SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3229. [2007 : 170, 172; 2007 : 179–181] *Proposed by Mihály Bencze, Brasov, Romania.*

(a) Let x and y be positive real numbers, and let n be a positive integer. Prove that

$$(x+y)^n \sum_{k=0}^{n} \frac{1}{\binom{n}{k} x^{n-k} y^k} \;\geq\; n+1+2\sum_{i=1}^{n}\sum_{k=0}^{n-i} \frac{\binom{n}{k}}{\binom{n}{k+i}} \;\geq\; (n+1)^2 .$$

(b)★ Let x_1, x_2, $\ldots$, x_k be positive real numbers, and let n be a positive integer. Determine the minimum value of

$$(x_1+x_2+\cdots+x_k)^n \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\,\ldots,\,i_k \geq 0}} \frac{i_1!i_2!\cdots i_k!}{n!x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k}} .$$

Solution to (b) *by Sergey Sadov, Memorial University of Newfoundland, St. John's, NL.*

We may assume that $x_1+x_2+\cdots+x_k=1$ without loss of generality. The minimum occurs when $x_1 = x_2 = \cdots = x_k = \frac{1}{k}$. We will obtain this as a consequence of a more general proposition. For convenience, let $X = (x_1, x_2, \ldots, x_k)$ be a vector in $\mathbb{R}_+^k$ (that is, $x_i > 0$ for all i) and let $Q = (q_1, q_2, \ldots, q_k)$ be a multi-index with non-negative integer entries q_i. Let

$$\begin{aligned} |Q| &= \sum_{i=1}^{n} q_i\,; \\ |X| &= \sum_{i=1}^{n} x_i\,; \\ Q! &= q_1!q_2!\cdots q_k!\,; \\ X^Q &= x_1^{q_1}x_2^{q_2}\cdots x_k^{q_k}\,. \end{aligned}$$

Finally, let $\Delta = \{X \in \mathbb{R}_+^k : |X| = 1\}$ be the positive unit simplex in $\mathbb{R}^k$ of dimension $k-1$.

Theorem 1 With the above notation, let

$$P(X) \;=\; \sum_{|Q|=n} \frac{c_Q}{X^Q}$$

be a homogeneous rational function of degree $-n$ in the variables x_i. Suppose that $c_Q \geq 0$ for each Q and that $P(X)$ is a symmetric function, that is, interchanging any x_i and x_j does not change the value of $P(X)$. Then the minimum value of $P(X)$ over the simplex Δ exists and is attained when $x_1 = x_2 = \cdots = x_k = \dfrac{1}{k}$.

Proof. If $P(X)$ is identically zero, then there is nothing to prove, so we assume at least one coefficient c_Q is not zero. Note that $P(x)$ has a minimum value, since $P(X)$ is continuous on Δ and tends to infinity as any of the x_i approaches 0. Suppose that the minimum occurs at a point with at least two unequal coordinates. Without loss of generality (due to the symmetry of $P(X)$) we may assume that $x_1 > x_2$. We will show that by slightly decreasing x_1 and slightly increasing x_2 (while keeping all other variables and the sum $x_1 + x_2$ unchanged) the value of $P(X)$ will become smaller, contrary to our assumption. Fixing x_3, x_4, $\ldots$, x_n causes $P(X)$ to become a symmetric function of two variables

$$F(u,v) \;=\; P(u,v,x_3,\ldots,x_n) \;=\; \sum_{j+s\leq n} \frac{A_{j,s}}{u^j v^s}\,,$$

where the coefficients $A_{j,s}$ depend on x_3, x_4, $\ldots$, x_n. Note that each $A_{j,s}$ is non-negative and at least one of these is positive, and that $A_{j,s} = A_{s,j}$. Thus, $F(u,v)$ is a linear combination with non-negative coefficients, not all zero, of functions of the form

$$F_{j,s}(u,v) \;=\; \frac{1}{u^j v^s} \;+\; \frac{1}{u^s v^j}\,,$$

where j and s are non-negative integers with $j+s \leq n$. Now let $u(t) = x_1 - t$ and $v(t) = x_2 + t$, so that $\dfrac{du}{dt} = -1$ and $\dfrac{dv}{dt} = 1$. It suffices to prove that the "time derivative" $\dfrac{d}{dt}F_{j,s}(u(t),v(t))$ is negative at $t = 0$. We have

$$\begin{aligned}\frac{d\,F_{j,s}(u(t),v(t))}{dt} \;&=\; \frac{\partial F_{j,s}(u,v)}{\partial v} \;-\; \frac{\partial F_{j,s}(u,v)}{\partial u}\\ &=\; \left(\frac{j}{u}-\frac{s}{v}\right)\frac{1}{u^j v^s} \;+\; \left(\frac{s}{u}-\frac{j}{v}\right)\frac{1}{u^s v^j}\,.\end{aligned}$$

Differentiating once again, we obtain

$$\frac{d^2 F_{j,s}(u(t),v(t))}{dt^2} \;=\;$$
$$\left(\frac{j}{u^2}+\frac{s}{v^2}+\left(\frac{j}{u}-\frac{s}{v}\right)^2\right)\frac{1}{u^j v^s} \;+\; \left(\frac{j}{v^2}+\frac{s}{u^2}+\left(\frac{j}{v}-\frac{s}{u}\right)^2\right)\frac{1}{u^s v^j}\,.$$

