only if $x = y = z$, that is $a = b = c$.

C1. There are $n \geq 2$ lamps L_1, L_2, ..., L_n arranged in a row. Each of them is either *on* or *off*. Initially the lamp L_1 is on and all of the other lamps are off. Each second the state of each lamp changes as follows : if the lamp L_i and its neighbours (L_1 and L_n each have one neighbor, any other lamp has two neighbours) are in the same state, then L_i is switched off ; otherwise, L_i is switched on. Prove that there are

(a) infinitely many n for which all of the lamps will eventually be off,

(b) infinitely many n for which the lamps will never be all off.

Solved by Oliver Geupel, Brühl, NRW, Germany.

First, we prove by mathematical induction that, if $n = 2^k$ where k is a positive integer, all lamps are on after $n - 1$ steps, while not all lamps have equal states after each of the steps 1, 2, ..., $n - 2$.

This is clear for $k = 1$. Let $n = 2^k$ and assume the leftmost $m = \frac{n}{2}$ lamps are on after $m - 1$ steps. This is valid, as the m rightmost lamps do not change during the first $m - 1$ steps. After step m the two middle lamps are on and the other lamps are off. Afterwards, the on/off-pattern is symmetric about the centre, and from then on the two middle lamps will have the same state. Hence, the state of L_m is only affected by L_{m-1}, so the state of $(L_m, L_{m-1}, \ldots, L_1)$ after step $m+j$ coincides with that of $(L_1, L_2, \ldots, L_m)$ after step j. Hence, after $n-1$ steps the m leftmost lamps are again on, and by symmetry so are the m rightmost lamps. Moreover, not all the lamps are in the same state before then. The induction is complete.

Thus, for $n = 2^k$, all lamps are off after n steps, completing part (a).

For part (b), we claim that the lamps will never all be in the same state if $n = 2^k + 1$. To see this, note that after $n - 2$ steps the lamps L_1, L_2, ..., L_{n-1} are on and L_n is off. Moreover, by the analysis for part (a), the lamps are never all in the same state before then. After step $n - 1$, the lamps L_{n-1} and L_n are on and the rest are off. At this moment, the state of $(L_n, L_{n-1}, \ldots, L_1)$ coincides with that of $(L_1, L_2, \ldots, L_n)$ after the first step. Therefore, the state of $(L_1, L_2, \ldots, L_n)$ after step $j \geq 1$ is the same as that of $(L_n, L_{n-1}, \ldots, L_1)$ after step $n - 2 + j$. Hence, the sequence of states is eventually periodic with minimal period $2(n-2)$, and the lamps are never all in the same state.

C6. Let $\mathcal{P}$ be a convex polyhedron with no parallel edges and no edge parallel to a face other than the two faces it borders. A pair of points on $\mathcal{P}$ are *antipodal* if there exist two parallel planes each containing one of the points and such that $\mathcal{P}$ lies between them. Let A be the number of antipodal pairs of vertices and let B be the number of antipodal pairs of midpoints of edges. Express $A - B$ in terms of the numbers of vertices, edges, and faces of $\mathcal{P}$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let F_1, F_2, $\ldots$, F_f be the faces of $\mathcal{P}$ and n_1, n_2, $\ldots$ n_f the respective outward normal unit vectors. Consider a graph $\mathcal{G}$ on the unit sphere $\mathcal{S}$ centred at the origin O. The vertices of $\mathcal{G}$ are the end points V_1, V_2, $\ldots$, V_f of the position vectors n_1, n_2, $\ldots$, n_f. An edge V_iV_j is drawn as an arc on a great circle of $\mathcal{S}$ if and only if the faces F_i and F_j of $\mathcal{P}$ have a common edge. The graph $\mathcal{G}$ is dual to $\mathcal{P}$, in that each vertex V, each edge E, and each face F of $\mathcal{P}$ corresponds to a unique face $d(V)$, edge $d(E)$, and vertex $d(F)$, respectively, of $\mathcal{G}$. Let $\mathcal{P}$ have f faces, e edges, and v vertices, then $\mathcal{G}$ has v faces, e edges, and f vertices.

Let σ be the reflection of $\mathcal{S}$ with respect to the point O. Then $\mathcal{G}$ is mapped to another graph $\sigma(\mathcal{G}) = \mathcal{G}'$. Finally we merge $\mathcal{G}$ and $\mathcal{G}'$ into a new graph, $\tilde{\mathcal{G}}$. The vertices of $\tilde{\mathcal{G}}$ are the vertices of $\mathcal{G} \cup \mathcal{G}'$ and the points of intersection of edges of $\mathcal{G}$ with edges of $\mathcal{G}'$. The edges of $\tilde{\mathcal{G}}$ are all segments of edges of $\mathcal{G} \cup \mathcal{G}'$.

Consider the planes π that contain an edge E of $\mathcal{P}$ bordering faces F_i and F_j. The outward normal unit vectors of these planes π all lie on the great circle of $\mathcal{S}$ containing $d(E)$ in $\mathcal{G}$. The plane π does not intersect the interior of $\mathcal{P}$ if and only if its outward normal unit vector is on the edge (that is, arc) $d(E)$. Parallel edges of $\mathcal{P}$ correspond to arcs on the same great circle of $\mathcal{S}$. An edge E is parallel to a face F in $\mathcal{P}$ if and only if $d(E)$ and the vertex $d(F)$ are on the same great circle of $\mathcal{S}$. By hypothesis, all edges of $\tilde{\mathcal{G}}$ are on distinct great circles, and no vertex of $\tilde{\mathcal{G}}$ lies on the same great circle as a non-adjacent edge.

The edges E_i and E_j of $\mathcal{P}$ have antipodal midpoints if and only if there are planes π_i and π_j containing E_i and E_j, respectively, and with opposite normal vectors, that is, if $d(E_i)$ and $\sigma\big(d(E_j)\big)$ intersect on $\mathcal{S}$. Hence, $\tilde{\mathcal{G}}$ has a total of $\tilde{v} = 2f + 2B$ vertices. Each of the $2B$ vertices splits 2 edges ; hence $\tilde{\mathcal{G}}$ has $\tilde{e} = 2e + 4B$ vertices.

Consider a plane π containing a vertex V of $\mathcal{P}$ if and only if its outward normal unit vector is in the face of $\mathcal{G}$ on $\mathcal{S}$ bordered by the edges $d(E_1)$, $d(E_2)$, $\ldots$, $d(E_k)$. Thus, vertices V_i and V_j of $\mathcal{P}$ are antipodal if and only if the faces $d(V_i)$ and $\sigma\big(d(V_j)\big)$ are non-disjoint on $\tilde{\mathcal{G}}$. Therefore, the number of faces of $\tilde{\mathcal{G}}$ is $\tilde{f} = 2A$.

By Euler's polyhedral formula, we obtain

$$\begin{aligned} 0 &= \tilde{v} + \tilde{f} - \tilde{e} - 2 \;=\; 2f + 2B + 2A - 2e - 4B - 2 \\ &= 2(A - B) + 2(f - e - 1) \;=\; 2(A - B) + 2(1 - v)\,. \end{aligned}$$

Consequently, $A - B = v - 1$ and $A - B$ depends only on the number of vertices of $\mathcal{P}$.

That completes the *Corner* for this issue. Send me your nice solutions and generalizations.

BOOK REVIEWS

Amar Sodhi

When Less is More : Visualizing Basic Inequalities
By Claudi Alsina and Roger Nelsen,
Mathematical Association of America, 2009
ISBN 978-0-88385-342-9, hardcover, 164 pages, US$58.95
Reviewed by ***Bruce Shawyer****, Memorial University of Newfoundland, St. John's, NL*

I have always (at least, since my High School days) been a great believer in the power of Geometry to lead one into understanding Mathematics.

Let no one ignorant of Geometry enter here.

Tradition has it that this phrase was engraved at the door of Plato's Academy, the school he had founded in Athens. Proclus tells us about 750 years later that Ptolemy Soter, the first King of Egypt and the founder of the Alexandrian Museum, patronized the Museum by studying geometry there under Euclid. He found the subject difficult and one day asked his teacher if there was not some easier way to learn the material. To this Euclid replied,

> *Oh King, in the real world there are two kinds of roads, roads for the common people to travel upon and roads reserved for the King to travel upon. In Geometry there is no royal road.*

Claudi Alsina and Roger Nelsen's book expounds the principle that many inequalities become much more apparent when they can be visualized. Of course, Roger Nelsen is famous for his "Proofs without Words" that are wonderful examples of this principle.

