

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a **Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crux Mathematicorum with Mathematical Mayhem*.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Veillez nous transmettre vos solutions aux problèmes du présent numéro avant le 15 Mars 2009. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

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M369. *Proposé par l'Équipe de Mayhem.*

Soit $A(0, 0)$, $B(6, 0)$, $C(6, 4)$ et $D(0, 4)$ les sommets d'un rectangle. Par le point $P(4, 3)$, on trace d'une part une droite horizontale coupant BC en M et AD en N et d'autre part une droite verticale coupant AB en Q et CD en R . Montrer que AP , DM et BR passent toutes par le même point.

M370. *Proposé par l'Équipe de Mayhem.*

- Montrer que $\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$ pour tous les angles A et B .
- Montrer que $\cos C + \cos D = 2 \cos\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$ pour tous les angles C et D .
- Trouver la valeur exacte de $\cos 20^\circ + \cos 60^\circ + \cos 100^\circ + \cos 140^\circ$, sans l'aide d'une calculatrice.

M371. *Proposé par Panagioté Ligouras, École Secondaire Léonard de Vinci, Noci, Italie.*

Un segment AB de longueur 3 contient un point C tel que $AC = 2$. On construit d'un même côté de AB deux triangles équilatéraux ACF et CBE . Déterminer l'aire du triangle AKE si K est le point milieu de FC .

M372. *Proposé par l'Équipe de Mayhem.*

Soit x un nombre réel satisfaisant $x^3 = x + 1$. Trouver des entiers a, b et c de sorte que $x^7 = ax^2 + bx + c$.

M373. *Proposé par Kunal Singh, étudiant, Kendriya Vidyalaya School, Shillong, Inde.*

Les côtés d'un triangle sont mesurés par trois nombres entiers consécutifs et le plus grand angle est le double du plus petit. Déterminer la longueur des côtés du triangle.

M374. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit p un nombre premier fixé, avec $p \geq 3$. Trouver le nombre de solutions de $x^3 + y^3 = x^2y + xy^2 + p^{2009}$, où x et y sont des entiers.

M375. *Proposé par Neculai Stanciu, École Technique Supérieure de Saint Mucenic Sava, Berca, Roumanie.*

Déterminer toutes les solutions réelles du système d'équations

$$\frac{1}{x^2} + \frac{4}{y^2} + \frac{9}{z^2} = 4; \quad x^2 + y^2 + z^2 = 9; \quad xyz = \frac{9}{2}.$$

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M369. *Proposed by the Mayhem Staff.*

A rectangle has vertices $A(0, 0)$, $B(6, 0)$, $C(6, 4)$, and $D(0, 4)$. A horizontal line is drawn through $P(4, 3)$, meeting BC at M and AD at N . A vertical line is drawn through P , meeting AB at Q and CD at R . Prove that AP , DM , and BR all pass through the same point.

M370. *Proposed by the Mayhem Staff.*

- (a) Prove that $\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$ for all angles A and B .
- (b) Prove that $\cos C + \cos D = 2 \cos \left(\frac{C + D}{2}\right) \cos \left(\frac{C - D}{2}\right)$ for all angles C and D .
- (c) Determine the exact value of $\cos 20^\circ + \cos 60^\circ + \cos 100^\circ + \cos 140^\circ$, without using a calculator.

M371. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Suppose that the line segment AB has length 3 and C is on AB with $AC = 2$. Equilateral triangles ACF and CBE are constructed on the same side of AB . If K is the midpoint of FC , determine the area of $\triangle AKE$.

M372. *Proposed by the Mayhem Staff.*

A real number x satisfies $x^3 = x + 1$. Determine integers a , b , and c so that $x^7 = ax^2 + bx + c$.

M373. *Proposed by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.*

The side lengths of a triangle are three consecutive positive integers and the largest angle in the triangle is twice the smallest one. Determine the side lengths of the triangle.

M374. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that p is a fixed prime number with $p \geq 3$. Determine the number of solutions to $x^3 + y^3 = x^2y + xy^2 + p^{2009}$, where x and y are integers.

M375. *Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.*

Determine all real solutions to the system of equations

$$\frac{1}{x^2} + \frac{4}{y^2} + \frac{9}{z^2} = 4; \quad x^2 + y^2 + z^2 = 9; \quad xyz = \frac{9}{2}.$$

Mayhem Solutions

M332. *Proposed by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.*

A closed right circular cylinder has an integer radius and an integer height. The numerical value of the volume is four times the numerical value of its total surface area (including its top and bottom). Determine the smallest possible volume for the cylinder.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Let r and h be the radius and the height of the closed right circular cylinder. The volume of such a cylinder is $V = \pi r^2 h$ and the surface area is $A = 2\pi r^2 + 2\pi r h$.