Hence, $\frac{d^2F}{dt^2} > 0$. Since $\frac{dF}{dt} = 0$ when $u = v$ (this occurs when $t = \frac{u-v}{2}$), it follows that $\frac{dF}{dt} < 0$ when $t = 0$. We have obtained a contradiction by assuming $x_1 > x_2$ at a point achieving the minimum. This proves that the minimum occurs when all the x_i are equal. ∎

It follows that the minimum sought in part (b) is

$$k^n \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k \geq 0}} \frac{i_1!i_2!\ldots i_k!}{n!}\,.$$

No other solutions to part (b) *were received.*
Regarding part (a) *Sadov remarks that*

$$\min_{x+y=1} \sum_{k=0}^{n} \frac{1}{\binom{n}{k} x^{n-k} y^k} \;=\; 2^n \sum_{j=0}^{n} \frac{1}{\binom{n}{j}} \;=\; n+1+2\sum_{j=0}^{n-1}\sum_{i=0}^{n-j} \frac{\binom{n}{j}}{\binom{n}{i+j}}\,,$$

where the first equality follows Theorem 1 and the last expression is the minimum obtained by Bataille [2007 : 179–181]. He notes that the first equality yields a minimum of at least 2^{n+1}*, considerably improving the lower bound of* $(n+1)^2$ *if* $n > 4$*. He observes that if* $b_i = \binom{n}{i} x^{n-i} y^i$*, then by the AM–HM Inequality*

$$\left(\frac{1}{n+1}\sum \frac{1}{b_i}\right)^{-1} \leq \frac{1}{n+1}\sum b_i \;=\; \frac{(x+y)^n}{n+1}\,,$$

hence $(x+y)^2 \sum b_i^{-1} \geq (n+1)^2$*, which yields a quick proof of part* (a).

Sadov comments that the function $P(X)$ *in Theorem 1 is* Schur-convex, *referring to* [1] *for the definition of this term and applications. He indicates that the AM-GM Inequality and the AM–HM Inequality can be obtained by taking* $P(X) = (x_1x_2\cdots x_n)^{-1}$ *and* $P(X) = \sum x^i$ *in Theorem 1, respectively.*

He mentions that particular cases (and other theorems of a more general nature) of Theorem 1 can be found in [2]*, Chapter 3, Section G, Examples G.1.k and G.1.m; though he believes that Theorem 1 is present somewhere in the existing literature.*

Finally, he refers to [3]*, Section 2.18, for a treatment of (the related) Muirhead's Inequality.*

References

[1] M.L. Clevenson and W. Watkins, Majorization and the Birthday Inequality, *Math. Magazine*, vol. 64, No. 3 (1991), pp. 183–188.

[2] A. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, 1979.

[3] G.H. Hardy, J.E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, 1952.

3289. [2007 : 485, 487] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let ABC be a triangle for which there exists a point D in its interior such that $\angle DAB = \angle DCA$ and $\angle DBA = \angle DAC$. Let E and F be points on the lines AB and CA, respectively, such that $AB = BE$ and $CA = AF$. Prove that the points A, E, D, and F are concyclic.

A composite of similar solutions by John G. Heuver, Grande Prairie, AB and George Tsapakidis, Agrinio, Greece.

Triangles ADC and BDA are similar (because their angles are assumed to be equal), whence

$$\frac{CD}{AD} = \frac{CA}{AB} = \frac{2CA}{2AB} = \frac{CF}{AE}.$$

It follows that $\triangle DCF \sim \triangle DAE$ ($\angle DCF = \angle DCA = \angle DAB = \angle DAE$, while the adjacent sides are proportional from the previous step), so that $\angle AFD$ ($= \angle CFD$) $= \angle AED$. Noting that E and F both lie on the same side of AD, we conclude that A, E, F, and D lie on the same circle, as desired.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; GEOFFREY A. KANDALL, Hamden, CT, USA; ANDREA MUNARO, student, University of Trento, Trento, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Bataille comments that D can be constructed as the point inside $\triangle ABC$ where the circle FAE intersects the circle through B, C, and the circumcentre O. It lies on the first circle by the result of this problem. It lies on the second circle because $\angle BOC = \angle BDC = 2\angle BAC$ as follows: on the one hand the angle at A is inscribed in the circle BAC that is centred at O and is therefore $\frac{1}{2}\angle BOC$; on the other hand the similar triangles ADC and BDA fit together at A and at D in such a way that the two exterior angles at D (that form $\angle BDC$) sum to twice the sum of the two interior angles at A (that form $\angle BAC$).

3290. [2007 : 485, 487] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let $ABCD$ be a trapezoid with $AD \parallel BC$. Denote the lengths of AD and BC by a and b, respectively. Let M be the mid-point of CD, and let P and Q be the mid-points of AM and BM, respectively. If N is the intersection of DP and CQ, prove that N belongs to the interior of $\triangle ABM$ if and only if $\frac{1}{3} < \frac{a}{b} < 3$.

Solution by Joel Schlosberg, Bayside, NY, USA.

There is a misleading subtlety in the statement of the problem: we shall see that the conclusion fails should $AD = BC$; in other words, our trapezoid $ABCD$ must not be a parallelogram.