To quote Charles Ashbacher :

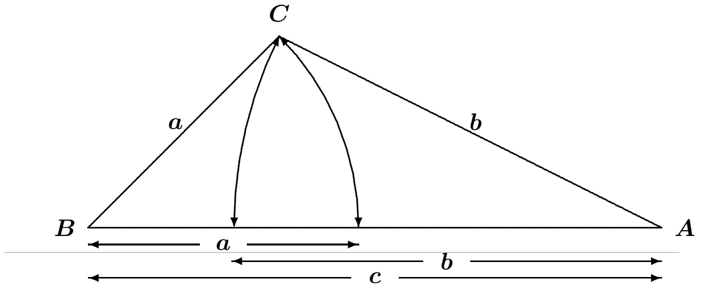
> *Proofs without words will not work everywhere, but when they do, it can be the difference that makes the light bulb of understanding burn bright.*

This book takes a range of inequalities from the simplest and best known to the more complicated and somewhat obscure. The exposition is well presented with easy to follow diagrams and there are challenges to the reader in every section.

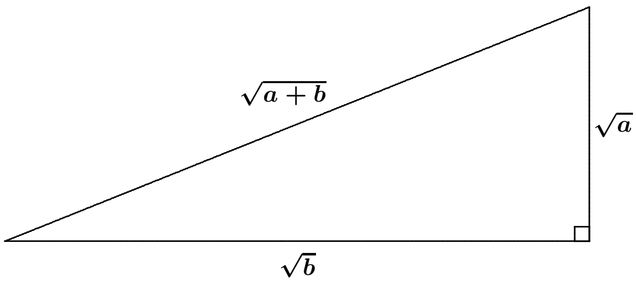
I should like to illustrate an example to give a flavour of what to expect.

The subadditive property of the square root function.

First, we show that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides.



Now, we take a right triangle with legs of lengths $\sqrt{a}$ and $\sqrt{b}$. Use of the Pythagorean Theorem gives the length of the hypotenuse as $\sqrt{a+b}$.



An application of the first result shows that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$.

This book should be in the personal library of everyone who teaches Mathematics where inequalities come into the curriculum. And this means that almost everyone who teaches Mathematics should own this book.

I Want to be a Mathematician, A Conversation with Paul Halmos
A DVD produced and directed by George Csicsery, Zala Films,
for the Mathematical Association of America, 2009
ISBN 978-0-88385-909-4, Running time : 44 minutes, US$39.95
Reviewed by ***Brenda Davison****, Simon Fraser University, Burnaby, BC*

In the DVD, 'I Want to be a Mathematician', Paul Halmos (1916-2006) is interviewed late in his career by Peter Renz, primarily about his teaching. Interspersed are interviews with several people who worked or studied with Halmos, notably an emotional Don Sarason whom Halmos considered to be his best student.

The title of the DVD follows from the title of Halmos' book *I Want to be a Mathematician, an Automathography in Three Parts* but the content does not. Where the book is really an excellent meditation on one person's journey from chemical engineering to philosophy, ending in mathematics, the DVD focuses primarily on the teaching of mathematics. For Halmos, the best teaching consisted of the Moore method with softened edges. The Moore method is a discovery based teaching technique where the teacher does not tell but rather asks and leads students to discover the ideas for themselves. Perhaps the reason for this emphasis is that, in the end, Halmos considers himself to be primarily an expositor of mathematics — his teaching and his textbooks are what he is most proud of.

The technical aspects of the DVD are good : the lighting, camera motion, the speed of the presentation, the Bach violin background music make for an easy to watch, easy to follow introduction to Halmos.

The DVD should appeal to two groups of people : 1) those starting their teaching career, after having already decided to make mathematics their subject and who are reflecting on how best to convey mathematical content to their students, and 2) those who are deciding whether or not to take on mathematics as an undergraduate or graduate major.

This second group will not be served solely by the DVD. For this group, the function of the DVD will be to provide a short, easy introduction to Halmos which can then be followed by reading some or all of the book of the same title. The transition is made easy by the inclusion of several long excerpts from the book on the DVD. These can be read on a Windows or Macintosh based computer as DVD-ROM content and will allow a quick and inexpensive look into Halmos' book prior to committing to purchasing it. I would most definitely recommend that someone considering mathematics as a career buy and read the book. Halmos is honest about himself, his profession, the people around him and he is particularly careful not to present himself as a finished, polished package. This provides the reader with the opportunity to glimpse a mathematician in the making.

Someone interested in studying in the United States will also benefit from discussion that spans the last 70 years of American mathematical activity in such geographical dispersed areas as the Institute for Advanced Study in Princeton, Syracuse, Indiana and Santa Clara Universities, and the Universities of Michigan, Hawaii and Illinois.

The well-written book has the added advantage that the mathematics that Halmos produced is discussed in an accessible way. Furthermore, many different topics are touched upon as Halmos changed his focus from one area of mathematics to another several times throughout his career.

So, watch the DVD and allow it to lead you to the book.

On an Inequality from the IMO 2008

Nikolai Nikolov and Svilena Hristova

The following problem is from the IMO 2008 :

Problem 2(a) (IMO 2008) Prove that $\dfrac{x^2}{(1-x)^2} + \dfrac{y^2}{(1-y)^2} + \dfrac{z^2}{(1-z)^2} \geq 1$ for all real numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

Replacing x, y, z respectively by $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, the inequality becomes

$$\frac{1}{(1-x)^2} + \frac{1}{(1-y)^2} + \frac{1}{(1-z)^2} \geq 1 .$$

The aim of this note is to show that this inequality remains true for three or more variables. More precisely, we have the following.

Proposition 1 Let $n \geq 2$ be an integer and let $x_1, x_2, \dots, x_n$ be real numbers, each different from 1, and satisfying $x_1 x_2 \cdots x_n = 1$. Let $S_n = \sum\limits_{i=1}^{n} \dfrac{1}{(1-x_i)^2}$.

(a) If $n = 2$, then $S_n \geq \frac{1}{2}$, with equality if and only if $x = y = -1$.

(b) If $n = 3$, then $S_n \geq 1$, with equality if and only if $x + y + z = 3$.

(c) If $n = 4$, then $S_n \geq 1$, with equality if and only if $x = y = z = t = -1$.

(d) If $n \geq 5$, then $S_n > 1$. The inequality is sharp.

Proof : Clearing the fractions in (a), the inequality becomes $x^2 + y^2 \geq 2xy$, that is, $(x - y)^2 \geq 0$.

Clearing the fractions in (b), the inequality becomes $(x+y+z-3)^2 \geq 0$.

To prove (c) and (d), we shall use the following result (see [2], and also the remark at the end of [1]) :

> If $y_1, y_2, \dots, y_n$ are positive real numbers, $1 - n \leq \alpha < 0$, and $\prod\limits_{i=1}^{n} y_i = \lambda^n$, then $\sum\limits_{i=1}^{n} (1 + y_i)^\alpha \geq \min\{1,\ n(1+\lambda)^\alpha\}$. The inequality is sharp, with equality if and only if $n(1+\lambda)^\alpha \leq 1$ and $y_1 = y_2 = \cdots = y_n = \lambda$.

This result implies (c) and (d) by setting $\alpha = -2$, $\lambda = 1$, $y_i = |x_i|$, and using the fact that $\dfrac{1}{(1-x_i)^2} \geq \dfrac{1}{(1+y_i)^2}$. ∎

A more direct approach for proving (c) and (d) is to use the inequality $\dfrac{1}{(1+a)^2} + \dfrac{1}{(1+b)^2} \geq \dfrac{1}{1+ab}$ for a, $b \geq 0$. This inequality holds since it is equivalent to the obvious inequality $(ab-1)^2+ab(a-b)^2 \geq 0$. Then (c) follows immediately :

$$\sum_{i=1}^{4} \frac{1}{(1-x_i)^2} \geq \frac{1}{1+|x_1x_2|} + \frac{1}{1+|x_3x_4|} = 1\,.$$

To prove (d), it is enough to observe that

$$\sum_{i=1}^{n} \frac{1}{(1-x_i)^2} \geq \sum_{i=1}^{n} \frac{1}{\big(1+|x_i|\big)^2} \geq$$

$$\frac{1}{1+|x_1x_2|} + \sum_{i=3}^{n} \frac{1}{\big(1+|x_i|\big)^2} \geq \frac{1}{\big(1+|x_1x_2|\big)^2} + \sum_{i=3}^{n} \frac{1}{\big(1+|x_i|\big)^2}$$

and then apply induction on n.