From the hypotheses, $\pi r^2 h = 4(2\pi r^2 + 2\pi r h)$, or $r h = 8r + 8h$, or $r h - 8r - 8h + 64 = 64$, or $(r - 8)(h - 8) = 64$. Note that $r - 8 > -8$ and $h - 8 > -8$. This gives us the following possibilities:

$r - 8$	1	2	4	8	16	32	64
$h - 8$	64	32	16	84	4	2	1
r	9	10	12	16	24	40	72
h	72	40	24	16	12	10	9
V	5832π	4000π	3456π	4096π	6912π	16000π	46656π

Thus, the smallest possible volume for the cylinder is 3456π .

Also solved by DENISE CORNWELL, student, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; BILLY SUANDITO, Palembang, Indonesia; and TITU ZVONARU, Comănești, Romania. There was 1 incomplete solution submitted.

M333. Proposed by the Mayhem Staff.

Anne and Brenda play a game which begins with a pile of n toothpicks. They alternate turns with Anne going first. On each player's turn, she must remove 1, 3, or 6 toothpicks from the pile. The player who removes the last toothpick wins the game. For which of the values of n from 36 to 40 inclusive does Brenda have a winning strategy?

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA, modified by the editor.

We can build a table of winning and losing positions for Anne. Her winning positions start with 1, 3, or 6, since she can immediately win by removing all of the toothpicks.

Starting with 2 toothpicks, Anne must remove 1 toothpick, leaving Brenda with 1, and so Brenda wins. Starting with 4 toothpicks, Anne must remove 1 or 3 toothpicks, leaving Brenda with 3 or 1 (respectively), and so Brenda wins by removing all of the toothpicks.

Starting with 5 toothpicks, Anne can remove 3 toothpicks, thus leaving Brenda with 2 toothpicks. Since 2 is a losing position for whoever goes first, then Brenda loses, so Anne wins.

So far, 1, 3, 5, and 6 are winning positions for Anne, while 2 and 4 are losing positions for Anne.

Starting with a pile of size n , Anne must reduce the pile to one of size $n - 1$, $n - 3$, or $n - 6$ and pass to Brenda. If the person who goes first has a winning strategy starting with a pile of each of these sizes, then Anne will lose. In other words, if Anne has a winning strategy starting with piles of size $n - 1$, $n - 3$, and $n - 6$, then Anne will lose starting with a pile of size n , as Brenda can implement Anne's strategy for the smaller pile and win, no matter what Anne does. If one or more of these pile sizes are such that the first person does not have a winning strategy, then Anne should reduce to this size, thus preventing Brenda from being able to win, and so Anne herself will win.

We can examine the cases from $n = 7$ to $n = 40$, obtaining the following lists:

Winning positions for Anne: 1, 3, 5, 6, 7, 8, 10, 12, 14, 15, 16, 17, 19, 21, 23, 24, 25, 26, 28, 30, 32, 33, 34, 35, 37, 39.

Losing positions for Anne: 2, 4, 9, 11, 13, 18, 20, 22, 27, 29, 31, 36, 38, 40.

Therefore, Brenda wins for $n = 36, 38, 40$.

Also solved by JACLYN CHANG, student, Western Canada High School, Calgary, AB.

See the Problem of the Month column in [2007 : 15–17] for a similar problem with a more detailed explanation.

M334. Proposed by the Mayhem Staff.

(a) Determine all integers x for which $\frac{x-3}{3x-2}$ is an integer.

(b) Determine all integers y for which $\frac{3y^3+3}{3y^2+y-2}$ is an integer.

I. Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina.

Let A be an integer such that $A = \frac{x-3}{3x-2}$. Then $3A$ is an integer and $3A = \frac{3x-9}{3x-2} = \frac{3x-2-7}{3x-2} = 1 - \frac{7}{3x-2}$.

Thus, $\frac{7}{3x-2}$ is an integer; that is, $3x-2$ is a divisor of 7, so $3x-2$ is one of $\pm 1, \pm 7$. Since x is an integer, $x = 1$ or $x = 3$. This answers part (a).