Since all of the conditions of the problem are invariant under an affine transformation, we can assume without loss of generality that

$$AB \perp AD \quad \text{and} \quad AB = 4\,.$$

We therefore introduce a Cartesian coordinate system with the origin at A, and with B on the positive x–axis and D on the positive y–axis; then A, B,

C, and D have coordinates

$$A(0,0)\,,\quad B(4,0)\,,\quad C(4,b)\,,\quad D(0,a)\,,$$

where a and b are positive. It follows that the coordinates of M, P, and Q are

$$M\left(2,\frac{a+b}{2}\right)\,,\quad P\left(1,\frac{a+b}{4}\right)\,,\quad Q\left(3,\frac{a+b}{4}\right)\,,$$

so that the lines DP and CQ satisfy

$$y\;=\;\left(\frac{b-3a}{4}\right)x\;+\;a \qquad \text{and} \qquad y\;=\;\left(\frac{3b-a}{4}\right)x\;+\;a-2b\,,$$

whence $N = DP \cap CQ$ has coordinates

$$N\left(\frac{4b}{a+b},\frac{(a-b)^2}{a+b}\right)\,.$$

A point in the plane is on the same side of AB as M if and only if its y–coordinate is positive. The y–coordinate of N satisfies

$$\frac{(a-b)^2}{a+b}\;\geq\;0\,,$$

with equality if and only if $a = b$. Since $P = DP \cap AM$ has x–coordinate 1, a point on line DP is on the same side of AM as B if and only if its x–coordinate exceeds 1. Lastly, since $Q = CQ \cap BM$ has x–coordinate 3, a point on line CQ is on the same side of BM as A if and only if its x–coordinate is less than 3. Therefore, N is within the interior of $\triangle ABM$ if and only if $a \neq b$ and $1 < \frac{4b}{a+b} < 3$. This inequality is equivalent to

$$\frac{1}{3} < \frac{a}{b} < 3\,,$$

which is the desired inequality; it is equivalent to N belonging to the interior of $\triangle ABM$ as long as $a \neq b$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvarez Cubero, Priego de Córdoba, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3291. [2007 : 485, 487] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let ABC be an isosceles triangle with $AB = AC$. Find all points P such that the sum of the squares of the distances of the points A, B, and C from any line through P is constant.

Solution by Michel Bataille, Rouen, France.

We will introduce a Cartesian coordinate system. Label the points as $A(0,a)$, $B(-b,0)$, $C(b,0)$, and $P(u,v)$, where a and b are positive and u and v are variables. The equation of a line ℓ through P is $x\cos\theta + y\sin\theta = p$, where $p = u\cos\theta + v\sin\theta$ and θ is an arbitrary real number. Let $d(Q,\ell)$ denote the distance between a point Q and the line ℓ, and let

$$S \;=\; d(A,\ell)^2 \;+\; d(B,\ell)^2 \;+\; d(C,\ell)^2\,.$$

We then have $d(A,\ell)^2 = (a\sin\theta - p)^2$, $d(B,\ell)^2 = (p + b\cos\theta)^2$, and $d(C,\ell)^2 = (p - b\cos\theta)^2$. After a simple calculation, we obtain

$$\begin{aligned} S \;&=\; (\cos 2\theta)\left(\frac{3(u^2-v^2)}{2} + av - \frac{a^2}{2} + b^2\right) \\ &\quad+\; (\sin 2\theta)(3uv - au) + \frac{3(u^2+v^2)}{2} - av + \frac{a^2}{2} + b^2\,. \end{aligned}$$

The number S is independent of θ if and only if

$$3uv - au \;=\; 0 \quad\text{and}\quad 3(u^2-v^2) + 2av - a^2 + 2b^2 \;=\; 0\,,$$

which gives

$$u \;=\; 0 \quad\text{and}\quad -3v^2 + 2av - a^2 + 2b^2 \;=\; 0\,, \tag{1}$$

or

$$v \;=\; \frac{a}{3} \quad\text{and}\quad 3\left(u^2 - \frac{a^2}{9}\right) + \frac{2a^2}{3} - a^2 + 2b^2 \;=\; 0\,. \tag{2}$$

Now, (1) is satisfied if and only if $0 < a \le b\sqrt{3}$ (that is, $\angle B = \angle C \le 60^\circ$) and the coordinates of P are

$$\left(0,\; \frac{a + \sqrt{2(3b^2-a^2)}}{3}\right) \quad\text{or}\quad \left(0,\; \frac{a - \sqrt{2(3b^2-a^2)}}{3}\right)\,.$$

Similarly, (2) is satisfied if and only if $a \ge b\sqrt{3}$ (that is, $\angle B = \angle C \ge 60^\circ$) and the coordinates of P are

$$\left(\frac{\sqrt{2(a^2-3b^2)}}{3},\; \frac{a}{3}\right) \quad\text{or}\quad \left(-\frac{\sqrt{2(a^2-3b^2)}}{3},\; \frac{a}{3}\right)\,.$$

In conclusion, if ABC is equilateral, only the centre of ABC is a solution for P. Otherwise, there are two solutions for P:

$$P\left(0,\; \frac{a \pm \sqrt{2(3b^2-a^2)}}{3}\right) \qquad (\text{if } \angle A > 60^\circ)\,,$$

$$P\left(\pm\frac{\sqrt{2(a^2-3b^2)}}{3},\; \frac{a}{3}\right) \qquad (\text{if } \angle A < 60^\circ)\,.$$

In both cases, the two solutions are symmetric about the centroid of the triangle; in the first case, the two points lie on the median through A and in the second case the two points lie on a line parallel to BC and passing through the centroid of $\triangle ABC$.