More is true when all the variables are positive.

Proposition 2 With notation as in Proposition 1, if additionally $x_1, \ldots, x_n$ are positive, then $\sum\limits_{i=1}^{n} \frac{1}{(1-x_i)^2} > 1$. The inequality is sharp.

Proof : For $n = 2$, by clearing fractions, the inequality becomes $x + y > 2$, that is, $\left(\sqrt{x} - \sqrt{y}\right)^2 > 0$. It remains only to note that $x = y$ implies that $x = y = 1$.

For $n = 3$ we know that strict inequality holds in Proposition 1(b) when $(x+y+z-3)^2 > 0$. In the present case it then suffices to observe that $x + y + z \geq 3\sqrt[3]{xyz} = 3$, with equality if and only if $x = y = z = 1$.

If $n \geq 4$, the inequality follows from Proposition 1, parts (c) and (d).

Finally, to see that the inequality is sharp, set $x_1 = \cdots = x_{n-1} = j$, $x_n = \frac{1}{j^{n-1}}$, and let $j \to \infty$. ∎

References

[1] O. Mushkarov and N. Nikolov, Some generalizations of an inequality from IMO 2001, ***CRUX Mathematicorum with Mathematical Mayhem***, Vol. 28, No. 5 (2002) pp. 308-312.

[2] O. Mushkarov and N. Nikolov, Variations on an inequality from IMO 2001, *Mathematics and Education in Mathematics*, Vol. 32 (2003) pp. 323-327. (`arXiv:math.HO/0605380v1`)

Nikolai Nikolov — Svilena Hristova

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
1113 Sofia, Bulgaria

`nik@math.bas.bg` — `svilenajh@yahoo.com`

PROBLEMS

*Solutions to problems in this issue should arrive no later than **1 August 2010**. An asterisk $(\star)$ after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3501. *Proposed by Hassan A. ShahAli, Tehran, Iran.*

Let $\mathbb{N}$ be the set of positive integers, E the set of all even positive integers, and O the set of all odd positive integers. A set $S \subseteq \mathbb{N}$ is *closed* if $x + y \in S$ for all distinct x, $y \in S$, and *unclosed* if $x + y \notin S$ for all distinct x, $y \in S$. Prove that if $\mathbb{N}$ is partitioned into A and B, where A is closed and nonempty, and B is unclosed and infinite, then $A = E$ and $B = O$.

3502. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find all real solutions of the following system of equations

$$\begin{aligned} x_1^2 + \sqrt{x_2^2 + 21} &= \sqrt{x_2^2 + 77}\,, \\ x_2^2 + \sqrt{x_3^2 + 21} &= \sqrt{x_3^2 + 77}\,, \\ \cdots \quad &\cdots \quad \cdots \\ x_n^2 + \sqrt{x_1^2 + 21} &= \sqrt{x_1^2 + 77}\,. \end{aligned}$$

3503. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Given a triangle and the midpoints of its sides, with the use of a straight edge and only three uses of a pair of compasses, bisect all three angles of the triangle.

3504. *Proposed by Mariia Rozhkova, Kiev, Ukraine.*

Given triangle ABC, set $Q = a\cos^2 A + b\cos^2 B + c\cos^2 C$, and let ABC have area S and circumradius R. Prove that

(a) $Q \geq \dfrac{S}{R}$, with equality if and only if ABC is equilateral.

(b) $Q \leq \dfrac{S\sqrt{2}}{R}$ if ABC is not obtuse, with equality if and only if ABC is an isosceles right triangle.

3505. *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

The circles Γ_1 and Γ_2 have a common centre O, and Γ_1 lies inside Γ_2. The point $A \neq O$ lies inside Γ_1 and a ray through A intersects Γ_1 and Γ_2 at the points B and C, respectively. Let E be a point on the line BC such that DE is perpendicular to BC. Prove that $AB = EC$ if and only if OA is perpendicular to BC.

3506. *Proposed by Pedro Henrique O. Pantoja, UFRN, Brazil.*

Prove that $Q(n) + Q\big(n^2\big) + Q\big(n^3\big)$ is a perfect square for infinitely many positive integers n that are not divisible by 10, where $Q(n)$ is the sum of the digits of n.

3507. *Proposed by Pham Huu Duc, Ballajura, Australia.*

Let a, b, and c be positive real numbers. Prove that

$$\sqrt{\frac{a(b+c)}{a^2+bc}} + \sqrt{\frac{b(c+a)}{b^2+ca}} + \sqrt{\frac{c(a+b)}{c^2+ab}} \leq \sqrt{2(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)}\,.$$

3508. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, c, d be nonnegative real numbers such that $a+b+c+d = 4$. Prove that

$$a\sqrt{bc} + b\sqrt{cd} + c\sqrt{da} + d\sqrt{ab} \ \leq\ 2\big(1+\sqrt{abcd}\big)\,.$$

3509. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a + b + c = 3$. For each positive real number k, find the maximum value of

$$\big(a^2b+k\big)\,\big(b^2c+k\big)\,\big(c^2a+k\big)\ .$$

3510. *Proposed by Cosmin Pohoaţă, Tudor Vianu National College, Bucharest, Romania.*

Let d be a line exterior to a given circle Γ with centre O. Let A be the orthogonal projection of O on the line d, M be a point on Γ, and X, Y be the intersections of Γ, d with the circle Γ' of diameter AM. Prove that the line XY passes through a fixed point as M moves about Γ.

3511. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, and d be nonnegative real numbers. Prove that

$$\prod_{\text{cyclic}} \left(a^2+b^2+c^2\right) \;\le\; \frac{1}{64}(a+b+c+d)^8\,.$$

3512. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let α be a real number and let $p \ge 1$. Find

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{n^p+(\alpha-1)k^{p-1}}{n^p-k^{p-1}}\,.$$

3513. *Proposed by Hassan A. ShahAli, Tehran, Iran.*

Let α and β be positive real numbers, and r be a positive rational number. Prove that there exist infinitely many integers m and n such that

$$\frac{\lfloor m\alpha\rfloor}{\lfloor n\beta\rfloor} \;=\; r\,,$$

where $\lfloor x\rfloor$ is the greatest integer not exceeding x.

. . . .

3501. *Proposé par Hassan A. ShahAli, Téhéran, Iran.*

Soit $\mathbb{N}$ l'ensemble des nombres entiers positifs, $E \subseteq \mathbb{N}$ l'ensemble de ceux qui sont pairs, et $O \subseteq \mathbb{N}$ l'ensemble de ceux qui sont impairs. On dit qu'un ensemble $S \subseteq \mathbb{N}$ est *fermé* si $x+y \in S$ pour tous les x, $y \in S$ distincts, et *non-fermé* si $x+y \notin S$ pour tous les x, $y \in S$ distincts. Montrer que si $\mathbb{N}$ est partagé en A et B, où A est fermé et non vide, et B est non-fermé et infini, alors $A = E$ et $B = O$.

3502. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Trouver toutes les solutions réelles du système d'équations suivant

$$\begin{array}{rcl} x_1^2+\sqrt{x_2^2+21} & = & \sqrt{x_2^2+77}\,,\\ x_2^2+\sqrt{x_3^2+21} & = & \sqrt{x_3^2+77}\,,\\ \cdots & \cdots & \cdots\\ x_n^2+\sqrt{x_1^2+21} & = & \sqrt{x_1^2+77}\,. \end{array}$$

3503. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Etant donné un triangle et les points milieux de ses côtés, construire les bissectrices de ses trois angles avec la règle et trois utilisations du compas.

3504. *Proposé par Mariia Rozhkova, Kiev, l'Ukraine.*

Dans un triangle donné ABC, d'aire S et dont le rayon du cercle circonscrit est R, on pose $Q = a\cos^2 A + b\cos^2 B + c\cos^2 C$. Montrer que

(a) $Q \geq \frac{S}{R}$, l'égalité ayant lieu si et seulement si ABC est équilatéral.

(b) $Q \leq \frac{S\sqrt{2}}{R}$ si ABC est non obtus, l'égalité ayant lieu si et seulement si ABC est un triangle rectangle isocèle.

