MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga) and Eric Robert (Leo Hayes High School, Fredericton).

Mayhem Problems

Please send your solutions to the problems in this edition by **1 May 2009**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M376. *Proposed by the Mayhem Staff.*

Determine the value of x if $\left(10^{2009}+25\right)^2-\left(10^{2009}-25\right)^2=10^x$.

M377. *Proposed by the Mayhem Staff.*

An arithmetic sequence consists of 9 positive integers. The sum of the terms in the sequence is greater than 200 and less than 220. If the second term in the sequence is 12, determine the sequence.

M378. *Proposed by the Mayhem Staff.*

Points C and D are chosen on the semi-circle with diameter AB so that C is closer to A. Segments CB and DA intersect at P; segments AC and BD extended intersect at Q. Prove that QP extended is perpendicular to AB.

M379. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The integers $27+C$, $555+C$, and $1371+C$ are all perfect squares, the square roots of which form an arithmetic sequence. Determine all possible values of C.

M380. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Triangle ABC is right-angled at C and has $BC = a$ and $CA = b$, with $a \geq b$. Squares $ABDE$, $BCFG$, and $CAHI$ are drawn externally to triangle ABC. The lines through FI and EH intersect at P, the lines through FI and DG intersect at Q, and the lines through DG and EH intersect at R. If triangle PQR is right-angled, determine the value of $\frac{b}{a}$.

M381. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all solutions to the equation

$$\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-6} + \frac{1}{x-7} = x^2 - 4x - 4\,.$$

. . . .

M376. *Proposé par l'Équipe de Mayhem.*

Déterminer la valeur de x si $\left(10^{2009} + 25\right)^2 - \left(10^{2009} - 25\right)^2 = 10^x$.

M377. *Proposé par l'Équipe de Mayhem.*

Déterminer la suite arithmétique formée de 9 entiers positifs dont la somme se situe entre 200 et 220 et dont le second terme vaut 12.

M378. *Proposé par l'Équipe de Mayhem.*

A partir du point A, on choisit deux points C et D sur un demi-cercle de diamètre AB. Soit P l'intersection des droites CB et DA, et Q celle des droites AC et BD. Montrer que la droite PQ est perpendiculaire à AB.

M379. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

Les entiers $27 + C$, $555 + C$ et $1371 + C$ sont tous des carrés parfaits dont les racines carrées forment une suite arithmétique. Trouver toutes les valeurs possibles de C.

M380. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Dans un triangle ABC d'angle droit en C, soit $BC = a$ et $CA = b$, avec $a \geq b$. Extérieurement au triangle ABC, on construit les carrés $ABDE$, $BCFG$ et $CAHI$. Soit respectivement P, Q et R les intersections des droites FI et EH, FI et DG, DG et EH. Déterminer la valeur de $\frac{b}{a}$ pour que PQR soit un triangle rectangle.

M381. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Déterminer toutes les solutions de l'équation

$$\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-6} + \frac{1}{x-7} = x^2 - 4x - 4\,.$$

Mayhem Solutions

M338. *Proposed by the Mayhem Staff.*

Two students miscopy the quadratic equation $x^2 + bx + c = 0$ that their teacher writes on the board. Jim copies b correctly but miscopies c; his equation has roots 5 and 4. Vazz copies c correctly, but miscopies b; his equation has roots 2 and 4. What are the roots of the original equation?

Solution by Taylor Thetford, student, Lakeview High School, San Angelo, TX, USA.

The roots of the quadratic equation that Jim writes down are 5 and 4. His quadratic equation is thus $(x-5)(x-4) = x^2 - 9x + 20 = 0$. Since Jim copied b correctly, we can conclude that in the original quadratic equation, $b = -9$.

Similarly, since Vazz's roots are 2 and 4, his quadratic equation has the form $(x-2)(x-4) = x^2 - 6x + 8 = 0$. Since Vazz copied c correctly, then $c = 8$.