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3292. [2007 : 485, 488] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, c, and d be arbitrary real numbers. Show that

$$\begin{aligned} &11a^2 + 11b^2 + 221c^2 + 131d^2 + 22ab + 202cd + 48c + 6 \\ &\qquad \geq \; 98ac + 98bc + 38ad + 38bd + 12a + 12b + 12d\,. \end{aligned}$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

The proof is by contradiction. Assume that

$$\begin{aligned} &11a^2 + 11b^2 + 221c^2 + 131d^2 + 22ab + 202cd + 48c + 6 \\ &\qquad < \; 98ac + 98bc + 38ad + 38bd + 12a + 12b + 12d\,. \end{aligned}$$

We consider the quadratic $f(x) = 11x^2 + px + q$ with

$$\begin{aligned} p &= 22b - 98c - 38d - 12\,, \\ q &= 11b^2 + 221c^2 + 131d^2 + 202cd + 48c + 6 - 98bc - 38bd \\ &\quad - 12b - 12d\,. \end{aligned}$$

Then $f(a) < 0$, and the leading coefficient of f is positive, hence f has two distinct real roots; that is, the discriminant of f is positive. By computing the discriminant, we find $p^2 - 44q = -120(c+6d-1)^2 \leq 0$, a contradiction.

Also solved by APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, É-U; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; JOEL SCHLOSBERG, Bayside, NY, USA; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3293. [2007 : 485, 488] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\prod_{k=1}^{n} \frac{\arcsin\left(\dfrac{9k+2}{\sqrt{27k^3+54k^2+36k+8}}\right)}{\arctan\left(\dfrac{1}{\sqrt{3k+1}}\right)} \;=\; 3^n\,.$$

Composite of similar solutions by Michel Bataille, Rouen, France and Douglass L. Grant, Cape Breton University, Sydney, NS, modified by the editor.

For each $k = 1, 2, \ldots, n$ let P_k denote the corresponding factor under the product sign. We prove that in fact, $P_k = 3$ for each k.

Note first that $27k^3 + 54k^2 + 36k + 8 = (3k+2)^3$. For fixed k, let $\theta = \tan^{-1}\left(\dfrac{1}{\sqrt{3k+1}}\right)$. Then $\tan\theta = \dfrac{1}{\sqrt{3k+1}}$ implies $\sin\theta = \dfrac{1}{\sqrt{3k+2}}$.

Since $\tan^{-1}$ is an increasing function, we have

$$0 \;<\; \theta \;\le\; \tan^{-1}\left(\frac{1}{2}\right) \;<\; \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \;=\; \frac{\pi}{6}\,,$$

hence, $0 < 3\theta < \frac{\pi}{2}$.

From

$$\begin{aligned} \sin(3\theta) &= 3\sin\theta - 4\sin^3\theta \\ &= \frac{3}{\sqrt{3k+2}} - \frac{4}{\sqrt{(3k+2)^3}} \;=\; \frac{9k+2}{\sqrt{(3k+2)^3}}\,, \end{aligned}$$

we obtain

$$\begin{aligned} &\sin^{-1}\left(\frac{9k+2}{\sqrt{27k^3+54k^2+36k+8}}\right) \\ &= \sin^{-1}\left(\frac{9k+2}{\sqrt{(3k+2)^3}}\right) \;=\; \sin^{-1}(\sin 3\theta) \;=\; 3\theta\,, \end{aligned}$$

and it follows that $P_k = \dfrac{3\theta}{\theta} = 3$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, CHARLES DIMINNIE, and KARL HAVLAK, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3294. [2007 : 486, 488] *Proposed by Mihály Bencze, Brasov, Romania.*

For all positive integers m and n, show that

$$m(m+1)n^2(n+1)^2(2n^2+2n-1) - n(n+1)m^2(m+1)^2(2m^2+2m-1)$$

is divisible by 720.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Let $f(k) = k(k+1)\left(2k^2+2k-1\right)$. Write

$$\begin{aligned} A(m,n) &= m(m+1)\,n^2(n+1)^2\left(2n^2+2n-1\right) \\ &\quad - n(n+1)\,m^2(m+1)^2\left(2m^2+2m-1\right) \\ &= mn(m+1)(n+1)\left[f(n) - f(m)\right] . \end{aligned}$$

Let $C(m,n) = mn(m+1)(n+1)$ and $D(m,n) = f(n) - f(m)$, so that $A(m,n) = C(m,n)\,D(m,n)$. Since $720 = 2^4 3^2 5$, it suffices to show that $A(m,n)$ is divisible by 16, 9, and 5.

(a) Divisibility by 16. The residues of $f(n)$ modulo 4 are given in the following table.