3505. *Proposé par Yakub N. Aliyev, Université de Qafqaz, Khyrdalan, Azerbaïdjan.*

Les cercles Γ_1 et Γ_2 ont un centre commun O, et Γ_1 est à l'intérieur de Γ_2. Par le point $A \neq O$, situé à l'intérieur de Γ_1, on trace un rayon coupant respectivement Γ_1 et Γ_2 aux points B et C. Soit E un point de la droite BC tel que DE soit perpendiculaire à BC. Montrer que $AB = EC$ si et seulement si OA est perpendiculaire à BC.

3506. *Proposé par Pedro Henrique O. Pantoja, UFRN, Brésil.*

Montrer que $Q(n) + Q\left(n^2\right) + Q\left(n^3\right)$ est un carré parfait pour une infinité d'entiers positifs n qui ne sont pas divisibles par 10, où $Q(n)$ dénote la somme des chiffres de n.

3507. *Proposé par Pham Huu Duc, Ballajura, Australie.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\sqrt{\frac{a(b+c)}{a^2+bc}} + \sqrt{\frac{b(c+a)}{b^2+ca}} + \sqrt{\frac{c(a+b)}{c^2+ab}} \leq \sqrt{2(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)}\,.$$

3508. *Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit a, b, c et d des nombres réels non négatifs tels que $a+b+c+d = 4$. Montrer que

$$a\sqrt{bc} + b\sqrt{cd} + c\sqrt{da} + d\sqrt{ab} \;\leq\; 2\left(1+\sqrt{abcd}\right).$$

3509. *Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit a, b et c trois nombres réels non négatifs tels que $a+b+c=3$. Pour tout nombre réel positif k, trouver la valeur maximale de

$$\left(a^2b+k\right)\left(b^2c+k\right)\left(c^2a+k\right) \,.$$

3510. *Proposé par Cosmin Pohoaţă, Collège National Tudor Vianu, Bucarest, Roumanie.*

Soit d une droite extérieure à un cercle donné Γ de centre O. Soit A la projection orthogonale de O sur la droite d, M un point sur Γ, et X, Y les intersections de Γ et d avec le cercle Γ' de diamètre AM. Montrer que la droite XY passe par un point fixe lorsque M parcourt Γ.

3511. *Proposé par Pham Van Thuan, Université de Science de Hanoï, Hanoï, Vietnam.*

Soit a, b, c et d quatre nombres réels non négatifs. Montrer que

$$\prod_{\text{cyclique}}\left(a^2+b^2+c^2\right) \;\leq\; \frac{1}{64}(a+b+c+d)^8 \,.$$

3512. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit α un nombre réel et soit $p \geq 1$. Trouver

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{n^p+(\alpha-1)k^{p-1}}{n^p-k^{p-1}} \,.$$

3513. *Proposé par Hassan A. ShahAli, Tehran, Iran.*

Soit α et β deux nombres réel positifs, et soit r un nombre rationnel positif. Montre qu'il existe une infinité d'entiers positifs m et n tels que

$$\frac{\lfloor m\alpha\rfloor}{\lfloor n\beta\rfloor} \;=\; r \,,$$

où $\lfloor x\rfloor$ dénote le plus grand entier ne dépassant pas x.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3401. [2009 : 42, 44] *Proposed by Tigran Sloyan, Basic Gymnasium of SEUA, Yerevan, Armenia.*

Let $ABCDE$ be a convex pentagon such that $\angle BAC = \angle EAD$ and $\angle BCA = \angle EDA$, and let the lines CB and DE intersect in the point F. Prove that the midpoints of CD, BE, and AF are collinear.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor.

Convexity is not required here. We shall prove that the midpoints of CD, BE, and AF are collinear when ABC and AED are oppositely similar triangles that share the vertex A (and $F = CB \cap DE$). Our argument requires two easy lemmas.

Lemma 1 If $PQRS$ is a quadrilateral for which $\angle PQR = \angle PSR$ and the midpoint M of PR lies between Q and S on QS, then $PQRS$ is a parallelogram.

Proof. Let us call S' the point of the line QM for which M is the midpoint of QS', and show that $S' = S$. Because its diagonals bisect one another, $PQRS'$ is necessarily a parallelogram, whence, $\angle PS'R = \angle PQR = \angle PSR$. But there can only be one point on the ray from Q toward M that can be the vertex of this angle, whence $S' = S$ and $PQRS$ is a parallelogram. ∎

Lemma 2 When all the points P on BC are related by a similarity to all the points P' on $B'C'$ *(that is, $B'C' : BC = B'P' : BP$)*, then the midpoints of PP' are collinear.

Proof. Let B'', C'', P'' be the midpoints of the segments BB', CC', PP'. Translate B', C', P' to B, C_1', P_1' and denote by C_1'', P_1'' the midpoints of CC_1', PP_1'. Because PP_1' cuts the sides BC and BC_1' of triangle BCC_1' proportionally, it follows that $CC_1' \| PP_1'$; consequently, the midpoints C_1'' and P_1'' are collinear with the vertex B. Because $C_1''C''$ is parallel to and half the length of $C_1'C'$, which is parallel and equal to BB', it follows that $BC_1''C''B''$ is a parallelogram. Similarly for $BP_1''P''B''$. Since B, C_1'', P_1'' are collinear, so are B'', C'', P''. ∎

Comment. Lemma 2 is a special case of a classical theorem : *Given two directly similar figures in the plane, the points that divide the line segments joining corresponding points of the two figures in the same ratio form a figure that is directly similar to them.* See, for example, F. G.-M., Exercices de

Géométrie—comprenant l'exposé des méthodes géométriques et 2000 questions résolues, sixième édition, J. De Gigord, Paris, 1920, Paragraph 1146d, pages 473-474, whose proof was used above. In the lemma, our given figures are lines, and the ratio is $1 : 1$. Note that as an immediate consequence of the general theorem, one can continuously transform any figure into any directly similar figure in the plane in such a way that the shape never changes and corresponding points move along straight lines.

We turn now to the given oppositely similar triangles ABC and AED. We assume that the midpoints of CD and BE are distinct; otherwise there is nothing to prove. The lines BC and ED play the roles of BC and $B'C'$ of Lemma 2 — P will move along BC while P' moves along ED in such a way that the triangles ABP and AEP are directly similar; in particular, $\angle APF = \angle AP'F$ for all positions of P. By Lemma 2, the midpoint of PP' moves along the line joining the midpoints of CD and BE. There will be a unique position of P on BC where PP' contains the midpoint of AF. At this position $APFP'$ is a parallelogram by Lemma 1; there the midpoint of AF coincides with the midpoint of PP', and therefore it lies on the line joining the midpoints of CD and BE.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvarez Cubero, Priego de Córdoba, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There was one incorrect submission.

Most of the submitted solutions used coordinates. Geupel found the problem in the 2005 Mathlinks *internet forum,* `www.mathlinks.ro/viewtopic.php?t=38041`*, where there is a nice synthetic proof from someone who goes by the name of "Armo".*

3402. [2009 : 42, 44] *Proposed by Mihály Bencze, Brasov, Romania.*

Let D and E be the midpoints of the sides AB and AC in triangle ABC, respectively. Prove that CD is perpendicular to BE if and only if

$$5BC^2 \ = \ AC^2 + AB^2 \,.$$

I. *Solution by Dung Nguyen Manh, High School of HUS, Hanoi, Vietnam.*

Let $G = BE \cap CD$, and $a = BC$, $b = CA$, $c = AB$, $m_b = BE$, $m_c = CD$. Using the Pythagorean Theorem, its converse, and Stewart's theorem, we have

$$\begin{aligned} CD \perp BE &\iff BG^2 + CG^2 \ = \ BC^2 \\ &\iff \left(\frac{2}{3}m_b\right)^2 + \left(\frac{2}{3}m_c\right)^2 \ = \ a^2 \\ &\iff 4m_b^2 + 4m_c^2 = 9a^2 \\ &\iff 2\left(c^2 + a^2\right) - b^2 + 2\left(a^2 + b^2\right) - c^2 \ = \ 9a^2 \\ &\iff b^2 + c^2 \ = \ 5a^2 \,, \end{aligned}$$

as desired.

II. *Solution by Joe Howard, Portales, NM, USA.*

From Problem 5 of ***CRUX with MAYHEM*** [2003 : 375, 377], in quadrilateral $BCED$ the diagonals CD and BE are perpendicular if and only if $BC^2 + DE^2 = BD^2 + CE^2$. Since D and E are midpoints of their respective sides, the perpendicularity of the two lines is equivalent to

$$BC^2 + \left(\tfrac{1}{2}BC\right)^2 \;=\; \left(\tfrac{1}{2}AB\right)^2 + \left(\tfrac{1}{2}AC\right)^2 \,,$$

or $5BC^2 = AC^2 + AB^2$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania (two solutions); JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; ALBERT STADLER, Herrliberg, Switzerland; VASILE TEODOROVICI, Toronto, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Most of the submitted solutions were similar to one of the featured solutions.

