Problem of the Month

Ian VanderBurgh

Here is a problem that might seem to be not very interesting initially, but turns out to have a whole lot of unexpected solutions.

Problem (2005 Canadian Open Mathematics Challenge) In the grid shown, each row has a value assigned to it and each column has a value assigned to it. The number in each cell is the sum of its row and column values. For example, the "8" is the sum of the value assigned to the 3^{rd} row and the value assigned to the 4^{th} column. Determine the values of x and y.

3	0	5	6	-2
-2	-5	0	1	y
5	2	x	8	0
0	-3	2	3	-5
-4	-7	-2	-1	-9

It is tempting first of all to give labels to the values that are assigned to the rows and columns in order to be able to dive into some algebra. Let's label the values assigned to the five columns A, B, C, D, E and the values assigned to the five rows a, b, c, d, e.

Each entry in the table gives us an equation involving two of these variables. For example, the -3 in row 4, column 2 gives us $d + B = -3$, and the -9 in row 5, column 5, gives us $e + E = -9$. We could proceed and write down 25 equations, one for each entry in the table. These equations would include 12 variables – the 10 that label the rows and columns together with x and y. We could then spend pages and pages wading through algebra trying to come up with the answers. At this point, we would hope that there has to be a better way. Maybe we should have looked before we leapt!

Here are three neat ways to approach this. (As a point of interest, I was recently talking about this problem with a friend while driving and so neither of us really wanted to do any algebra, and so were forced to come up with better ways to do it.)

Solution 1. If we choose five entries from the table which include one from each row and one from each column, then the sum of these entries is constant no matter how we choose the entries, as it is always equal to

$$A + B + C + D + E + a + b + c + d + e\,.$$

Can you see why? Here are three ways in which this can be done (looking at the underlined numbers in the two grids below and the grid on the following page):

$\underline{3}$	0	5	6	-2
-2	$-\underline{5}$	0	1	y
5	2	x	$\underline{8}$	0
0	-3	$\underline{2}$	3	-5
-4	-7	-2	-1	$-\underline{9}$

3	0	5	6	$-\underline{2}$
-2	-5	0	$\underline{1}$	y
5	2	$\underline{x}$	8	0
0	$-\underline{3}$	2	3	-5
$-\underline{4}$	-7	-2	-1	-9

Therefore,

$$\begin{aligned} & 3+(-5)+2+8+(-9) \\ = \; & (-4)+(-3)+x+1+(-2) \\ = \; & 3+y+2+(-2)+3\,, \end{aligned}$$

$\underline{3}$	0	5	6	−2
−2	−5	0	1	$\underline{y}$
5	$\underline{2}$	x	8	0
0	−3	2	$\underline{3}$	−5
−4	−7	$-\underline{2}$	−1	−9

or $-1 = x - 8 = y + 6$. Thus, $x = 7$ and $y = -7$.

Solution 2. Consider the first two entries in row 1. From the labels above, we have $3 = A + a$ and $0 = B + a$. Subtracting these, we obtain the equation $3 = 3 - 0 = (A + a) - (B + a) = A - B$.

Notice that whenever we take entries in columns 1 and 2 from the same row, their difference will always equal $A - B$, which is equal to 3. Similarly, since the difference between the 0 and the 5 in the first row is 5, then every entry in column 3 will be 5 greater than the entry in column 2 from the same row. In row 3, we see that $x = 2 + 5 = 7$.

Also, since the difference between the 6 and the -2 in the first row is 8, then every entry in column 5 is 8 less than the entry in column 4 from the same row. In row 2, we see that $y = 1 - 8 = -7$. Thus, $x = 7$ and $y = -7$.

Solution 3. Consider the sub-grid

0	1
x	8

.

Since the 0 is in row 2 and column 3, then $0 = b + C$. Similarly, $1 = b + D$, $8 = c + D$, and $x = c + C$.

But then $0 + 8 = (b + C) + (c + D) = (c + C) + (b + D) = x + 1$, or $x = 7$.

In a similar way, by looking at the sub-grid

1	y
8	0

we can show that $1 + 0 = y + 8$, or $y = -7$. Thus, $x = 7$ and $y = -7$.

So there are three different but neat solutions to the problem. One footnote to the final solution is that in fact, in any sub-grid of the form

p	q
r	s

, we must have $p + s = q + r$. Can you see why?

Another interesting point about this problem is that it might be easier for those who know less! If we replaced the x and the y with "?" and gave it to someone who didn't know a lot of algebra, they might find the answers faster than those of us who go immediately to algebra. Sometimes, the extra machinery that we have can get in the way.

As 2008 draws to a close, the Mayhem Editor has three enormous sets of thanks to offer. First, to the Mayhem Staff, especially to Monika Khbeis and Eric Robert, for all of their help over the past year. Second, to the Editor-in-Chief of ***CRUX with MAYHEM***, Václav Linek, for all of his help and encouragement over the past year (as well as for his sharp eyes!). Third, to the Mayhem readership for their support. Please keep those problems and solutions coming!