$n \pmod 4$	0	1	2	3
$f(n) \pmod 4$	0	2	2	0

(i) If $n \equiv 0$ or $3 \pmod 4$, or $m \equiv 0$ or $3 \pmod 4$, then $C(m,n)$ is divisible by 8 and $D(m,n)$ is divisible by 2, so that $A(m,n)$ is divisible by 16.

(ii) Otherwise, $C(m,n)$ is divisible by 4 and $D(m,n)$ is divisible by 4, and therefore, $A(m,n)$ is divisible by 16.

(b) Divisibility by 9. The residues of $f(n)$ modulo 9 are given in the next table.

$n \pmod 9$	0	1	2	3	4	5	6	7	8
$f(n) \pmod 9$	0	6	3	6	6	6	3	6	0

(i) If $n \equiv 0$ or $8 \pmod 9$, or $m \equiv 0$ or $8 \pmod 9$, then $C(m,n) \equiv 0 \pmod 9$, so that $A(m,n)$ is divisible by 9.

(ii) If $n \equiv 2$ or $6 \pmod 9$, or $m \equiv 2$ or $6 \pmod 9$, then $C(m,n)$ and $D(m,n)$ are each divisible by 3, and therefore, $A(m,n)$ is divisible by 9.

(iii) Otherwise, $f(n) \equiv 6 \pmod 9$ and $f(m) \equiv 6 \pmod 9$, so that $D(m,n)$ is divisible by 9. Thus, $A(m,n)$ is divisible by 9.

(c) Divisibility by 5. The residues of $f(n)$ modulo 5 are given in the table below.

$n \pmod 9$	0	1	2	3	4
$f(n) \pmod 9$	0	1	1	1	0

(i) If $n \equiv 0$ or $4 \pmod 5$, or $m \equiv 0$ or $4 \pmod 5$, then $C(m,n)$ is divisible by 5, so that $A(m,n)$ is divisible by 5.

(ii) Otherwise, $D(m,n)$ is divisible by 5, and therefore, $A(m,n)$ is divisible by 5.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ANDREA MUNARO, student, University of Trento, Trento, Italy; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3295. [2007 : 486, 488] *Proposed by Michel Bataille, Rouen, France.*

Let $u : \mathbb{R} \to \mathbb{R}$ be a bounded function. For $x > 0$, let

$$\begin{aligned} f(x) &= \sup\{u(t) : t > \ln(1/x)\} \\ \text{and} \quad g(x) &= \sup\{u(t) - xe^{-t} : t \in \mathbb{R}\}\,. \end{aligned}$$

Prove that $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to\infty} g(x)$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor.

The proof consists of showing that both limits are equal to the limit $L = \limsup\limits_{t\to\infty} u(t)$, which exists since the function $u(t)$ is bounded. Let K be such that $|u(t)| < K$ for all $t \in \mathbb{R}$. Given any $\epsilon > 0$ and any $t_0 > 0$, we have

(i) there exists some $t > t_0$ such that $L - \epsilon < u(t)$, and

(ii) there exists some $t_1 > t_0$ such that for all $t > t_1$, $u(t) < L + \epsilon$.

Let $\epsilon > 0$ be given. For a fixed $x > 0$ we have $\lim\limits_{t\to\infty} xe^{-t} = 0$, so there exists $t_0 > 0$ such that $xe^{-t} < \epsilon$ for any $t > t_0$. By part (i), there exists $t > t_0$ such that $L - \epsilon < u(t)$, hence

$$L - 2\epsilon < u(t) - xe^{-t} \;\le\; g(x)\,.$$

Thus, $g(x) > L - 2\epsilon$ for each $x > 0$.

On the other hand, there exists $t_1 \in \mathbb{R}$ such that $u(t) < L + \epsilon$ for all $t > t_1$. Since $e^{-t_1} > 0$, let $M > 0$ be such that $K - Me^{-t_1} < L + \epsilon$. We claim that if $x > M$, then $g(x) \le L + \epsilon$. For this it suffices to show that $u(t) - xe^{-t} < L + \epsilon$ whenever $x > M$ and $t \in \mathbb{R}$.

Indeed, if $x > M$ and $t \le t_1$, then $u(t) - xe^{-t} < K - Me^{-t_1} < L + \epsilon$, while if $x > M$ and $t > t_1$, then $u(t) - xe^{-t} < u(t) < L + \epsilon$.

Therefore, for $x > M$ we have $L - 2\epsilon < g(x) < L + 2\epsilon$, hence $\lim\limits_{x \to \infty} g(x) = L$.

Finally, writing $S(v) = \sup\{u(t) : t > v\}$ and making two changes of variable in the limit yields

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} S\big(\ln(1/x)\big) = \lim_{y \to \infty} S(\ln y) = \lim_{z \to \infty} S(z) = L\,.$$

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

3296. [2007 : 486, 488] *Proposed by Michel Bataille, Rouen, France.*

Find the greatest constant K such that

$$\frac{b^2c^2}{a^2(a-b)(a-c)} + \frac{c^2a^2}{b^2(b-c)(b-a)} + \frac{a^2b^2}{c^2(c-a)(c-b)} > K$$

for all distinct positive real numbers a, b, and c.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, expanded by the editor.

We prove that $K = 10$.

Let L denote the left side of the given inequality. Since L is completely symmetric in a, b, and c, we may assume without loss of generality that $a < b < c$.