3403. [2009 : 42, 44] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

The circles Γ_1 and Γ_2 intersect at P and Q. A line ℓ through P intersects Γ_1 and Γ_2 for the second time at A and B, respectively. The tangents to Γ_1 and Γ_2 at A and B intersect at C. If O is the circumcentre of $\triangle ABC$ determine the locus of O when ℓ rotates about P.

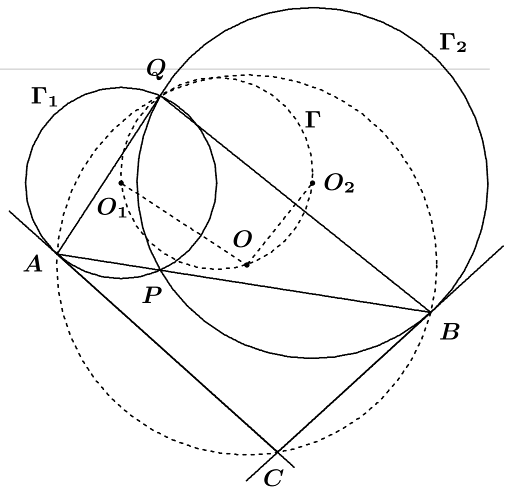
Solution by Michel Bataille, Rouen, France.

Let O_1 and O_2 be the centres of Γ_1 and Γ_2, respectively, and Γ be the circumcircle of the triangle QO_1O_2. We show that the required locus is $\Gamma - \{Q, O_1, O_2\}$. (See the figure on the next page.)

Let σ denote the spiral similarity with centre Q transforming O_1 into O_2. Then, $\sigma(\Gamma_1) = \Gamma_2$ and it follows that $\sigma(A) = B$ (a known property). We exclude the cases when $\triangle ABC$ is degenerate, that is, when ℓ either is the line PQ (in which case $A = B = Q$) or is tangent to Γ_1 or Γ_2 at P (in which case $B = C$ or $A = C$).

Since the lines CA and CB are perpendicular to O_1A and O_2B, we also have $\sigma(CA) = CB$.

Thus, we have $\angle(CA, CB) = \theta = \angle(QA, QB) \pmod{\pi}$, where θ is the angle of σ, so that Q is on the circumcircle of $\triangle ABC$. Note that we certainly have $O \neq O_1$, O_2 (if $O = O_1$, say, then $O_1B = O_1A$, hence $B = P$ or Q, which has been excluded) and the lines OO_1 and OO_2 are the perpendicular bisectors of QA and QB, respectively. Thus,



$$\angle(OO_1, OO_2) = \angle(QA, QB) = \angle(QO_1, QO_2) \pmod{\pi}$$

and finally, O, Q, O_1, O_2 are concyclic.

Conversely, let $O \neq Q$, O_1, O_2 be any point on the circle (QO_1O_2). Let the perpendicular to OO_1 through Q meet Γ_1 again at A and the perpendicular to OO_2 through Q meet Γ_2 again at B. Then, $\angle(QA, QB) = \angle(OO_1, OO_2) \pmod{\pi}$, hence $\sigma(A) = B$ (since $\sigma(\Gamma_1) = \Gamma_2$) and it follows that A, P, B are collinear on a line ℓ. The circumcentre of $\triangle QAB$ is O (because OO_1 and OO_2 are the perpendicular bisectors of QA and QB; note that $O_1A = O_1Q$ and $O_2B = O_2Q$). Moreover $\sigma(CA) = CB$ (since $\sigma(AO_1) = BO_2$ and $CA \perp AO_1$, $CB \perp BO_2$), hence $\angle(QA, QB) = \angle(CA, CB)$ and the circle (QAB) passes through C. Thus, O is the circumcentre of $\triangle ABC$.

Also solved by RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Geupel noted that problem G4 on the IMO 2002 short list essentially generalizes the given problem (he gave the reference D. Djukić et al., The IMO Compendium, *Springer 2006, pages 319 and 692 for the problem and solution, respectively).*

3404. [2009 : 42, 45] *Proposed by Michel Bataille, Rouen, France.*

Let Q be a cyclic quadrilateral. The perpendiculars to each diagonal through its endpoints form a parallelogram, P. Characterize the centre of P and show that opposite sides of Q intersect on a diagonal of P.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let A, B, C, and D be the vertices of Q in cyclic order.

First, we prove that the centre of P is the circumcentre of Q. The centre of P is the point of intersection of the perpendicular bisectors of the line segments AC and BD. But AC and BD are chords of the circumcircle of Q. Hence, their perpendicular bisectors intersect at the centre of this circle.

It remains to prove that opposite sides of Q intersect on a diagonal of P. Let E, F, G, and H be the vertices of P such that A, B, C, and D lie on HE, EF, FG, and GH, respectively. We show that the lines AD, BC, and EG are concurrent. (The proof that the lines AB, CD, and FH are concurrent is similar.)

Since $\angle AHD = \angle BFC$ and

$$\angle DAH = 90^\circ - \angle CAD = 90^\circ - \angle CBD = \angle CBF,$$

we conclude that triangles ADH and BCF are similar. Hence,

$$\frac{DH}{AH} = \frac{CF}{BF}.$$

Similarly,

$$\frac{AE}{DG} = \frac{BE}{CG}.$$

Let the lines AD and BC meet the line EG at points I and J, respectively. Using Menelaus' thereom for $\triangle EGH$ and $\triangle EGF$, we obtain

$$\frac{IE}{IG} = \frac{DH \cdot AE}{DG \cdot AH} = \frac{CF \cdot BE}{CG \cdot BF} = \frac{JE}{JG}.$$

Consequently, $I = J$, which shows that the lines AD, BC, and EG are concurrent.

If we assume that parallel lines intersect at infinity, then the result remains true when one or both pairs of opposite sides of Q are parallel (Q is then an isosceles trapezium or rectangle, respectively). If opposite sides of Q are parallel, then they are also parallel to a diagonal of P, in which case they both intersect the diagonal at infinity.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3405. [2009 :42, 45] *Proposed by Michel Bataille, Rouen, France.*

Find the minimum value of

$$|\cos\alpha| + |\cos\beta| + |\cos\gamma| + |\cos(\alpha-\beta)| + |\cos(\beta-\gamma)| + |\cos(\gamma-\alpha)|,$$

where α, β, and γ are real numbers.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We prove that the minimum value is 2. Suppose that x, y, and z are real numbers such that $x+y+z=0$. Then using the triangle inequality as well as the trigonometric addition formulas, we obtain

$$\begin{aligned} |\cos x| \;+\; |\cos y| \;+\; |\cos z| & \\ \geq\; & |\cos x| \;+\; |\cos y \sin z + \sin y \cos z| \\ =\; & |\cos x| \;+\; |\sin(y+z)| \\ \geq\; & |\cos x \cos(y+z) - \sin x \sin(y+z)| \\ =\; & |\cos(x+y+z)| \;=\; 1 \end{aligned}$$

Choosing successively $(-\alpha, \beta, \alpha-\beta)$, $(-\beta, \gamma, \beta-\gamma)$, $(-\gamma, \alpha, \gamma-\alpha)$, and $(\alpha-\beta, \beta-\gamma, \gamma-\alpha)$ for (x, y, z), yields

$$\begin{aligned} |\cos\alpha| \;+\; |\cos\beta| \;+\; |\cos(\alpha-\beta)| \;&\geq\; 1\,, \\ |\cos\beta| \;+\; |\cos\gamma| \;+\; |\cos(\beta-\gamma)| \;&\geq\; 1\,, \\ |\cos\gamma| \;+\; |\cos\alpha| \;+\; |\cos(\gamma-\alpha)| \;&\geq\; 1\,, \\ |\cos(\alpha-\beta)| \;+\; |\cos(\beta-\gamma)| \;+\; |\cos(\gamma-\alpha)| \;&\geq\; 1\,. \end{aligned}$$

By adding up, we conclude that

$$|\cos\alpha| + |\cos\beta| + |\cos\gamma| + |\cos(\alpha-\beta)| + |\cos(\beta-\gamma)| + |\cos(\gamma-\alpha)| \geq 2\,.$$

The minimum is achieved when $(\alpha, \beta, \gamma) = \left(0, \frac{\pi}{2}, \frac{3\pi}{2}\right)$, among other values.

