

M372. *Proposed by the Mayhem Staff.*

A real number x satisfies $x^3 = x + 1$. Determine integers a , b , and c so that $x^7 = ax^2 + bx + c$.

M373. *Proposed by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.*

The side lengths of a triangle are three consecutive positive integers and the largest angle in the triangle is twice the smallest one. Determine the side lengths of the triangle.

M374. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that p is a fixed prime number with $p \geq 3$. Determine the number of solutions to $x^3 + y^3 = x^2y + xy^2 + p^{2009}$, where x and y are integers.

M375. *Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.*

Determine all real solutions to the system of equations

$$\frac{1}{x^2} + \frac{4}{y^2} + \frac{9}{z^2} = 4; \quad x^2 + y^2 + z^2 = 9; \quad xyz = \frac{9}{2}.$$

Mayhem Solutions

M332. *Proposed by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.*

A closed right circular cylinder has an integer radius and an integer height. The numerical value of the volume is four times the numerical value of its total surface area (including its top and bottom). Determine the smallest possible volume for the cylinder.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Let r and h be the radius and the height of the closed right circular cylinder. The volume of such a cylinder is $V = \pi r^2 h$ and the surface area is $A = 2\pi r^2 + 2\pi r h$.

From the hypotheses, $\pi r^2 h = 4(2\pi r^2 + 2\pi r h)$, or $rh = 8r + 8h$, or $rh - 8r - 8h + 64 = 64$, or $(r - 8)(h - 8) = 64$. Note that $r - 8 > -8$ and $h - 8 > -8$. This gives us the following possibilities:

$r - 8$	1	2	4	8	16	32	64
$h - 8$	64	32	16	84	4	2	1
r	9	10	12	16	24	40	72
h	72	40	24	16	12	10	9
V	5832π	4000π	3456π	4096π	6912π	16000π	46656π

Thus, the smallest possible volume for the cylinder is 3456π .

Also solved by DENISE CORNWELL, student, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; BILLY SUANDITO, Palembang, Indonesia; and TITU ZVONARU, Comănești, Romania. There was 1 incomplete solution submitted.

M333. Proposed by the Mayhem Staff.

Anne and Brenda play a game which begins with a pile of n toothpicks. They alternate turns with Anne going first. On each player's turn, she must remove 1, 3, or 6 toothpicks from the pile. The player who removes the last toothpick wins the game. For which of the values of n from 36 to 40 inclusive does Brenda have a winning strategy?

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA, modified by the editor.

We can build a table of winning and losing positions for Anne. Her winning positions start with 1, 3, or 6, since she can immediately win by removing all of the toothpicks.

Starting with 2 toothpicks, Anne must remove 1 toothpick, leaving Brenda with 1, and so Brenda wins. Starting with 4 toothpicks, Anne must remove 1 or 3 toothpicks, leaving Brenda with 3 or 1 (respectively), and so Brenda wins by removing all of the toothpicks.

Starting with 5 toothpicks, Anne can remove 3 toothpicks, thus leaving Brenda with 2 toothpicks. Since 2 is a losing position for whoever goes first, then Brenda loses, so Anne wins.

So far, 1, 3, 5, and 6 are winning positions for Anne, while 2 and 4 are losing positions for Anne.

Starting with a pile of size n , Anne must reduce the pile to one of size $n - 1$, $n - 3$, or $n - 6$ and pass to Brenda. If the person who goes first has a winning strategy starting with a pile of each of these sizes, then Anne will lose. In other words, if Anne has a winning strategy starting with piles of size $n - 1$, $n - 3$, and $n - 6$, then Anne will lose starting with a pile of size n , as Brenda can implement Anne's strategy for the smaller pile and win, no matter what Anne does. If one or more of these pile sizes are such that the first person does not have a winning strategy, then Anne should reduce to this size, thus preventing Brenda from being able to win, and so Anne herself will win.

We can examine the cases from $n = 7$ to $n = 40$, obtaining the following lists:

Winning positions for Anne: 1, 3, 5, 6, 7, 8, 10, 12, 14, 15, 16, 17, 19, 21, 23, 24, 25, 26, 28, 30, 32, 33, 34, 35, 37, 39.

Losing positions for Anne: 2, 4, 9, 11, 13, 18, 20, 22, 27, 29, 31, 36, 38, 40.

Therefore, Brenda wins for $n = 36, 38, 40$.

Also solved by JACLYN CHANG, student, Western Canada High School, Calgary, AB.

See the Problem of the Month column in [2007 : 15–17] for a similar problem with a more detailed explanation.