Note first that $L = \dfrac{P}{Q}$ where $Q = a^2b^2c^2(b-a)(c-a)(c-b)$ and $P = b^4c^4(c-b) - c^4a^4(c-a) + a^4b^4(b-a)$.

Observing that $P = 0$ when $c = b$ or $b = a$ or $c = a$, we find by straightforward but tedious computations that

$$\begin{aligned} P &= b^4c^4(c-b) - a^4\left(c^5 - b^5\right) + a^5\left(c^4 - b^4\right) \\ &= (c-b)\Big(b^4c^4 - a^4\left(c^4 + c^3b + c^2b^2 + cb^3 + b^4\right) \\ &\quad + a^5\left(c^3 + c^2b + cb^2 + b^3\right)\Big) \end{aligned}$$

$$\begin{aligned}
&= (c-b)\Big(c^4\left(b^4-a^4\right) \;+\; a^4(a-b)\left(c^3+c^2b+cb^2+b^3\right)\Big) \\
&= (c-b)(b-a)\Big(c^4\left(b^3+b^2a+ba^2+a^3\right) \\
&\qquad -\, a^4\left(c^3+c^2b+cb^2+b^3\right)\Big) \\
&= (c-b)(b-a)\Big(b^3\left(c^4-a^4\right) \;+\; cb^2a\left(c^3-a^2\right) \\
&\qquad +\; c^2ba^2\left(c^2-a^2\right)+c^3a^3(c-a)\Big) \\
&= (c-b)(b-a)(c-a)\Big(b^3\left(c^3+c^2a+ca^2+a^3\right) \\
&\qquad +\; cb^2a\left(c^2+ca+a^2\right) \;+\; c^2ba^2(c+a)+c^3a^3\Big)\,.
\end{aligned}$$

Hence, $\dfrac{P}{Q} = \dfrac{W}{a^2b^2c^2}$, where

$$\begin{aligned}
W &= b^3\left(c^3+c^2a+ca^2+a^3\right) \;+\; cb^2a\left(c^2+ca+a^2\right) \\
&\qquad +\; c^2ba^2(c+a)+c^3a^3\,.
\end{aligned}$$

By writing W as a sum of 10 terms and using the AM–GM Inequality, we readily see that $W \geq 10\left(a^{20}b^{20}c^{20}\right)^{1/10} = 10a^2b^2c^2$, from which it follows that $L \geq 10$. Since a, b, and c are distinct, equality cannot hold. Thus, $L > 10$.

Finally, if we set $a = 1$, $b = 1+\varepsilon$, and $c = 1+2\varepsilon$ and let $\varepsilon \to 0^+$, then the value of L can be made arbitrarily close to 10 from the right. Hence, $K = 10$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and the proposer. There were one incorrect and three incomplete solutions (which only showed that $L > 10$ and then concluded immediately that $K = 10$).

3297. [2007 : 486, 488] *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

If A, B, and C are the angles of a triangle, prove that

$$\sin A + \sin B \sin C \;\leq\; \frac{1+\sqrt{5}}{2}\,.$$

When does equality hold?

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

The following are equivalent

$$\begin{aligned} \sin A \;+\; \sin B \sin C &\;\le\; \frac{1+\sqrt{5}}{2}\,; \\ 2\sin A \;+\; \cos(B-C) \;-\; \cos(B+C) &\;\le\; 1+\sqrt{5}\,; \\ 2\sin A \;+\; \cos A \;+\; \cos(B-C) &\;\le\; 1+\sqrt{5}\,. \end{aligned}$$

Let φ be the first quadrant angle with $\cos\varphi = \frac{2}{\sqrt{5}}$ and $\sin\varphi = \frac{1}{\sqrt{5}}$ (the angle φ is approximately 26.6°). The last inequality then becomes

$$\sqrt{5}\sin(A+\varphi) \;+\; \cos(B-C) \;\le\; 1+\sqrt{5}\,.$$

However, $\sin(A+\varphi) \le 1$ and $\cos(B-C) \le 1$, so the last inequality is true.

Equality holds when $A = \arcsin\frac{2}{\sqrt{5}} \approx 63.4^\circ$ and $B = C \approx 58.3^\circ$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incorrect solution submitted.

Janous proved a more general result, namely, for $\lambda > 0$

$$\lambda\sin A + \sin B\sin C \;\le\; \frac{1+\sqrt{4\lambda^2+1}}{2}\,,$$

with equality if and only if $A = \arccos\left(\frac{1}{\sqrt{4\lambda^2+1}}\right)$ *and* $B = C$.

3298. [2007 : 486, 489] *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Let ABC be a triangle of area $\frac{1}{2}$ in which a is the length of the side opposite vertex A. Prove that

$$a^2 + \csc A \;\ge\; \sqrt{5}\,.$$

[*Ed.:* The proposer's only proof of this is by computer. He is hoping that some ***CRUX with MAYHEM*** reader will find a simpler solution.]

Solution by Kee-Wai Lau, Hong Kong, China.

Let $b = AC$, $c = AB$ and let S denote the area of triangle ABC. Since $S = \frac{1}{2}bc\sin A = \frac{1}{2}$, we obtain $bc = \csc A \geq 1$.