Also solved by ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer

3406. [2009 : 43, 45] *Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find

$$\lim_{n\to\infty} \ln\left[\frac{1}{2^n}\prod_{k=1}^{n}\left(2+\frac{k}{n^2}\right)\right]\,.$$

Independent solutions by Michel Bataille, Rouen, France and Alberto Arenas Gómez, student, University of La Rioja, Logroño, Spain.

Let $A_n = \ln\left[\frac{1}{2^n}\prod_{k=1}^{n}\left(2+\frac{k}{n^2}\right)\right]$. Then we have

$$A_n \;=\; \ln\left(\prod_{k=1}^{n}\left(1+\frac{k}{2n^2}\right)\right) \;=\; \sum_{k=1}^{n}\ln\left(1+\frac{k}{2n^2}\right)\,.$$

Using the known inequalities $\frac{x}{1+x} \leq \ln(1+x) \leq x$ valid for positive x, and setting $x = \frac{k}{2n^2}$, we obtain

$$\frac{k}{2n^2+k} \;=\; \frac{\frac{k}{2n^2}}{1+\frac{k}{2n^2}} \;\leq\; \ln\left(1+\frac{k}{2n^2}\right) \;\leq\; \frac{k}{2n^2}\,. \tag{1}$$

For each $k = 1, 2, \ldots, n$ we have

$$1 + \frac{k}{2n^2} \leq 1 + \frac{n}{2n^2} = \frac{2n+1}{2n}$$

and (1) yields

$$\frac{2n}{2n+1}\left(\frac{k}{2n^2}\right) \leq \ln\left(1 + \frac{k}{2n^2}\right) \leq \frac{k}{2n^2}\,.$$

Summing over k and using the identity $\sum\limits_{k=1}^{n} k = \dfrac{n(n+1)}{2}$ yields

$$\frac{n+1}{2(2n+1)} \leq A_n \leq \frac{1}{4}\cdot\frac{n+1}{n}\,.$$

Finally, by the Squeeze Theorem, we have that $\lim\limits_{n\to\infty} A_n = \dfrac{1}{4}$.

Also solved by ARKADY ALT, San Jose, CA, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There was one incomplete solution submitted.

Stan Wagon, Macalester College, St. Paul, MN, USA, submitted a computer generated solution.

3407. [2009 : 43, 45] *Proposed by Roy Barbara, Lebanese University, Fanar, Lebanon.*

Let S be a set of positive integers containing the integer 2007 and such that

(a) If x, $y \in S$ and $x \neq y$, then $|x - y| \in S$, and

(b) If $x \in S$, then $\left(x^3 - 1007x + 3007\right) \in S$.

Prove that S is the set of all positive integers.

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Since $2007 \in S$, we have that $2007^3 - 1007 \cdot 2007 + 3007 \in S$. By subtracting 2007 enough times, we find that $1000 \in S$.

Thus, by property (a), we also find that $1007 = 2007 - 1000 \in S$ and that $7 = 1007 - 1000 \in S$.

Since $6 = 1000 - 7 \cdot 142$, we also obtain $6 \in S$. Thus $1 = 7 - 6 \in S$.

We define $x_1 = 2007$ and $x_{i+1} = x_i^3 - 1007x_i + 3007$. Then

$$\lim_{i\to\infty} x_i = \infty\,.$$

[*Ed.: Since* $x_1 = 2007$*, it is obvious that* $x_{i+1} > x_i > 2007$*, hence* x_i *is an increasing sequence of integers.*]

Now let n be any positive integer. We know that there exists an i such that $x_i \in S$ and $x_i \geq n$. Then, by subtracting 1 from x_i enough times, we find $n \in S$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; JOEL SCHLOSBERG, Bayside, NY, USA; DIGBY SMITH, Mount Royal College, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; VASILE TEODOROVICI, Toronto, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Hess, Smith, and the proposer pointed out that part (b) contained a minor error, since for $x = 4, 5, \ldots, 30$ *the cubic* $x^3 - 1007x + 3007$ *takes negative integer values. This can be remedied by rephrasing (b) as: If* $x \in S$*, then* $\left|x^3 - 1007x + 3007\right| \in S$.

3408. [2009 : 43, 45] *Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia.*

Let $\{c_i\}_{i=1}^{\infty}$ be a sequence of distinct positive integers, and let $|q| < 1$. Prove that the inequality

$$\frac{\displaystyle\sum_{i=1}^{\infty} c_i q^{c_i}}{1 + \displaystyle\sum_{i=1}^{\infty} q^{c_i}} \leq \frac{q}{1-q}$$

holds for all such sequences $\{c_i\}_{i=1}^{\infty}$ if and only if $q \in \left[0, \frac{1}{2}\right]$.

Solution by Michel Bataille, Rouen, France, modified by the editor.

First, suppose that $q \in \left[0, \frac{1}{2}\right]$ and let $\{c_i\}_{i=1}^{\infty}$ be a sequence of distinct positive integers. The inequality is equivalent to

$$(1-q)\sum_{i=1}^{\infty} c_i q^{c_i} \;\leq\; q + \sum_{i=1}^{\infty} q^{c_i+1}\,. \tag{1}$$

Both sides are 0 if $q = 0$, so we suppose that $0 < q \leq \frac{1}{2}$. Henceforth, we will also suppose that $c_1 < c_2 < c_3 < \cdots$, since the series involved are absolutely convergent and therefore may be rearranged. Rewriting the left side of (1) as $c_1 q^{c_1} + \sum_{i=1}^{\infty}\left(c_{i+1}q^{c_{i+1}} - c_i q^{c_i+1}\right)$, we see it suffices to prove

(a) $c_1 q^{c_1} \leq q$ and (b) $c_{i+1}q^{c_{i+1}} - c_i q^{c_i+1} \leq q^{c_i+1}$ $(i = 1, 2, \ldots)$.

Now, (a) is equivalent to $q^{c_1-1} \le \frac{1}{c_1}$, which holds since $q^{c_1-1} \le \frac{1}{2^{c_1-1}}$ and $2^{n-1} \ge n$ for each positive integer n. As for inequality (b), we rewrite it as

$$q^{c_{i+1}-c_i-1} \;\le\; \frac{c_i+1}{c_{i+1}}\,.$$

We have $c_{i+1} - c_i - 1 \ge 0$, so it suffices to prove that

$$\begin{aligned} \frac{1}{2^{c_{i+1}-c_i-1}} &\le \frac{c_i+1}{c_{i+1}}\,, \quad \text{or} \\ \frac{c_{i+1}}{2^{c_{i+1}}} &\le \frac{c_i+1}{2^{c_i+1}}\,. \end{aligned}$$

The latter holds because $c_{i+1} \ge c_i + 1 \ge 2 > \frac{1}{\ln 2}$, and $f(x) = \frac{x}{2^x}$ is decreasing on the interval $\left[\frac{1}{\ln 2}, \infty\right)$.

Next, suppose $-1 < q < 0$. Let $c_i = 2i$, for each $i = 1, 2, \ldots$. Observe that the left-hand side of the original inequality is positive, while the right-hand side of the original inequality is negative, a contradiction. Thus, the inequality does not hold in this case for all admissible sequences $\{c_i\}_{i=1}^{\infty}$.

Lastly, suppose $\frac{1}{2} < q < 1$. Let $c_i = i+1$ for each $i = 1, 2, \ldots$. Then, $c_1 q^{c_1} = 2q^2 > q$. Observing that $c_{i+1} = c_i + 1$ for each i, we deduce that

$$c_1 q^{c_1} + \sum_{i=1}^{\infty} c_{i+1} q^{c_{i+1}} \;>\; q + \sum_{i=1}^{\infty} (c_i+1) q^{c_i+1}\,,$$

and so (1) does not hold. This completes the proof.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; KEE-WAI LAU, Hong Kong, China; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There were two incomplete solutions submitted.

