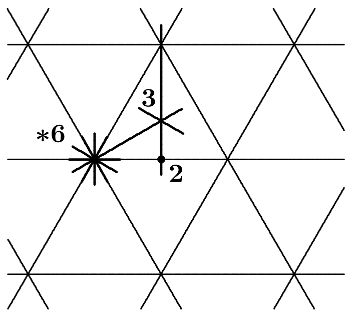
BOOK REVIEW

John Grant McLoughlin

The Symmetries of Things
By John H. Conway, Heidi Burgiel, and Chaim Goodman–Strauss, published by AK Peters, Wellesley, MA, 2008
ISBN 978-1-56881-220-5, hardcover, 423+xxv pages, US$75.00
Reviewed by **J. Chris Fisher**, *University of Regina, Regina, SK*

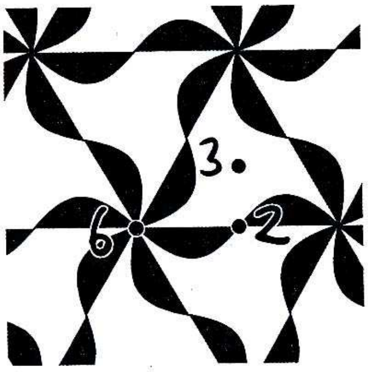
The authors set themselves the ambitious goal of producing a book that appeals to everybody. As far as I can tell from a single reading, they have succeeded admirably. The first thing anybody would notice about the book is that it is filled with beautiful and fascinating photographs and computer drawings. No special knowledge is required for admiring beauty; this book would be as much at home on the living room coffee table as on the office shelf. Of course, it is primarily a mathematics book.

The contents have been organized into three parts. The first of them describes and enumerates the symmetries found in repeating patterns on surfaces; it is written at a level suitable for *Crux with Mayhem* readers. This part might well serve as a textbook for a geometry course directed at university students specializing in mathematics, education, physical science, or computer science. What makes the authors' approach both novel and elementary is the introduction of what they call the *orbifold signature notation*. Groups are not needed here; the concept can be easily described and quickly mastered. Here is the idea. A point in a pattern where two mirrors of symmetry meet at an angle of $\frac{\pi}{m}$ is called *kaleidoscopic* and is denoted by $*m$; points having rotational symmetry of order m (but no kaleidoscopic symmetry) are called *gyrational* and are denoted by m (with no asterisk). If a region has an oppositely oriented image in the pattern that is not explained by mirrors, then these two regions must be related by a glide reflection, which here is called a *miracle* (short for "mirrorless crossing", they say), denoted by $\times$. To identify the signature of any repeating plane pattern one writes down the symbols starting from the middle and working outward. First locate mirror lines and each kind of kaleidoscopic point, if any (where two points are of the same kind if they are related by a symmetry of the pattern); list them after the asterisk in decreasing order. Next locate any gyrational points and order them before any asterisk. Then look for miracles. Typical signatures are $*\mathbf{632}$ for a pattern whose symmetry is explained by three kinds of mirrors that meet in pairs at angles of $\frac{\pi}{6}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$; $\mathbf{632}$ (having no asterisk) for a pattern with



$*\mathbf{632}$

6–fold, 3–fold, and 2–fold gyrational points but no reflections or miracles; 2∗22 for a pattern with two kinds of kaleidoscopic points where a pair of mirrors intersect at right angles, and one point where there is a half-turn symmetry but no mirror.



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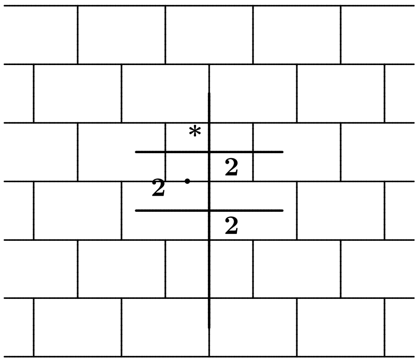