Thus, the original equation was $x^2 - 9x + 8 = 0$. Factoring, we obtain $(x-1)(x-8) = 0$. Therefore, the roots of the original equation are 1 and 8.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; PETER CHIEN, student, Central Elgin Collegiate, St. Thomas, ON; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; JOHAN GUNARDI, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; BILLY SUANDITO, Palembang, Indonesia; LUYAN ZHONG-QIAO, Columbia International College, Hamilton, ON; and TITU ZVONARU, Comănești, Romania.

M339. *Proposed by the Mayhem Staff.*

(a) Determine the number of integers between 100 and 199, inclusive, which contain *exactly* two equal digits.

(b) An integer between 1 and 999 is chosen at random, with each integer being equally likely to be chosen. What is the probability that the integer has *exactly* two equal digits?

Solutions by Taylor Thetford, student, Lakeview High School, San Angelo, TX, USA.

(a) Each of the integers in the given range can be written in the form $1xy$. There are three cases to consider.

Case 1. The first and last digits are the same. Here, we are looking for integers $1x1$ where the middle digit can take any value except 1. This yields 9 possibilities.

Case 2. The first and second digits are the same. Here, we are looking for integers $11y$ where the last digit can take any value except 1. This again yields 9 possibilities.

Case 3. The second and last digits are the same. Here, we are looking for integers $1xx$ where x is not 1. Again, there are 9 possibilities.

Adding the results from our three cases, we find that there are 27 numbers between 100 and 199, inclusive, that contain exactly two equal digits.

(b) We count the number of integers in the range 1 to 999, inclusive, that have exactly two equal digits.

First, between 1 and 99, there are 9 of these, namely, 11, 22, $\ldots$, 99. Next, between 100 and 199, we have counted 27 in part (a). Using the same argument as in (a), we can show that there are 27 numbers between 200 and 299, inclusive, and for every other interval of one hundred numbers up to the range of 900 to 999.

There are therefore $9 + 9 \cdot 27 = 252$ numbers between 1 and 999 which contain exactly two equal digits.

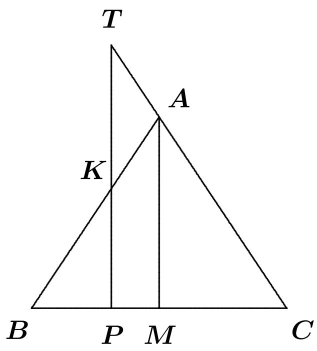
The probability that a randomly selected integer between 1 and 999 has exactly two equal digits is thus $\frac{252}{999} = \frac{28}{111}$.

Also solved by PETER CHIEN, student, Central Elgin Collegiate, St. Thomas, ON; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and LUYAN ZHONG-QIAO, Columbia International College, Hamilton, ON. There were 3 incorrect solutions and 1 partial solution submitted.

M340. *Proposed by the Mayhem Staff.*

Let ABC be an isosceles triangle with $AB = AC$, and let M be the mid-point of BC. Let P be any point on BM. A perpendicular is drawn to BC at P, meeting BA at K and CA extended at T. Prove that $PK + PT$ is independent of the position of P (that is, the value of $PK + PT$ is always the same, no matter where P is placed).

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.



Since $\triangle ABC$ is isosceles with sides AB and AC of equal length, we have $MA \perp BC$. Also, since $PT \perp BC$, then $MA||PT$.

Since $MA||PK$, then $\triangle MBA$ is similar to $\triangle PBK$ since each is right-angled and they share the angle at B. From this, we obtain $\frac{PK}{PB} = \frac{MA}{MB}$, hence $PK = \frac{PB \cdot MA}{MB}$.

Similarly, since $MA||PT$, then $\triangle CPT$ is similar to $\triangle CMA$, whence $\frac{PT}{PC} = \frac{MA}{MC}$ and so $PT = \frac{PC \cdot MA}{MC}$.