3409. [2009 : 43, 45] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, c, and d be positive real numbers. Prove that

$$\begin{aligned} &\frac{ab+bc+ca}{a^3+b^3+c^3} + \frac{ab+bd+da}{a^3+b^3+d^3} + \frac{ac+cd+da}{a^3+c^3+d^3} + \frac{bc+cd+db}{b^3+c^3+d^3} \\ &\quad\le\; \min\left\{ \frac{a^2+b^2}{(ab)^{3/2}} + \frac{c^2+d^2}{(cd)^{3/2}},\, \frac{a^2+c^2}{(ac)^{3/2}} + \frac{b^2+d^2}{(bd)^{3/2}},\, \frac{a^2+d^2}{(ad)^{3/2}} + \frac{b^2+c^2}{(bc)^{3/2}} \right\}. \end{aligned}$$

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

By the AM–GM Inequality, $a^3+b^3+c^3 \geq 3abc$, so that

$$\frac{ab+bc+ca}{a^3+b^3+c^3} \leq \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

with analogous inequalities holding for the other three terms on the left side of the claimed inequality. Hence,

$$\frac{ab+bc+ca}{a^3+b^3+c^3}+\frac{ab+bd+da}{a^3+b^3+d^3}+\frac{ac+cd+da}{a^3+c^3+d^3}+\frac{bc+cd+db}{b^3+c^3+d^3}$$
$$\leq \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right). \qquad (1)$$

The AM–GM and AM–QM inequalities imply that $(ab)^{1/2} \leq \sqrt{\frac{a^2+b^2}{2}}$ and $a+b \leq 2\sqrt{\frac{a^2+b^2}{2}}$, respectively. Multiplying across these inequalities yields $(ab)^{1/2}(a+b) \leq a^2+b^2$. Hence,

$$\frac{1}{a}+\frac{1}{b} = \frac{a+b}{ab} \leq \frac{a^2+b^2}{(ab)^{3/2}}.$$

Analogous inequalities again hold for the other pairs of variables, thus

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$
$$\leq \min\left\{\frac{a^2+b^2}{(ab)^{3/2}}+\frac{c^2+d^2}{(cd)^{3/2}}, \frac{a^2+c^2}{(ac)^{3/2}}+\frac{b^2+d^2}{(bd)^{3/2}}, \frac{a^2+d^2}{(ad)^{3/2}}+\frac{b^2+c^2}{(bc)^{3/2}}\right\}.$$

The desired inequality now follows from the above inequality and (1).

Also solved by MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer. One incorrect solution was submitted.

A computer generated solution totaling 228 pages was submitted, which due to its length and complexity could not be verified in the available time.

3410. [2009 : 43, 46] *Proposed by Joe Howard, Portales, NM, USA.*

Let a, b, and c be the sides of triangle ABC, let R be its circumradius, and let F be its area. Prove that

$$\sum_{\text{cyclic}} \frac{bc\sin^2 A/2}{b+c} \geq \frac{F}{2R}.$$

Similar solutions by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let r and s denote the inradius and semiperimeter of ΔABC, respectively. By the Law of Cosines, $\dfrac{b^2+c^2-a^2}{2bc} = \cos A = 2\cos^2\dfrac{A}{2} - 1$, hence

$$bc\sin^2\frac{A}{2} \;=\; s(s-a)\tan^2\frac{A}{2}\,.$$

Since $F = rs$, it follows that the proposed inequality is equivalent to

$$\sum_{\text{cyclic}} \frac{b+c-a}{b+c}\tan^2\frac{A}{2} \;\geq\; \frac{r}{R}\,.$$

Using the Law of Sines, we have

$$\begin{aligned} \frac{b+c-a}{b+c} &= \frac{\sin B + \sin C - \sin A}{\sin B + \sin C} = \frac{4\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}}{2\cos\frac{A}{2}\cos\frac{B-C}{2}} \\ &\geq 2\sin\frac{B}{2}\sin\frac{C}{2}\,. \end{aligned}$$

Now, using the last inequality and the well-known and easy to prove identity $r = 4R\sin\dfrac{B}{2}\sin\dfrac{C}{2}\sin\dfrac{A}{2}$, we have

$$\frac{b+c-a}{b+c}\tan^2\frac{A}{2} \;\geq\; \left(2\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}\right)\cdot\frac{\sin\frac{A}{2}}{\cos^2\frac{A}{2}} \;=\; \frac{r}{2R}\cdot\frac{\sin\frac{A}{2}}{\cos^2\frac{A}{2}}\,.$$

Thus, it suffices to prove that

$$\sum_{\text{cyclic}} \frac{\sin\frac{A}{2}}{\cos^2\frac{A}{2}} \;\geq\; 2\,.$$

To this end, consider $f(x) = \dfrac{\sin x}{\cos^2 x}$ for $x \in \left(0, \frac{\pi}{2}\right)$. An easy calculation yields $f''(x) = (\cos x)^{-4}(\sin x)\left(5 + \sin^2 x\right) > 0$, so that f is a convex function on the interval $\left(0, \frac{\pi}{2}\right)$. From Jensen's inequality,

$$f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right) \;\geq\; 3f\left(\frac{A/2+B/2+C/2}{3}\right)\,,$$

that is,

$$\sum_{\text{cyclic}} \frac{\sin\frac{A}{2}}{\cos^2\frac{A}{2}} \;\geq\; 3\cdot\frac{\sin\left(\frac{\pi}{6}\right)}{\cos^2\left(\frac{\pi}{6}\right)} \;=\; 2\,,$$

which completes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvarez Cubero, Priego de Córdoba, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; TITU ZVONARU, Comănești, Romania; and the proposer.

3411. [2009 :44, 46] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers such that

$$a^6+b^6+c^6 \;<\; \frac{32}{33}\left(a^3+b^3+c^3\right)^2 \,.$$

Prove that at least one of the quadratics ax^2+bx+c, bx^2+cx+a, or cx^2+ax+b has no real roots.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA, modified by the editor.

We prove that in fact the conclusion holds for all positive real numbers a, b, and c. Suppose that each quadratic has at least one real root. Then we have $a^2 \ge 4bc$, $b^2 \ge 4ac$, and $c^2 \ge 4ab$. Multiplying these inequalities we obtain $a^2b^2c^2 \ge 64a^2b^2c^2$, which is a contradiction.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; D.J. SMEENK, Zaltbommel, the Netherlands; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3412. [2009 :44, 46] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers such that $abc = 1$. Prove that

$$\sum_{\text{cyclic}} \frac{1}{\sqrt{a^3+2b^3+6}} \;\le\; 1\,.$$

Solution by Dung Nguyen Manh, High School of HUS, Hanoi, Vietnam.

Let x, y, z be positive real numbers such that $\frac{x}{y}=a$, $\frac{y}{z}=b$, $\frac{z}{x}=c$. By using the AM–GM Inequality we obtain

$$\begin{aligned} a^3+2b^3+6 &= \left(a^3+b^3+1\right)+\left(b^3+1+1\right)+3 \\ &\ge 3(ab+b+1) \;=\; 3\left(\frac{x}{z}+\frac{y}{z}+1\right) \;=\; \frac{3(x+y+z)}{z}\,. \end{aligned}$$

Hence, $\dfrac{1}{\sqrt{a^3+2b^3+6}} \le \dfrac{\sqrt{z}}{\sqrt{3(x+y+z)}}$. Adding up the cyclic variants of this inequality, we obtain

$$\sum_{\text{cyclic}} \frac{1}{\sqrt{a^3+2b^3+6}} \;\le\; \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{3(x+y+z)}}\,. \qquad (1)$$

On the other hand, by the Cauchy–Schwarz Inequality we have

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \;\le\; \sqrt{3(x+y+z)}\,. \qquad (2)$$

The result now follows from (2) and (1).

Equality holds if and only if $a=b=c=1$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3413. [2009 : 44, 46] *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Let a, b, c, and d be real numbers in the interval $[1,2]$. Prove that

$$\frac{a+b}{c+d}+\frac{c+d}{a+b}-\frac{a+c}{b+d} \le \frac{3}{2}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany, modified by the editor.

Since the inequality is invariant under the permutation

$$\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix},$$

then without loss of generality we can also assume that $b \le d$.

Let $f(a,b,c,d) = \dfrac{a+b}{c+d}+\dfrac{c+d}{a+b}-\dfrac{a+c}{b+d}$. We need to prove that $f(a,b,c,d) \le \dfrac{3}{2}$ for all a, b, c, d in the interval $[1,2]$ with $b \le d$. We will prove more generally that $f(a,b,c,d) \le \dfrac{3}{2}$ in the region

$$D \;=\; \left\{(a,b,c,d)\in\mathbb{R}^4 \;:\; 1\le b\le d\le 2 \text{ and } \frac{d}{2}\le a,c\le 2b\right\}.$$

Since the function f is continuous and the region D is compact, f attains a maximum in D.