Now let B be an integer such that

$$\begin{aligned} B &= \frac{3y^3+3}{3y^2+y-2} = y - \frac{y^2-2y-3}{3y^2+y-2} \\ &= y - \frac{(y-3)(y+1)}{(y+1)(3y-2)} = y - \frac{y-3}{3y-2}. \end{aligned}$$

Since y is an integer, $\frac{y-3}{3y-2}$ is an integer. From the solution to part (a), $y = 1$ or $y = 3$, which answers part (b).

II. Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

We show that the only integer solutions to part (a) are $x = 1$ and $x = 3$.

Let $f(x) = \frac{x-3}{3x-2}$. Then $f(0) = \frac{3}{2}$, $f(1) = -2$, $f(2) = -\frac{1}{4}$, and $f(3) = 0$. Of these, only $f(1)$ and $f(3)$ are integers.

If $x > 3$, then $f(x)$ is not an integer, since $3x-2 > x-3 > 0$ for $x > 3$ and so $0 < \frac{x-3}{3x-2} < 1$.

If $x \leq -1$, let $x = -s$ where $s \geq 1$. Then $f(x) = f(-s) = \frac{s+3}{3s+2}$.

Since $3s+2 > s+3 > 0$ for $s \geq 1$, $f(-s)$ is not an integer by a similar argument so, $f(x)$ is not an integer.

Therefore, $f(x)$ is an integer for integer values of x if and only if $x = 1$ or $x = 3$.

Also solved by RICARD PEIRO, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and TITU ZVONARU, Comănești, Romania. There was one incorrect and one incomplete solution submitted.

M335. Proposed by the Mayhem staff.

In a sequence of four numbers, the second number is twice the first number. Also, the sum of the first and fourth numbers is 9, the sum of the second and third is 7, and the sum of the squares of the four numbers is 78. Determine all such sequences.

Solution by Denise Cornwell, student, Angelo State University, San Angelo, TX, USA.

Let a , b , c , and d represent the first, second, third and fourth number, respectively. We can now write the given information as $b = 2a$, $a + d = 9$, $b + c = 7$ and $a^2 + b^2 + c^2 + d^2 = 78$.

The first three equations allow us to rewrite b , c , and d in terms of a , obtaining $b = 2a$, $c = 7 - b = 7 - 2a$, and $d = 9 - a$.

Therefore,

$$\begin{aligned} a^2 + (2a)^2 + (7 - 2a)^2 + (9 - a)^2 &= 78, \\ a^2 + 4a^2 + 49 - 28a + 4a^2 + 81 - 18a + a^2 - 78 &= 0, \\ 5a^2 - 23a + 26 &= 0, \\ (5a - 13)(a - 2) &= 0, \end{aligned}$$

hence $a = \frac{13}{5}$ or $a = 2$.

Therefore, the sequences are $a = \frac{13}{5}$, $b = \frac{26}{5}$, $c = \frac{9}{5}$, $d = \frac{32}{5}$ and $a = 2$, $b = 4$, $c = 3$, $d = 7$. Both sequences satisfy the given requirements.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; JACLYN CHANG, student, Western Canada High School, Calgary, AB; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; BILLY SUANDITO, Palembang, Indonesia; and TITU ZVONARU, Comănești, Romania. There was one incorrect and one incomplete solution submitted.

M336. Proposed by the Mayhem Staff.

A lattice point is a point (x, y) in the coordinate plane with each of x and y an integer. Suppose that n is a positive integer. Determine the number of lattice points inside the region $|x| + |y| \leq n$.

Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina, modified by the editor.

We can rewrite the given inequality as the equations $|x| + |y| = 0$ and $|x| + |y| = k$ for $1 \leq k \leq n$, where $x, y \in \mathbb{Z}$.

The equation $|x| + |y| = 0$ has one integer solution only, namely $(x, y) = (0, 0)$.

Consider next $|x| + |y| = k$, for an integer k with $1 \leq k \leq n$. We can remove the absolute values by splitting into four cases:

Case 1. The integers x and y satisfy $x + y = k$, where $x \geq 0$ and $y \geq 0$. This has solutions $(k, 0), (k-1, 1), \dots, (1, k-1), (0, k)$, for a total of $k+1$ solutions.

Case 2. The integers x and y satisfy $x - y = k$, where $x \geq 0$ and $y < 0$. This has solutions $(k-1, -1), (k-2, -2), \dots, (1, -(k-1)), (0, -k)$, for a total of k solutions.

Case 3. The integers x and y satisfy $-x + y = k$, where $x < 0$ and $y \geq 0$. This case is the same as Case 2, but with the roles of x and y switched, so there are a total of k solutions here as well.

