A Limit of an Improper Integral Depending on One Parameter

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In this article we will calculate the following limit of an improper integral that depends on one parameter $\lambda \in \mathbb{R}^+$

$$\lim_{\lambda \to 0^+} \int_0^\infty \exp(-\lambda \mathcal{K}(x+l)^n) \lambda \mathcal{K} n x^{n-1}\, dx\,,$$

where $\mathcal{K}$ and l are positive real numbers and n is a positive integer.

This is an interesting example where one cannot interchange the order of the limit and the integral. Through a nice application of elementary tools such as change of variables in integrals and the Binomial Theorem, one is able to obtain this limit.

This problem arose in the calculation of a lower bound for a probability of theoretical interest in the study of multidimensional Poisson point processes, namely

$$\lim_{\lambda \to 0^+} \int_0^\infty \exp\bigl(-\lambda v_n(1)\bigl[(r+l)^n - r^n\bigr]\bigr) \lambda v_n(1) n r^{n-1} \exp\bigl(-\lambda v_n(1) r^n\bigr)\, dr\,,$$

where $v_n(1)$ is the volume of the n–dimensional unit ball, λ is the Poisson intensity, l is the distance between two distinguished points in $\mathbb{R}^n$, and r is the distance from the first point to the closest occurrence in the Poisson point process. We shall show that this limit is equal to 1.

Proposition For all positive real numbers $\mathcal{K}$ and l, and for each positive integer n we have

$$\lim_{\lambda \to 0^+} \int_0^\infty \exp\bigl(-\lambda \mathcal{K}\bigl[(x+l)^n\bigr]\bigr) \lambda \mathcal{K} n x^{n-1}\, dx \;=\; 1\,.$$

Proof: The case $n = 1$ is a straightforward calculation:

$$\begin{aligned} & \lim_{\lambda \to 0^+} \int_0^\infty \exp\bigl(-\lambda \mathcal{K}(x+l)\bigr) \lambda \mathcal{K}\, dx \\ = \;& \lim_{\lambda \to 0^+} \Bigl(-\exp\bigl(-\lambda \mathcal{K}(x+l)\bigr)\Big|_0^\infty\Bigr) \\ = \;& \lim_{\lambda \to 0^+} \exp\left(-\lambda \mathcal{K} l\right) \;=\; 1\,. \end{aligned}$$

Henceforth, we take $n > 1$.

Making first the change of variable

$$v \;=\; \mathcal{K}x^n\,, \qquad dv \;=\; \mathcal{K}nx^{n-1}\,dx\,,$$

in the integral we obtain

$$\int_0^\infty \lambda \exp\left[-\lambda\left(v^{\frac{1}{n}} + (l^n\mathcal{K})^{\frac{1}{n}}\right)^n\right]\,dv\,.$$

Changing variables again, this time according to

$$u^{\frac{1}{n}} \;=\; v^{\frac{1}{n}} + (l^n\mathcal{K})^{\frac{1}{n}}\;, \qquad dv \;=\; \left(\frac{u}{v}\right)^{\left(\frac{1}{n}-1\right)}\,du\,,$$

we obtain

$$\begin{aligned}
&\lim_{\lambda\to 0^+}\int_0^\infty \lambda \exp\left[-\lambda\left(v^{\frac{1}{n}} + (l^n\mathcal{K})^{\frac{1}{n}}\right)^n\right]\,dv \\
=\;& \lim_{\lambda\to 0^+}\int_{l^n\mathcal{K}}^\infty \lambda \exp\left(-\lambda u\right)\left(\frac{u}{v}\right)^{\left(\frac{1}{n}-1\right)}\,du \\
=\;& \lim_{\lambda\to 0^+}\int_{l^n\mathcal{K}}^\infty \lambda \exp\left(-\lambda u\right)\left[1-\left(\frac{l^n\mathcal{K}}{u}\right)^{\frac{1}{n}}\right]^{n-1}\,du\,,
\end{aligned}$$

which by the Binomial Theorem becomes

$$\begin{aligned}
&\lim_{\lambda\to 0^+}\int_{l^n\mathcal{K}}^\infty \lambda \exp\left(-\lambda u\right)\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\left(\frac{l^n\mathcal{K}}{u}\right)^{\frac{k}{n}}\,du \\
=\;& 1+\lim_{\lambda\to 0^+}\sum_{k=1}^{n-1}\binom{n-1}{k}\int_{l^n\mathcal{K}}^\infty \lambda \exp\left(-\lambda u\right)(-1)^k\left(\frac{l^n\mathcal{K}}{u}\right)^{\frac{k}{n}}\,du\,.
\end{aligned}$$

After yet another change of variable,

$$w \;=\; u^{\frac{1}{n}}\,, \qquad dw \;=\; \frac{1}{n}u^{\left(\frac{1}{n}-1\right)}\,du\,,$$

the preceding expression becomes

$$1+\lim_{\lambda\to 0^+}\sum_{k=1}^{n-1}\binom{n-1}{k}n\left(-l\mathcal{K}^{\frac{1}{n}}\right)^k\lambda\int_{l\mathcal{K}^{\frac{1}{n}}}^\infty \exp(-\lambda w^n)w^{n-1-k}\,dw\,.$$

It remains to show that

$$\lim_{\lambda\to 0^+}\lambda\int_{l\mathcal{K}^{\frac{1}{n}}}^\infty \exp\left(-\lambda w^n\right)w^{n-1-k}\,dw \tag{1}$$

is zero for each $k \in \{1\,,\,2,\,\ldots\,,\,n-1\}$.

To do this, we make a final change of variables

$$z = \lambda^{\frac{1}{n}} w\,, \qquad dz = \lambda^{\frac{1}{n}}\, dw\,,$$

which yields

$$\begin{aligned} & \lambda \int_{l\mathcal{K}^{\frac{1}{n}}}^{\infty} \exp\left(-\lambda w^n\right) w^{n-1-k}\, dw \\ = \;& \lambda^{\frac{k}{n}} \int_{(\lambda l^n \mathcal{K})^{\frac{1}{n}}}^{\infty} \exp\bigl(-z^n\bigr) z^{n-1-k}\, dz \\ < \;& \lambda^{\frac{k}{n}} \int_0^{\infty} \exp\bigl(-z^n\bigr) z^{n-1-k}\, dz\,. \end{aligned} \tag{2}$$

It now suffices for us to show that the last integral in (2) is finite for each $k \in \{1, 2, \ldots, n-1\}$.

Setting

$$C = \int_0^1 \exp\left(-z^n\right) z^{n-1-k}\, dz$$

and noting that if $z \geq 1$, then $z^{n-1-k} \leq z^{n-1}$, we finally obtain

$$\begin{aligned} \int_0^{\infty} \exp\bigl(-z^n\bigr) z^{n-1-k}\, dz &\leq C + \int_1^{\infty} \exp\bigl(-z^n\bigr) z^{n-1}\, dz \\ &= C + \frac{1}{ne}\,, \end{aligned}$$

which is a finite number. This completes the proof. ■

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