M334. Proposed by the Mayhem Staff.

(a) Determine all integers x for which $\frac{x-3}{3x-2}$ is an integer.

(b) Determine all integers y for which $\frac{3y^3+3}{3y^2+y-2}$ is an integer.

I. Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina.

Let A be an integer such that $A = \frac{x-3}{3x-2}$. Then $3A$ is an integer and $3A = \frac{3x-9}{3x-2} = \frac{3x-2-7}{3x-2} = 1 - \frac{7}{3x-2}$.

Thus, $\frac{7}{3x-2}$ is an integer; that is, $3x-2$ is a divisor of 7, so $3x-2$ is one of $\pm 1, \pm 7$. Since x is an integer, $x = 1$ or $x = 3$. This answers part (a).

Now let B be an integer such that

$$\begin{aligned} B &= \frac{3y^3+3}{3y^2+y-2} = y - \frac{y^2-2y-3}{3y^2+y-2} \\ &= y - \frac{(y-3)(y+1)}{(y+1)(3y-2)} = y - \frac{y-3}{3y-2}. \end{aligned}$$

Since y is an integer, $\frac{y-3}{3y-2}$ is an integer. From the solution to part (a), $y = 1$ or $y = 3$, which answers part (b).

II. Solution by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, ON.

We show that the only integer solutions to part (a) are $x = 1$ and $x = 3$.

Let $f(x) = \frac{x-3}{3x-2}$. Then $f(0) = \frac{3}{2}$, $f(1) = -2$, $f(2) = -\frac{1}{4}$, and $f(3) = 0$. Of these, only $f(1)$ and $f(3)$ are integers.

If $x > 3$, then $f(x)$ is not an integer, since $3x-2 > x-3 > 0$ for $x > 3$ and so $0 < \frac{x-3}{3x-2} < 1$.

If $x \leq -1$, let $x = -s$ where $s \geq 1$. Then $f(x) = f(-s) = \frac{s+3}{3s+2}$.

Since $3s+2 > s+3 > 0$ for $s \geq 1$, $f(-s)$ is not an integer by a similar argument so, $f(x)$ is not an integer.

Therefore, $f(x)$ is an integer for integer values of x if and only if $x = 1$ or $x = 3$.

Also solved by RICARD PEIRO, IÉS "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and TITU ZVONARU, Comănești, Romania. There was one incorrect and one incomplete solution submitted.

M335. Proposed by the Mayhem staff.

In a sequence of four numbers, the second number is twice the first number. Also, the sum of the first and fourth numbers is 9, the sum of the second and third is 7, and the sum of the squares of the four numbers is 78. Determine all such sequences.

Solution by Denise Cornwell, student, Angelo State University, San Angelo, TX, USA.

Let a , b , c , and d represent the first, second, third and fourth number, respectively. We can now write the given information as $b = 2a$, $a + d = 9$, $b + c = 7$ and $a^2 + b^2 + c^2 + d^2 = 78$.

The first three equations allow us to rewrite b , c , and d in terms of a , obtaining $b = 2a$, $c = 7 - b = 7 - 2a$, and $d = 9 - a$.

Therefore,

$$\begin{aligned} a^2 + (2a)^2 + (7 - 2a)^2 + (9 - a)^2 &= 78, \\ a^2 + 4a^2 + 49 - 28a + 4a^2 + 81 - 18a + a^2 - 78 &= 0, \\ 5a^2 - 23a + 26 &= 0, \\ (5a - 13)(a - 2) &= 0, \end{aligned}$$

hence $a = \frac{13}{5}$ or $a = 2$.

Therefore, the sequences are $a = \frac{13}{5}$, $b = \frac{26}{5}$, $c = \frac{9}{5}$, $d = \frac{32}{5}$ and $a = 2$, $b = 4$, $c = 3$, $d = 7$. Both sequences satisfy the given requirements.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; JACLYN CHANG, student, Western Canada High School, Calgary, AB; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; BILLY SUANDITO, Palembang, Indonesia; and TITU ZVONARU, Comănești, Romania. There was one incorrect and one incomplete solution submitted.

M336. Proposed by the Mayhem Staff.

A lattice point is a point (x, y) in the coordinate plane with each of x and y an integer. Suppose that n is a positive integer. Determine the number of lattice points inside the region $|x| + |y| \leq n$.

Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina, modified by the editor.

We can rewrite the given inequality as the equations $|x| + |y| = 0$ and $|x| + |y| = k$ for $1 \leq k \leq n$, where $x, y \in \mathbb{Z}$.