By the Law of Cosines we have (regardless of the sign of $\cos A$) that

$$\begin{aligned} a^2 + \csc A &= a^2 + bc = b^2 + c^2 - 2bc\cos A + bc \\ &\geq b^2 + c^2 - 2bc\sqrt{1 - \sin^2 A} + bc \\ &= b^2 + c^2 - 2\sqrt{b^2c^2 - (bc\sin A)^2} + bc \\ &= \left(b^2 + bc + c^2\right) - 2\sqrt{b^2c^2 - 1} \\ &\geq 3bc - 2\sqrt{b^2c^2 - 1}\,. \end{aligned}$$

For $x \geq 1$, let $y = 3x - 2\sqrt{x^2 - 1}$. Then y is positive and from $(y - 3x)^2 = 4(x^2 - 1)$ we get $5x^2 - 6xy + y^2 + 4 = 0$. Since x is real, the discriminant of the quadratic polynomial above must be non-negative.

Thus, $(-6y)^2 - 20(y^2 + 4) \geq 0$, or $16y^2 - 80 \geq 0$, from which we obtain $y \geq \sqrt{5}$. The result now follows by setting $x = bc$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece (two solutions); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece (two solutions); OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SOTIRIS LOURIDAS, Aegaleo, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incorrect solution submitted.

From the proof given above, it is easy to deduce that equality holds if and only if $b = c$ *and* $a = \sqrt[4]{\frac{4}{5}}$*, in which case* $A = \cos^{-1}\left(\frac{2}{3}\right) \approx 48.19^\circ$. *This was pointed out by Barbara, Geupel, Hess, Konečný, Tsapakidis, and Zvonaru.*

Both Barbara and Demis generalized to an arbitrary triangle of area $k > 0$*, proving that* $a^2 + \csc A \geq \sqrt{8k + 1}$ *and the lower bound* $\sqrt{8k + 1}$ *is the best possible.*

3299. [2007 : 487, 489] *Proposed by Victor Oxman, Western Galilee College, Israel.*

Given positive real numbers a, b, and w_b, show that

(a) if a triangle ABC exists with $BC = a$, $CA = b$, and the length of the interior bisector of angle B equal to w_b, then it is unique up to isomorphism;

(b) for the existence of such a triangle in (a), it is necessary and sufficient that

$$b > \frac{2a\,|a - w_b|}{2a - w_b} \geq 0\,;$$

(c) if h_a is the length of the altitude to side BC in such a triangle in (a), we have $b > |a - w_b| + \frac{1}{2}h_a$.

Solution by Michel Bataille, Rouen, France.

(a) For convenience we write $w = w_b$. Let the interior bisector of $\angle B$ meet AC at W. We assume that $\triangle ABC$ exists with $BC = a, CA = b$, and $BW = w$, and shall produce a (implicitly defined) formula for the third side $c = AB$. We recall that $w = \frac{2ac\cos\frac{B}{2}}{a+c}$, so that

$$2a\cos\frac{B}{2} - w \;=\; \frac{aw}{c}\,.$$

By the Law of Cosines we have $WC^2 = a^2 + w^2 - 2aw\cos\frac{B}{2}$; using the standard formula $WC = \frac{ab}{a+c}$, we therefore have

$$\frac{a^2b^2}{(a+c)^2} \;=\; a^2 - w\left(2a\cos\frac{B}{2} - w\right) \;=\; a^2 - \frac{aw^2}{c}\,.$$

It follows that c must be the unique positive solution of $f(x) = a$, where f is the decreasing function on $(0, \infty)$ given by

$$f(x) \;=\; \frac{ab^2}{(a+x)^2} + \frac{w^2}{x}\,.$$

Thus, if a triangle ABC with the given parameters does exist, then its side lengths are uniquely determined, and (a) is proved.

(b) First, it is easy to see that the two given inequalities are equivalent to the conjunction of the following three inequalities

$$w \;<\; 2a\,; \quad (2a+b)w \;<\; 2a(a+b)\,; \quad 2a(a-b) \;<\; (2a-b)w\,. \qquad (1)$$

Second, the existence of a suitable triangle ABC is equivalent to the fact that the solution c of $f(x) = a$ satisfies $|a-b| < c < a+b$; that is,

$$f(|a-b|) \;>\; a \;>\; f(a+b)\,. \qquad (2)$$

To show that (1) and (2) are equivalent, we first suppose that the inequalities in (1) hold. The inequality $f(a+b) < a$ reduces to the equivalent inequality $(2a+b)w < 2a(a+b)$, which holds by (1). As for $f(|a-b|) > a$, it is equivalent to

$$ab^2\,|a-b| \;+\; w^2\left(a+|a-b|\right)^2 \;>\; a\,|a-b|\left(a+|a-b|\right)^2.$$

This reduces to $w^2b^2 > 0$ when $a \le b$, so it holds then; but it also holds if $b < a$ since then it becomes $w(2a-b) > 2a(a-b)$, which holds by (1). We have proved that (1) implies (2).

Conversely, we assume that (2) holds (that is, that a triangle ABC with the given parameters exists). Then $\frac{w}{2a} = \frac{c}{a+c}\cdot\cos\frac{B}{2}$; hence $w < 2a$.