Case 4. The integers x and y satisfy $-x - y = k$, where $x < 0$ and $y < 0$. This has solutions $(-1, -(k-1)), (-2, -(k-2)), \dots, (-(k-2), -2), (-(k-1), -1)$, for a total of $k-1$ solutions.

Thus, for each k with $1 \leq k \leq n$, the equation $|x| + |y| = k$ has $(k+1) + k + k + (k-1) = 4k$ solutions.

Therefore, the number of lattice points inside the region is

$$\begin{aligned} 1 + \sum_{k=1}^n 4k &= 1 + 4 \sum_{k=1}^n k = 1 + 4 \cdot (1 + 2 + \dots + n) \\ &= 1 + 4 \cdot \frac{n(n+1)}{2} = 2n^2 + 2n + 1. \end{aligned}$$

Also solved by RICARD PEIRO, IES "Abastos", Valencia, Spain. There were one incorrect and two incomplete solutions submitted.

M337. *Proposed by the Mayhem Staff.*

On sides AB and CD of rectangle $ABCD$ with $AD < AB$, points F and E are chosen so that $AFCE$ is a rhombus.

- (a) If $AB = 16$ and $BC = 12$, determine EF .
- (b) If $AB = x$ and $BC = y$, determine EF in terms of x and y .

Solution by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.

We present the solution to (b), which is a general version of the specific case in (a).

Suppose that $AF = FC = CE = EA = m$. Let O be the point of intersection of diagonals AC and EF of rhombus $AFCE$. Note that AC and EF are perpendicular and bisect each other at O .

By the Pythagorean Theorem,

$$\begin{aligned} CF^2 - FB^2 &= CB^2, \\ m^2 - (x - m)^2 &= y^2, \\ m^2 - x^2 + 2mx - m^2 &= y^2, \\ 2mx &= x^2 + y^2, \\ m &= \frac{x^2 + y^2}{2x}. \end{aligned}$$

Now, $AF = m = \frac{x^2 + y^2}{2x}$ and $AC = \sqrt{AB^2 + BC^2} = \sqrt{x^2 + y^2}$. Also, $OA = \frac{1}{2}AC$. Thus, by the Pythagorean Theorem again,

$$\begin{aligned} OF^2 &= AF^2 - OA^2 \\ &= \left(\frac{x^2 + y^2}{2x}\right)^2 - \left(\frac{\sqrt{x^2 + y^2}}{2}\right)^2 \\ &= \left(\frac{x^4 + y^4 + 2x^2y^2}{4x^2}\right) - \left(\frac{x^2 + y^2}{4}\right) \\ &= \frac{x^4 + y^4 + 2x^2y^2 - x^4 - x^2y^2}{4x^2} \\ &= \frac{y^4 + x^2y^2}{4x^2}. \end{aligned}$$

Therefore,

$$OF = \sqrt{\frac{y^4 + x^2y^2}{4x^2}} = \frac{\sqrt{y^2(y^2 + x^2)}}{2x} = \frac{y\sqrt{x^2 + y^2}}{2x},$$

and

$$EF = 2OF = \frac{2y\sqrt{x^2 + y^2}}{2x} = \frac{y\sqrt{x^2 + y^2}}{x}.$$

In part (a), this yields

$$EF = \frac{12\sqrt{16^2 + 12^2}}{16} = \frac{12(20)}{16} = 15.$$

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; JACLYN CHANG, student, Western Canada High School, Calgary, AB (part (a) only); RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; BILLY SUANDITO, Palembang, Indonesia; LUYAN ZHONG-QIAO, Columbia International College, Hamilton, ON; and TITU ZVONARU, Comănești, Romania.

Problem of the Month

Ian VanderBurgh

Here is a problem that might seem to be not very interesting initially, but turns out to have a whole lot of unexpected solutions.

Problem (2005 Canadian Open Mathematics Challenge) In the grid shown, each row has a value assigned to it and each column has a value assigned to it. The number in each cell is the sum of its row and column values. For example, the “8” is the sum of the value assigned to the 3rd row and the value assigned to the 4th column. Determine the values of x and y .

3	0	5	6	-2
-2	-5	0	1	y
5	2	x	8	0
0	-3	2	3	-5
-4	-7	-2	-1	-9

It is tempting first of all to give labels to the values that are assigned to the rows and columns in order to be able to dive into some algebra. Let's label the values assigned to the five columns A, B, C, D, E and the values assigned to the five rows a, b, c, d, e .