The equation $|x| + |y| = 0$ has one integer solution only, namely $(x, y) = (0, 0)$.

Consider next $|x| + |y| = k$, for an integer k with $1 \leq k \leq n$. We can remove the absolute values by splitting into four cases:

Case 1. The integers x and y satisfy $x + y = k$, where $x \geq 0$ and $y \geq 0$. This has solutions $(k, 0), (k-1, 1), \dots, (1, k-1), (0, k)$, for a total of $k+1$ solutions.

Case 2. The integers x and y satisfy $x - y = k$, where $x \geq 0$ and $y < 0$. This has solutions $(k-1, -1), (k-2, -2), \dots, (1, -(k-1)), (0, -k)$, for a total of k solutions.

Case 3. The integers x and y satisfy $-x + y = k$, where $x < 0$ and $y \geq 0$. This case is the same as Case 2, but with the roles of x and y switched, so there are a total of k solutions here as well.

Case 4. The integers x and y satisfy $-x - y = k$, where $x < 0$ and $y < 0$. This has solutions $(-1, -(k-1)), (-2, -(k-2)), \dots, (-(k-2), -2), (-(k-1), -1)$, for a total of $k-1$ solutions.

Thus, for each k with $1 \leq k \leq n$, the equation $|x| + |y| = k$ has $(k+1) + k + k + (k-1) = 4k$ solutions.

Therefore, the number of lattice points inside the region is

$$\begin{aligned} 1 + \sum_{k=1}^n 4k &= 1 + 4 \sum_{k=1}^n k = 1 + 4 \cdot (1 + 2 + \dots + n) \\ &= 1 + 4 \cdot \frac{n(n+1)}{2} = 2n^2 + 2n + 1. \end{aligned}$$

Also solved by RICARD PEIRO, IES "Abastos", Valencia, Spain. There were one incorrect and two incomplete solutions submitted.

M337. *Proposed by the Mayhem Staff.*

On sides AB and CD of rectangle $ABCD$ with $AD < AB$, points F and E are chosen so that $AFCE$ is a rhombus.

- If $AB = 16$ and $BC = 12$, determine EF .
- If $AB = x$ and $BC = y$, determine EF in terms of x and y .

Solution by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.

We present the solution to (b), which is a general version of the specific case in (a).

Suppose that $AF = FC = CE = EA = m$. Let O be the point of intersection of diagonals AC and EF of rhombus $AFCE$. Note that AC and EF are perpendicular and bisect each other at O .

By the Pythagorean Theorem,

$$\begin{aligned} CF^2 - FB^2 &= CB^2, \\ m^2 - (x - m)^2 &= y^2, \\ m^2 - x^2 + 2mx - m^2 &= y^2, \\ 2mx &= x^2 + y^2, \\ m &= \frac{x^2 + y^2}{2x}. \end{aligned}$$

Now, $AF = m = \frac{x^2 + y^2}{2x}$ and $AC = \sqrt{AB^2 + BC^2} = \sqrt{x^2 + y^2}$. Also, $OA = \frac{1}{2}AC$. Thus, by the Pythagorean Theorem again,

$$\begin{aligned} OF^2 &= AF^2 - OA^2 \\ &= \left(\frac{x^2 + y^2}{2x}\right)^2 - \left(\frac{\sqrt{x^2 + y^2}}{2}\right)^2 \\ &= \left(\frac{x^4 + y^4 + 2x^2y^2}{4x^2}\right) - \left(\frac{x^2 + y^2}{4}\right) \\ &= \frac{x^4 + y^4 + 2x^2y^2 - x^4 - x^2y^2}{4x^2} \\ &= \frac{y^4 + x^2y^2}{4x^2}. \end{aligned}$$

Therefore,

$$OF = \sqrt{\frac{y^4 + x^2y^2}{4x^2}} = \frac{\sqrt{y^2(y^2 + x^2)}}{2x} = \frac{y\sqrt{x^2 + y^2}}{2x},$$

and

$$EF = 2OF = \frac{2y\sqrt{x^2 + y^2}}{2x} = \frac{y\sqrt{x^2 + y^2}}{x}.$$

In part (a), this yields

$$EF = \frac{12\sqrt{16^2 + 12^2}}{16} = \frac{12(20)}{16} = 15.$$

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; JACLYN CHANG, student, Western Canada High School, Calgary, AB (part (a) only); RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; BILLY SUANDITO, Palembang, Indonesia; LUYAN ZHONG-QIAO, Columbia International College, Hamilton, ON; and TITU ZVONARU, Comănești, Romania.