Since $MB = MC = \frac{1}{2}BC$, we can conclude that

$$\begin{aligned} PK + PT &= \frac{PB \cdot MA}{MB} + \frac{PC \cdot MA}{MC} = \frac{(PB + PC) \cdot MA}{MB} \\ &= \frac{BC \cdot MA}{MB} = 2MA\,. \end{aligned}$$

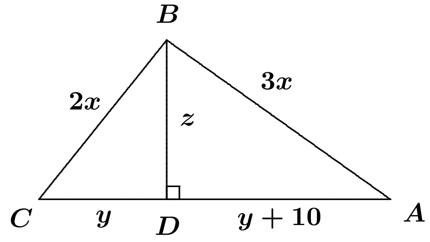
Thus, $PK + PT$ is independent of the position of P, since it depends only on the length of MA.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; BILLY SUANDITO, Palembang, Indonesia; and TITU ZVONARU, Comănești, Romania. There was 1 incorrect solution submitted.

M341. *Proposed by the Mayhem Staff.*

Let ABC be a right triangle with right angle at B. Sides BA and BC are in the ratio $3 : 2$. Altitude BD divides CA into two parts that differ in length by 10. What is the length of CA?

Solution by Taylor Thetford, student, Lakeview High School, San Angelo, TX, USA.



Let $2x$ and $3x$ be the lengths of CB and AB, respectively. Let y and $y + 10$ be the lengths of CD and DA, respectively. Let z be the length of BD. We wish to find $2y + 10$, which is the length of CA.

By applying the Pythagorean Theorem in $\triangle ABC$, we find that $(2x)^2 + (3x)^2 = (2y + 10)^2$ and so $13x^2 = 4y^2 + 40y + 100$.

Applying the Pythagorean Theorem to $\triangle BDC$ and $\triangle BDA$, we find that $y^2 + z^2 = 4x^2$ and $z^2 + (y+10)^2 = 9x^2$.

Eliminating z in the last two equations gives $4x^2 - y^2 = 9x^2 - (y+10)^2$. Therefore, $5x^2 = (y+10)^2 - y^2 = 20y + 100$ or $x^2 = 4y + 20$, and so $13x^2 = 52y + 260$.

Combining this result with $13x^2 = 4y^2 + 40y + 100$, we find that

$$\begin{aligned} 52y + 260 &= 4y^2 + 40y + 100\,; \\ 4y^2 - 12y - 160 &= 0\,; \\ y^2 - 3y - 40 &= 0\,; \\ (y+5)(y-8) &= 0\,. \end{aligned}$$

Since $y > 0$, then $y = 8$, and so $CA = 2y + 10 = 26$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; JACLYN CHANG, student, Western Canada High School, Calgary, AB; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; BILLY SUANDITO, Palembang, Indonesia; LUYAN ZHONG-QIAO, Columbia International College, Hamilton, ON; and TITU ZVONARU, Cománeşti, Romania. There was 1 incorrect solution submitted.

M342. *Proposed by the Mayhem Staff.*

Quincy and Celine have to move 10 small boxes and 10 large boxes. The chart below indicates the time that each person takes to move each type of box.

	Celine	Quincy
small box	1 min.	3 min.
large box	6 min.	5 min.

They start moving the boxes at 9:00 am. What is the earliest time at which they can be finished moving all of the boxes?

Solution by Mayhem Staff.

Let x represent the number of small boxes and y represent the number of large boxes that Celine moves. Since there are 10 small boxes and 10 large boxes, then Quincy moves $10 - x$ small boxes and $10 - y$ large boxes.

Given the lengths of time that each takes, it takes Celine $x + 6y$ minutes and it takes Quincy $3(10-x) + 5(10-y) = 80 - 3x - 5y$ minutes. If $x = 9$ and $y = 4$, then Celine takes 33 minutes and Quincy takes 33 minutes. We show that it cannot be done faster than this.

If Quincy and Celine finish in fewer than 33 minutes, then each takes at most 32 minutes, so the total working time is at most 64 minutes, so $x + 6y + (80 - 3x - 5y) = 80 - 2x + y \le 64$ or $2x - y \ge 16$.

