Sliding Down Inclines with Fixed Descent Time: a Converse to Galileo's Law of Chords

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Suppose that a vector is anchored at the origin and lies along the positive x-axis. Consider rotating the vector counter clockwise about the origin through an angle A, with $0 < A < \pi$. Consider a particle of mass m which is initially at rest and then slides, under gravity and without friction, from a starting point on the inclined vector down towards the origin. Let D_A denote the distance of the starting point from the origin. For each value of A, suppose that D_A is chosen to ensure that the particle requires exactly T seconds to reach the origin. Determine the curve characterized by the starting points of the particles.

For the particle under consideration, let $d_A(t)$ be the distance travelled along the vector at time t, $v_A(t) = d'_A(t)$ be the velocity along the vector at time t, and let $a_A(t) = v'_A(t)$ be the acceleration along the vector at time t.

Note that by definition, $D_A = d_A(T)$.

If $a_A(t)$ is some constant K, then

$$d_A(t) \;=\; \frac{K}{2}t^2\,. \tag{1}$$

This may be confirmed by integrating $a_A(t)$ twice with respect to time and applying the initial conditions $d_A(0) = 0$ and $v_A(0) = 0$ to obtain zero for both constants of integration.

Since the particle is sliding down a frictionless incline in the Earth's gravitational field, the component of acceleration along the incline is $K = g\sin A$, where g is the acceleration due to gravity at the Earth's surface.

If the descent time is fixed at T seconds, then

$$D_A \;=\; d_A(T) \;=\; \frac{g}{2}T^2\sin A\,. \tag{2}$$

A point (r,θ) in polar coordinates may be expressed in Cartesian coordinates as (x,y), where $x = r\cos\theta$ and $y = r\sin\theta$. Consider the following equation, which is expressed in polar coordinates as

$$r \;=\; 2c\sin\theta\,, \tag{3}$$

where $r > 0$ and $0 < \theta < \pi$. Multiplying both sides of equation (3) by r yields

$$r^2 \;=\; 2cr\sin\theta \tag{4}$$

By setting $x = r\cos\theta$ and $y = r\sin\theta$, equation (4) may be re-expressed in Cartesian coordinates as

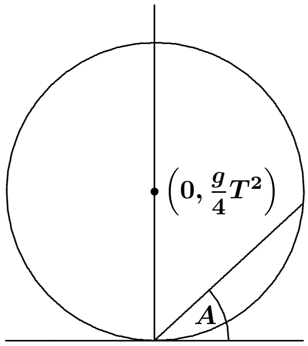
$$x^2 + y^2 \;=\; 2cy\,.$$

Subtracting $2cy$ from each side and completing the square on $y^2 - 2cy$ yields

$$x^2 + (y - c)^2 \;=\; c^2 \tag{5}$$

which is the equation of a circle of radius c centred at the point $(0, c)$.

Thus, as depicted in the figure at right, the locus of points defined by equation (2) is a circle resting upon the origin $(0, 0)$, centred at $\left(0, \frac{g}{4}T^2\right)$ on the vertical axis, and of radius $\frac{g}{4}T^2$.



On an historical note, the motivation for this problem arose while considering the Law of Chords, which was stated and proven by Galileo Galilei in his 1638 masterpiece *Dialogues Concerning Two New Sciences*. Galileo considered rates of descent along a vertical circle and, with his Proposition VI, established the Law of Chords (see [1], p. 212):

> If from the highest or lowest point in a vertical circle there be drawn any inclined planes meeting the circumference, the times of descent along these chords are each equal to the other.

Galileo's proof of the Law of Chords is presented via a series of geometric propositions, which require familiarity with many of Euclid's theorems.

The question the author wished to address was whether the vertical circle is the only curve with the property that descent time to the lowest point on the curve is constant for all chords. This paper demonstrates that the vertical circle, or one of its component arcs intersecting at the lowest point of the circle, is indeed the only such curve.

References

[1] Galilei, Galileo. 1638/1952 *Dialogues Concerning Two New Sciences*, translated by H. Crew and A. de Salvio, Encyclopaedia Britannica, Chicago, 1952.

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