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Unlike most other notation systems that have been devised for describing plane symmetry, these orbifold signatures can also be used to describe frieze patterns and spherical patterns. (We learn in Part III that they work equally well for describing hyperbolic patterns.) But how does one know that the resulting lists of 17 signatures for plane patterns, 7 signatures for frieze patterns, and 14 signatures for spherical patterns are correct and complete? There is a "Magic Theorem" that assigns a cost to every symbol in the signature in such a way that plane patterns and frieze patterns cost exactly $2 while spherical patterns cost a bit less. That theorem tells us immediately which signatures are feasible. To establish the Magic Theorem, a pattern on the surface is associated with a folded surface they call an *orbifold*. The orbifold is obtained by identifying points related by a symmetry of the pattern (whereby points of an *orbit* are *folded* atop of one another so that a single representative point of every orbit lives on the orbifold). This sounds a bit scary, but the authors manage to explain the details in a gentle way using suitable pictures and simple examples. They then state the Classification Theorem for Surfaces and provide Conway's elementary and intuitive Zip proof. They also prove that these surfaces can be distinguished using Euler's formula (involving the numbers of vertices, edges, and faces of a suitable map on the surface), which they also prove. Since the orbifolds are easily classified using Euler's formula, the corresponding patterns are thereby classified.

Remarkably, all the proofs should satisfy the professional mathematician even though they are directed at an elementary audience. The authors achieve this feat by repeatedly reducing technical difficulties down to problems that are postponed to the following chapter. This way they present one concept at a time, as compared to the typical textbook's initial barrage of poorly motivated definitions and lemmas. Their proofs are every bit as brilliant as their notation. The illustrations are not just beautiful, but they have been carefully chosen to clarify the exposition. I really appreciated the authors' decision to repeat pictures that they require for illustrating new ideas — instead of making the reader turn back to a picture on an earlier page, they reproduced a smaller version of it whenever needed. The authors clearly have fun coining whimsical new words; their terminology will not appeal to everybody, but the informal nature of their discussions makes for enjoyable reading. I rather liked the word *miracle* in place of the standard, but awkward and misleading term *glide reflection*; however I saw little need for *gyrational* in place of *rotational* or *wandering* in place of *translation*. We will have to wait to see which words catch on.

What I have described so far is the content of the 116 pages of the first nine chapters. Originally, according to the preface, this was all that the authors had intended to write. But they decided it was worthwhile to extend the signature to colour symmetry, and the book grew from there. For a careful reading of Part II the reader needs some group theory and a bit of mathematical maturity. The authors' main goal for this part is to present their analysis and notation for colour symmetry. They enumerate the p–fold colour types for plane, spherical, and frieze patterns (for all primes p). The complete classifications appear in a book for the first time. Along the way the authors show how their orbifold notation corresponds to previous classification systems, which gives them the opportunity to discuss the shortcomings of those systems. Also in this part, they enumerate the isohedral tilings of the sphere and plane, and they extend to $n = 2009$ the Besche-Eick-O'Brien table of the number of abstract groups of each order n.

The informative lists of Part II can probably be understood by readers who might not take an interest in the accompanying technical arguments. Similar comments apply to Part III, which the authors expect to be completely understood only by a few professional mathematicians. Still, as they point out, much of Part III can profitably be explored by other readers, while many more will enjoy inspecting the pretty pictures. Here, among other things, the authors discuss hyperbolic groups and Archimedean polyhedra and tilings; they list the 219 crystallographic space groups (and explain why chemists distinguish 230 groups), and they provide a complete list for the first time in print of the 4–dimensional Archimedean polytopes. Apparently they could have kept writing, but they decided to leave something for the rest of us to do. Their final words are, "A universe awaits — Go forth!"

I thank Bruce Shawyer for inviting me to serve as Book Review Editor. I am grateful to the late Jim Totten for guiding me during his tenure; Bruce Crofoot for insightful commentary; Václav Linek for recent support; Shawn Godin for leadership with Mayhem. Thanks go to all the reviewers, but especially this trio: Chris Fisher, a dependable source of thought provoking reviews usually concerning geometry; Ed Barbeau, an eclectic mathematician who is eager to help; and my successor, Amar Sodhi. Amar's passion for mathematics will shine as he assumes this role. Welcome Amar! Thanks to the ***CRUX with MAYHEM*** community for an enjoyable journey. — ***John Grant McLoughlin***