Moreover, from $f(a+b) < a$ we obtain $(2a+b)w < 2a(a+b)$. As for the condition $2a(a-b) < (2a-b)w$, it follows from $f(|a-b|) > a$ if $a > b$, and from $(2a-w)(a-b) < aw$ if $a \le b$ (because $2a > w$, the left side is negative). The desired equivalence follows.

(c) Note that $b > |a-w| + \frac{1}{2}h_a$ is equivalent to

$$ab \; > \; a\,|a-w| \; + \; \text{Area}(ABC)\,;$$

that is, to $ab > a\,|a-w| + \left(\frac{a+c}{2}\right) w \sin\frac{B}{2}$. Since $a\,|a-w| < \frac{(2a-w)b}{2}$ (from part (b)), the latter will certainly hold if

$$b \; \ge \; (a+c)\sin\frac{B}{2}\,.$$

This inequality is equivalent to

$$\sin B \; \ge \; (\sin A + \sin C)\sin\frac{B}{2}\,,$$

or to

$$2\cos\frac{B}{2} \; \ge \; 2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)\,,$$

or finally to

$$1 \ge \cos\left(\frac{A-C}{2}\right)\,,$$

which is certainly true. The result follows.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA (part (c) *only); and the proposer.*

Parts (a) *and* (b) *of our problem appear on page 11 of D.S. Mitrinović et al.,* Recent Advances in Geometric Inequalities, Kluwer Academic Publishers, *1989 as the first of 40 existence results from a 1952 paper (in Czech) by G. Petrov.*

In addition to his solution, Oxman also addressed the question of constructibility. Exercise 4 on page 142 of Günter Ewald's Geometry: An Introduction *(Wadsworth Publ., 1971) says that in general a triangle cannot be constructed by ruler and compass given the lengths a, b, and w_b, even when that triangle exists. The author suggests that the proof of his claim can be simplified by taking both the given side lengths equal to 1. The formula $f(x) = 1$ from part* (a) *of the featured solution (with $a = b = 1$, and w^2 chosen to be rational) is a cubic equation with rational coefficients. One simply has to choose a value of w for which the resulting cubic equation has no rational root. The theory of Euclidean constructions then tells us that the positive root, namely c, cannot be constructed by using ruler and compass.*

3300. [2007 : 487, 489] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let a, b, and c be positive real numbers. For any positive integer n define

$$F_n \; = \; \left(\frac{3(a^n+b^n+c^n)}{a+b+c} - \sum_{\text{cyclic}} \frac{b^n+c^n}{b+c}\right)\,.$$

(a) Prove that $F_n \ge 0$ for $n \le 5$.

(b)★ Prove or disprove that $F_n \ge 0$ for $n \ge 6$.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Since $F_1 = 0$, we take $n > 1$. We note that $\left(x^{n-1} - y^{n-1}\right)(x - y) \geq 0$ for all positive x and y, with equality if and only if $x = y$. We have

$$\begin{aligned} (a+b+c)F_n &= 3\left(a^n + b^n + c^n\right) - (a+b+c)\sum_{\text{cyclic}} \frac{b^n + c^n}{b+c} \\ &= \left(a^n + b^n + c^n\right) - \sum_{\text{cyclic}} \frac{a\left(b^n + c^n\right)}{b+c} \\ &= \sum_{\text{cyclic}} \left[a^n - \frac{a\left(b^n + c^n\right)}{b+c}\right] \\ &= \sum_{\text{cyclic}} \left[\frac{ab\left(a^{n-1} - b^{n-1}\right)}{(b+c)} + \frac{ac\left(a^{n-1} - c^{n-1}\right)}{(b+c)}\right] \\ &= \sum_{\text{cyclic}} \frac{ab\left(a^{n-1} - b^{n-1}\right)(a-b)}{(b+c)(c+a)} \;\geq\; 0\,. \end{aligned}$$

Equality holds if and only if $a = b = c$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; VASILE CÎRTOAJE, University of Ploiesti, Romania; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PANOS E. TSAOUSSOGLOU, Athens, Greece (part (a) only); STAN WAGON, Macalester College, St. Paul, MN, USA (part (a) only); TITU ZVONARU, Comănești, Romania; and the proposer.

Cîrtoaje mentioned that this problem was posted (together with a solution similar to the one featured above) by Wolfgang Berndt (Spanferkel) on the Mathlinks Forum website `http://www.mathlinks.ro/Forum/viewtopic.php?p=607167` *in August 2006. Barbara, Cîrtoaje, and Dergiades proved the following generalization: If $a_1, a_2, \ldots, a_m$ are positive real numbers, $m \geq 2$, and*

$$F_n = \frac{m(a_1^n + a_2^n + \cdots + a_m^n)}{a_1 + a_2 + \cdots + a_m} - \sum_{\text{cyclic}} \frac{a_2^n + \cdots + a_m^n}{a_2 + \cdots + a_m}\,,$$

then $F_n \geq 0$ for all $n \geq 1$. Alt ultimately proved that if a, b, c, p, and q are positive real numbers and

$$F(p,q) = \frac{3(a^p + b^p + c^p)}{a^q + b^q + c^q} - \sum_{\text{cyclic}} \frac{a^p + b^p}{a^q + b^q}\,,$$

then $(p-q)F(p,q) \geq 0$.