For fixed b and d, the function f has positive partial derivatives

$$\frac{\partial^2 f}{\partial a^2} = \frac{2(c+d)}{(a+b)^3}\,; \qquad \frac{\partial^2 f}{\partial c^2} = \frac{2(a+b)}{(c+d)^3}\,,$$

and therefore it is convex for a, $c > 0$. Thus, f attains its maximum in D at a point with $(a, c) \in \left\{ \left(\frac{d}{2}, \frac{d}{2}\right), \left(\frac{d}{2}, 2b\right), \left(2b, \frac{d}{2}\right), (2b, 2b)\right\}$.

By multiplying each side of the inequality by $8(c+d)(a+b)(b+d)$, we see that proving the inequality amounts to proving that

$$\begin{aligned} g(a,c) \;=\; & 8b^3 + 8d^3 + 8a^2b - 8a^2c + 16ab^2 - 8ac^2 \\ & - 12ad^2 - 12b^2c - 4b^2d - 4bd^2 + 8c^2d + 16cd^2 \\ & - 20abc - 4abd - 20acd - 4bcd \;\le\; 0 \end{aligned}$$

for $(a, c) \in \left\{ \left(\frac{d}{2}, \frac{d}{2}\right), \left(\frac{d}{2}, 2b\right), \left(2b, \frac{d}{2}\right), (2b, 2b) \right\}$ and $1 \le b \le d \le 2$.

We have

$$\begin{aligned} g\left(\frac{d}{2}, \frac{d}{2}\right) &= 8b^3 - 2b^2d - 11bd^2 + 5d^3 \;=\; (b-d)(2b-d)(4b+5d) \;\le\; 0\,, \\ g\left(\frac{d}{2}, 2b\right) &= -16b^3 - 8b^2d + 4bd^2 + 2d^3 \;=\; 2(d-2b)(2b+d)^2 \;\le\; 0\,, \\ g\left(2b, \frac{d}{2}\right) &= 72b^3 - 54b^2d - 54bd^2 + 18d^3 \\ &= 18\Big[(d^2(d-2b) + b(b-d)(4b+d)\Big] \;\le\; 0\,, \\ g(2b, 2b) &= -160b^3 - 68b^2d + 4bd^2 + 8d^3 \\ &= 8\Big[d^3 - (2b)^3\Big] + 4bd(d-2b) - 60b^2d - 96b^3 \;\le\; 0\,, \end{aligned}$$

and the inequality follows.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece, ALBERT STADLER, Herrliberg, Switzerland, and the proposer. There were three incomplete solutions submitted.

3414. [2009 : 108, 111] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

As triangle ABC varies, its circumcircle $\gamma_1(O, R)$ and its incircle $\gamma_2(I, r)$ are fixed, where O and I are the respective centres and R and r are the respective radii. Find the locus of the orthocentre H of triangle ABC.

Solution by Michel Bataille, Rouen, France.

We show that the locus of H is the circle $\gamma_3(J, R - 2r)$, where J is the reflection of O in the point I; note that O, I and, consequently, J are fixed while the triangle ABC varies.

(a) Let N be the centre of the nine-point (or Euler) circle, and F be the Feuerbach point (that is, the point where the nine-point circle is tangent to the incircle γ_2). Since the radii NF and IF have respective lengths $\frac{R}{2}$ and r, N must lie on the circle with centre I and radius $\frac{R}{2} - r$. Since N is the midpoint of OH while I is the midpoint of OJ, we must have JH parallel to and twice the length of IN; in other words, H must lie on the circle with centre J and radius $2\left(\frac{R}{2} - r\right) = R - 2r$.

(b) Conversely, let H be an arbitrary point of the circle $\gamma_3(J, R-2r)$. We shall use complex numbers to show that there exist points A, B, C on the circumcircle $\gamma_1(O, R)$ for which the triangle ABC has incircle $\gamma_2(I, r)$ and orthocentre H. Without loss of generality we assume that γ_1 is the unit circle (that is, $R = 1$ and O is represented by the complex number 0); moreover, we will take I on the real axis. Because Euler's formula gives $OI^2 = R^2 - 2Rr$, I will be represented by the real number $u := \sqrt{1 - 2r}$. Because we assume that H is on γ_3, it is represented by the complex number $h := 2u + u^2 e^{i\theta}$ for some real number θ. Now, let z_1, z_2, z_3 denote the complex roots of the polynomial

$$P(z) \;=\; z^3 - \left(2u + u^2 e^{i\theta}\right) z^2 + (u^2 + 2ue^{i\theta}) z - e^{i\theta} . \qquad (1)$$

Since $P(z) = z(z-u)^2 - e^{i\theta}(1 - uz)^2$, we have for $j = 1, 2, 3$,

$$\left(\frac{z_j - u}{1 - uz_j}\right)^2 \;=\; \frac{e^{i\theta}}{z_j} . \qquad (2)$$

Note that $\left|\frac{z-u}{1-uz}\right|$ is less than or greater than 1 according as $|z| < 1$ or $|z| > 1$, while $\left|\frac{e^{i\theta}}{z}\right| = \frac{1}{|z|}$; consequently, equation (2) implies that $|z_j| = 1$. Thus, the points A, B, C that correspond respectively to z_1, z_2, z_3 are on the circle $\gamma_1 = \gamma_1(O, 1)$. In addition, since $z_1 + z_2 + z_3 = h$ (from (1)), we have $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$. We recognize this last equation to be the vector formulation of the theorem that the segment AH is parallel to and twice as long as the segment joining O to the midpoint of BC, which implies that H is the orthocentre of $\triangle ABC$. It remains to prove that I is its incentre. We denote by L the second point where the line AI intersects γ_1, and represent it by the complex number ℓ; the complex equation of AI is then, $z + z_1 \ell \bar{z} = z_1 + \ell$. Because u (representing the point I) must satisfy this equation, we have

$$\ell \;=\; \frac{u - z_1}{1 - uz_1} ,$$

so that from (2), $\ell^2 = \frac{e^{i\theta}}{z_1} = \frac{z_1 z_2 z_3}{z_1} = z_2 z_3$. Thus, $2 \arg \ell = \arg z_2 + \arg z_3$, which tells us that the perpendicular bisector of BC meets γ_1 in L; it follows that $AI \equiv AL$ is one of the bisectors of $\angle BAC$. Similarly, BI and CI

are bisectors of $\angle CBA$ and $\angle ACB$, respectively. Since I is interior to the circumcircle γ_1 of $\triangle ABC$, I must be the incentre of $\triangle ABC$.

Comment. I came across the problem in the references listed below; however, the converse, treated above in part (b), was either absent from these sources or very incomplete.

References

[1] William Gallatly, *The Modern Geometry of the Triangle*, 2nd ed. Hodgson (1913).

[2] Jos.E. Hofmann, Zur elementaren Dreiecksgeometrie in der komplexen Ebene. *L'Enseignement mathématique* 4 (1958) pp. 197-199. This article has been translated into French by Lisiane Nivelle, *L'Ouvert*, **98** (2000) pp. 1-22; it is available at `http://irem.u-strasbg.fr/php/articles/98_Nivelle.pdf`.

[3] T. Lalesco, *La géometrie du triangle*. J. Gabay (2003), p. 21.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Observe that as a consequence of the argument in part (b), each point H on its circle γ_3 determines three positions for the vertex A: at z_1, z_2, and z_3; in other words, as the vertex A of the moving triangle travels once around γ_1, H will travel three times around γ_3. Woo observed that because the centroid G of $\triangle ABC$ is on the Euler line two-thirds of the way from O to N, the argument of part (a) also shows that the locus of G is a circle whose radius is two-thirds that of the locus of N, namely $\frac{2}{3}\left(\frac{R}{2} - r\right)$; its centre is the point two-thirds of the way from O to I.

Almost all solvers showed that the locus of orthocentre H was a subset of the circle $\gamma_3(J, R-2r)$ using an argument similar to part (a) of our featured solution. Geupel, however, used complex numbers; he provided the only solution other than Bataille's to address satisfactorily the converse problem of showing the locus to be all of γ_3. It would be nice if somebody could treat this converse using elementary geometry in the spirit of part (a). The proposer found the problem in the December 1912 issue of the Journal de mathématiques élémentaires *with a "rather long" proof; he does not mention if the converse problem was addressed there.*