Each entry in the table gives us an equation involving two of these variables. For example, the -3 in row 4, column 2 gives us $d + B = -3$, and the -9 in row 5, column 5, gives us $e + E = -9$. We could proceed and write down 25 equations, one for each entry in the table. These equations would include 12 variables – the 10 that label the rows and columns together with x and y . We could then spend pages and pages wading through algebra trying to come up with the answers. At this point, we would hope that there has to be a better way. Maybe we should have looked before we leapt!

Here are three neat ways to approach this. (As a point of interest, I was recently talking about this problem with a friend while driving and so neither of us really wanted to do any algebra, and so were forced to come up with better ways to do it.)

Solution 1. If we choose five entries from the table which include one from each row and one from each column, then the sum of these entries is constant no matter how we choose the entries, as it is always equal to

$$A + B + C + D + E + a + b + c + d + e.$$

Can you see why? Here are three ways in which this can be done (looking at the underlined numbers in the two grids below and the grid on the following page):

<u>3</u>	0	5	6	-2
-2	<u>-5</u>	0	1	y
5	2	x	<u>8</u>	0
0	-3	<u>2</u>	3	-5
-4	-7	-2	-1	<u>-9</u>

3	0	5	6	<u>-2</u>
-2	-5	0	<u>1</u>	y
5	2	<u>x</u>	8	0
0	<u>-3</u>	2	3	-5
<u>-4</u>	-7	-2	-1	-9

Therefore,

$$\begin{aligned} & 3 + (-5) + 2 + 8 + (-9) \\ &= (-4) + (-3) + x + 1 + (-2) \\ &= 3 + y + 2 + (-2) + 3, \end{aligned}$$

or $-1 = x - 8 = y + 6$. Thus, $x = 7$ and $y = -7$.

<u>3</u>	0	5	6	-2
-2	-5	0	1	<u>y</u>
5	<u>2</u>	<i>x</i>	8	0
0	-3	2	<u>3</u>	-5
-4	-7	- <u>2</u>	-1	-9

Solution 2. Consider the first two entries in row 1. From the labels above, we have $3 = A + a$ and $0 = B + a$. Subtracting these, we obtain the equation $3 = 3 - 0 = (A + a) - (B + a) = A - B$.

Notice that whenever we take entries in columns 1 and 2 from the same row, their difference will always equal $A - B$, which is equal to 3. Similarly, since the difference between the 0 and the 5 in the first row is 5, then every entry in column 3 will be 5 greater than the entry in column 2 from the same row. In row 3, we see that $x = 2 + 5 = 7$.

Also, since the difference between the 6 and the -2 in the first row is 8, then every entry in column 5 is 8 less than the entry in column 4 from the same row. In row 2, we see that $y = 1 - 8 = -7$. Thus, $x = 7$ and $y = -7$.

Solution 3. Consider the sub-grid $\begin{array}{|c|c|} \hline 0 & 1 \\ \hline x & 8 \\ \hline \end{array}$.

Since the 0 is in row 2 and column 3, then $0 = b + C$. Similarly, $1 = b + D$, $8 = c + D$, and $x = c + C$.

But then $0 + 8 = (b + C) + (c + D) = (c + C) + (b + D) = x + 1$, or $x = 7$.

In a similar way, by looking at the sub-grid $\begin{array}{|c|c|} \hline 1 & y \\ \hline 8 & 0 \\ \hline \end{array}$ we can show that $1 + 0 = y + 8$, or $y = -7$. Thus, $x = 7$ and $y = -7$.

So there are three different but neat solutions to the problem. One footnote to the final solution is that in fact, in any sub-grid of the form

$\begin{array}{|c|c|} \hline p & q \\ \hline r & s \\ \hline \end{array}$, we must have $p + s = q + r$. Can you see why?

Another interesting point about this problem is that it might be easier for those who know less! If we replaced the x and the y with "?" and gave it to someone who didn't know a lot of algebra, they might find the answers faster than those of us who go immediately to algebra. Sometimes, the extra machinery that we have can get in the way.

As 2008 draws to a close, the Mayhem Editor has three enormous sets of thanks to offer. First, to the Mayhem Staff, especially to Monika Khbeis and Eric Robert, for all of their help over the past year. Second, to the Editor-in-Chief of **CRUX with MAYHEM**, Václav Linek, for all of his help and encouragement over the past year (as well as for his sharp eyes!). Third, to the Mayhem readership for their support. Please keep those problems and solutions coming!