On an Inequality from the IMO 2008

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The following problem is from the IMO 2008 :

Problem 2(a) (IMO 2008) Prove that $\frac{x^2}{(1-x)^2} + \frac{y^2}{(1-y)^2} + \frac{z^2}{(1-z)^2} \geq 1$ for all real numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

Replacing x, y, z respectively by $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, the inequality becomes

$$\frac{1}{(1-x)^2} + \frac{1}{(1-y)^2} + \frac{1}{(1-z)^2} \geq 1\,.$$

The aim of this note is to show that this inequality remains true for three or more variables. More precisely, we have the following.

Proposition 1 Let $n \geq 2$ be an integer and let $x_1, x_2, \ldots, x_n$ be real numbers, each different from 1, and satisfying $x_1 x_2 \cdots x_n = 1$. Let $S_n = \sum\limits_{i=1}^{n} \frac{1}{(1-x_i)^2}$.

(a) If $n = 2$, then $S_n \geq \frac{1}{2}$, with equality if and only if $x = y = -1$.

(b) If $n = 3$, then $S_n \geq 1$, with equality if and only if $x + y + z = 3$.

(c) If $n = 4$, then $S_n \geq 1$, with equality if and only if $x = y = z = t = -1$.

(d) If $n \geq 5$, then $S_n > 1$. The inequality is sharp.

Proof : Clearing the fractions in (a), the inequality becomes $x^2 + y^2 \geq 2xy$, that is, $(x-y)^2 \geq 0$.

Clearing the fractions in (b), the inequality becomes $(x+y+z-3)^2 \geq 0$.

To prove (c) and (d), we shall use the following result (see [2], and also the remark at the end of [1]) :

If $y_1, y_2, \ldots, y_n$ are positive real numbers, $1 - n \leq \alpha < 0$, and $\prod\limits_{i=1}^{n} y_i = \lambda^n$, then $\sum\limits_{i=1}^{n} (1+y_i)^\alpha \geq \min\{1,\ n(1+\lambda)^\alpha\}$. The inequality is sharp, with equality if and only if $n(1+\lambda)^\alpha \leq 1$ and $y_1 = y_2 = \cdots = y_n = \lambda$.

This result implies (c) and (d) by setting $\alpha = -2$, $\lambda = 1$, $y_i = |x_i|$, and using the fact that $\frac{1}{(1-x_i)^2} \geq \frac{1}{(1+y_i)^2}$. ■

A more direct approach for proving (c) and (d) is to use the inequality $\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \geq \frac{1}{1+ab}$ for a, $b \geq 0$. This inequality holds since it is equivalent to the obvious inequality $(ab-1)^2+ab(a-b)^2 \geq 0$. Then (c) follows immediately :

$$\sum_{i=1}^{4} \frac{1}{(1-x_i)^2} \geq \frac{1}{1+|x_1x_2|} + \frac{1}{1+|x_3x_4|} = 1\,.$$

To prove (d), it is enough to observe that

$$\sum_{i=1}^{n} \frac{1}{(1-x_i)^2} \geq \sum_{i=1}^{n} \frac{1}{\bigl(1+|x_i|\bigr)^2} \geq$$

$$\frac{1}{1+|x_1x_2|} + \sum_{i=3}^{n} \frac{1}{\bigl(1+|x_i|\bigr)^2} \geq \frac{1}{\bigl(1+|x_1x_2|\bigr)^2} + \sum_{i=3}^{n} \frac{1}{\bigl(1+|x_i|\bigr)^2}$$

and then apply induction on n.

More is true when all the variables are positive.

Proposition 2 With notation as in Proposition 1, if additionally $x_1, \ldots, x_n$ are positive, then $\sum\limits_{i=1}^{n} \frac{1}{(1-x_i)^2} > 1$. The inequality is sharp.

Proof : For $n=2$, by clearing fractions, the inequality becomes $x+y>2$, that is, $\left(\sqrt{x}-\sqrt{y}\right)^2 > 0$. It remains only to note that $x=y$ implies that $x=y=1$.

For $n=3$ we know that strict inequality holds in Proposition 1(b) when $(x+y+z-3)^2>0$. In the present case it then suffices to observe that $x+y+z \geq 3\sqrt[3]{xyz} = 3$, with equality if and only if $x=y=z=1$.

If $n \geq 4$, the inequality follows from Proposition 1, parts (c) and (d).

Finally, to see that the inequality is sharp, set $x_1 = \cdots = x_{n-1} = j$, $x_n = \frac{1}{j^{n-1}}$, and let $j \to \infty$. ■

References

[1] O. Mushkarov and N. Nikolov, Some generalizations of an inequality from IMO 2001, ***CRUX Mathematicorum with Mathematical Mayhem***, Vol. 28, No. 5 (2002) pp. 308-312.

[2] O. Mushkarov and N. Nikolov, Variations on an inequality from IMO 2001, *Mathematics and Education in Mathematics*, Vol. 32 (2003) pp. 323-327. (arXiv:math.HO/0605380v1)

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