Since x and y are nonnegative integers and each is less than 10, then the possible pairs (x, y) that satisfy this inequality are $(8,0)$, $(9,0)$, $(9,1)$, $(9,2)$, $(10,0)$, $(10,1)$, $(10,2)$, $(10,3)$, and $(10,4)$.

Since we want each of Celine's time and Quincy's time to be at most 32 minutes, then we need $x + 6y \leq 32$ and $80 - 3x - 5y \leq 32$. The first inequality eliminates the pair $(10, 4)$ from the list of possible pairs. The second inequality simplifies to $3x + 5y \geq 48$; none of the remaining pairs satisfy this inequality.

Thus, none of these possibilities take any less time than 33 minutes. Therefore, the earliest possible finishing time is 9:33 a.m.

There were 4 incorrect and 3 incomplete solutions submitted.

An expanded treatment of a similar problem appeared in the Problem of the Month column in ***CRUX with MAYHEM****, volume 34, number 2.*

M343. *Proposed by the Mayhem Staff.*

The Fibonacci numbers are defined by $f_1 = f_2 = 1$ and, for $n \geq 2$, by $f_{n+1} = f_n + f_{n-1}$. The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, $\ldots$. Find the sum of the first 100 even Fibonacci numbers.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

Since $f_1 = f_2 = 1$, $f_3 = 2$, and $f_n = f_{n-1} + f_{n-2}$, then f_m is even if and only if m is a multiple of 3. (This is because the parities of the terms will form the pattern Odd, Odd, Even, Odd, Odd, Even, and so on.)

If $S_n = \sum\limits_{k=1}^{n} f_{3k}$, then

$$S_n = \frac{1}{2}\sum_{k=1}^{n}(f_{3k}+f_{3k}) = \frac{1}{2}\sum_{k=1}^{n}\big((f_{3k-2}+f_{3k-1})+f_{3k}\big) = \frac{1}{2}\sum_{k=1}^{3n} f_k\,. \quad (1)$$

Next, we have $f_1 = f_3 - f_2$, and $f_2 = f_4 - f_3$, and also $f_3 = f_5 - f_4$, and so on until $f_{r-1} = f_{r+1} - f_r$ and $f_r = f_{r+2} - f_{r+1}$.

Since the right side of the sum of the n equations above "telescopes", it follows that

$$\sum_{k=1}^{r} f_k = f_{r+2} - f_2 = f_{r+2} - 1\,. \quad (2)$$

From (1) and (2), we find that $S_n = \frac{1}{2}(f_{3n+2} - 1)$. In our particular case, $S_{100} = \frac{1}{2}(f_{302} - 1)$. Maple computes the value of S_{100} to be exactly 290905784918002003245752779317049533129517076702883498623284700.

For the record, by Binet's formula for Fibonacci numbers we have that $f_m = \dfrac{1}{\sqrt{5}}(\alpha^m - \beta^m)$, where $\alpha = \dfrac{1}{2}(1 + \sqrt{5})$ and $\beta = \dfrac{1}{2}(1 + \sqrt{5})$. Hence the required sum is also given by $S_{100} = \dfrac{1}{2\sqrt{5}}\left(\alpha^{302} - \beta^{302}\right) - \dfrac{1}{2}$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; DIVYANSHU RANJAN, Delhi, India; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; and TITU ZVONARU, Comănești, Romania.

Problem of the Month

Ian VanderBurgh

Approximation is one of the most important concepts in mathematics.

Problem (2006 Canadian Open Mathematics Challenge) Determine, with justification, the fraction $\frac{p}{q}$, where p and q are positive integers and $q < 100$, that is closest to, but not equal to, $\frac{3}{7}$.

While it is tempting to get out your calculator, it can initially only help so much. If you calculate $\frac{3}{7}$, you'll obtain $0.428571\ldots$. This doesn't help in any obvious way to answer the question.

A first approach after the calculator might be to go for the fraction with the largest possible denominator. This makes a lot of sense in some ways, as the fractions with the largest denominators will be closest together and so would seem to have the best chance of being closest to $\frac{3}{7}$. In our case, the largest possible denominator is $q = 99$. The given fraction, $\frac{3}{7}$, is between $\frac{42}{99} = 0.424242\ldots$ and $\frac{43}{99} = 0.434343\ldots$. After a quick look, we can tell that $\frac{3}{7}$ is closer to $\frac{42}{99}$. From the decimal approximations, $\frac{3}{7}$ and $\frac{42}{99}$ differ by about 0.004. Is this the closest of all possible fractions?

Another idea is to try to convert $\frac{3}{7}$ into the equivalent fraction with the largest possible denominator and then adjust from there. Multiplying numerator and denominator by 14, we obtain $\frac{42}{98}$. We could then add 1 or -1 to the numerator to obtain $\frac{41}{98}$ or $\frac{43}{98}$, which differ from $\frac{3}{7}$ by $\frac{1}{98}$. But this means that the difference is bigger than 0.01, which is worse than before, so this approach doesn't give a closer fraction.

Can we do better than $\frac{42}{99}$? It is possible that, even though fractions with smaller denominators are further apart, they can be between some of the other fractions that we've looked at, for example between $\frac{42}{99}$ and $\frac{3}{7}$ or between $\frac{43}{99}$ and $\frac{3}{7}$.

Solution We want to use the fraction $\frac{p}{q}$ to approximate $\frac{3}{7}$. Let's calculate their difference, which is what we want to minimize:

$$\left|\frac{p}{q} - \frac{3}{7}\right| = \left|\frac{7p-3q}{7q}\right| = \frac{|7p-3q|}{7q}\,.$$

What can we do to make this as small as possible? Two approaches would be to make the numerator of the difference as small as possible or to make the denominator of the difference as large as possible.

Let's focus initially on the numerator. The numerator cannot equal 0 because the fractions $\frac{p}{q}$ and $\frac{3}{7}$ are not equal. Thus, the smallest possible value for the numerator is 1, because p and q are integers. So let's try to find values of p and q for which the numerator equals 1. In this case, the difference equals $\frac{1}{7q}$ which is minimized when q is largest.

For the numerator to equal 1, we need $7p - 3q = \pm 1$. Since we also want to maximize q, we rewrite this as $7p = 3q \pm 1$ and work from the largest possible integer values of q to see when we also get an integer value for p.

If $q = 99$, the equation becomes $7p = 3(99) \pm 1 = 297 \pm 1$. Neither possibility is a multiple of 7.

If $q = 98$, the equation becomes $7p = 3(98) \pm 1 = 294 \pm 1$. Neither possibility is a multiple of 7.

If $q = 97$, the equation becomes $7p = 3(97) \pm 1 = 291 \pm 1$. Neither possibility is a multiple of 7.

If $q = 96$, the equation becomes $7p = 3(96) \pm 1 = 288 \pm 1$. Since 287 is a multiple of 7, then taking $q = 96$ and $p = 41$ gives a difference with numerator 1.

So we have $\left|\frac{41}{96} - \frac{3}{7}\right| = \frac{1}{7 \cdot 96} = \frac{1}{672}$ and this is the smallest possible difference with the numerator equal to 1.

If the numerator equalled 2 or something larger, then the smallest possible difference occurs when the numerator is as small as possible and the denominator is as large as possible, so is $\frac{2}{7 \cdot 99} = \frac{2}{693}$. This is the smallest possible difference with numerator at least 2.

Combining the cases, the smallest possible difference is indeed $\frac{1}{672}$, and so the closest fraction to $\frac{3}{7}$ of all of the fractions under consideration is $\frac{p}{q} = \frac{41}{96}$.

The approximation of functions with polynomials is often seen in first-year university calculus courses. As part of these investigations, we learn how to estimate the amount of error when approximating, for example, $\sin x$ with $x - \frac{1}{6}x^3 + \frac{1}{120}x^5$. The techniques used to estimate this type of error are not dissimilar to what we have seen above, and are very useful in many types of calculations.