THE OLYMPIAD CORNER

No. 274

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With the Winter break coming up, I have decided to focus this issue mainly on providing problems for your puzzling pleasure, and to give some time for the mails to deliver the solutions to problems from 2008 numbers of the *Corner* to restore the readers' solutions file, which is particularly thin for the February 2008 number, as you will see later in the column.

To start you off we have the problems proposed but not used at the 47^{th} IMO in Slovenia 2006. My thanks go to Robert Morewood, Canadian Team Leader at the IMO for collecting them for our use.

47^{th} INTERNATIONAL MATHEMATICAL OLYMPIAD SLOVENIA 2006
Problems Proposed But Not Used

Contributing Countries. Argentina, Australia, Brazil, Bulgaria, Canada, Colombia, Czech Republic, Estonia, Finland, France, Georgia, Greece, Hong Kong, India, Indonesia, Iran, Ireland, Italy, Japan, Republic of Korea, Luxembourg, Netherlands, Poland, Peru, Romania, Russia, Serbia and Montenegro, Singapore, Slovakia, South Africa, Sweden, Taiwan, Ukraine, United Kingdom, United States of America, Venezuela.

Problem Selection Committee. Andrej Bauer, Robert Geretschläger, Géza Kós, Marcin Kuczma, Sventoslav Savchev.

Algebra

A1. Given an arbitrary real number a_0, define a sequence of real numbers a_0, a_1, a_2, ... by the recursion

$$a_{i+1} = \lfloor a_i \rfloor \cdot \{a_i\}, \qquad i \geq 0,$$

where $\lfloor a_i \rfloor$ is the greatest integer not exceeding a_i, and $\{a_i\} = a_i - \lfloor a_i \rfloor$. Prove that $a_i = a_{i+2}$ for sufficiently large i.

A2. Let $a_0 = -1$ and define the sequence of real numbers a_0, a_1, a_2, ... by the recursion

$$\sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0$$

for $n \geq 1$. Show that $a_n > 0$ for $n \geq 1$.

A3. Let $c_0 = 1$, $c_1 = 0$ and define the sequence c_0, c_1, c_2, $\ldots$ by the recursion $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Let S be the set of ordered pairs (x, y) such that

$$x = \sum_{j \in J} c_j \quad \text{and} \quad y = \sum_{j \in J} c_{j-1}$$

for some finite set J of positive integers. Prove that there exist real numbers α, β, m, and M with the property that an ordered pair of non-negative integers (x, y) satisfies the inequality

$$m < \alpha x + \beta y < M$$

if and only if $(x, y) \in S$. (By convention an empty sum is 0.)

A4. Let a_1, a_2, $\ldots$, a_n be positive real numbers. Prove that

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \sum_{i<j} a_i a_j .$$

A5. Let a, b, and c be the lengths of the sides of a triangle. Prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3 .$$

Combinatorics

C1. There are $n \geq 2$ lamps L_1, L_2, $\ldots$, L_n arranged in a row. Each of them is either *on* or *off*. Initially the lamp L_1 is on and all of the other lamps are off. Each second the state of each lamp changes as follows: if the lamp L_i and its neighbours (L_1 and L_n each have one neighbor, any other lamp has two neighbours) are in the same state, then L_i is switched off; otherwise, L_i is switched on. Prove that there are

(a) infinitely many n for which all of the lamps will eventually be off,

(b) infinitely many n for which the lamps will never be all off.

C2. Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside of P. Prove that for every real number x

$$\sum_{P} x^{a(P)}(1-x)^{b(P)} = 1 ,$$

where the sum is taken over all convex polygons with vertices in S. (A line segment, a point, and the empty set are convex polygons of 2, 1, and 0 vertices, respectively.)

C3. A cake is shaped as an $n \times n$ square with n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathcal{A}$.

Let $\mathcal{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathcal{B}$ than of arrangement $\mathcal{A}$. Prove that the arrangement $\mathcal{B}$ can be obtained from $\mathcal{A}$ by performing a sequence of *swaps*, where a *swap* consists of selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and then moving these two strawberries to the other two corners of that rectangle.

C4. An (n,k)-*tournament* is a competition with n players held in k rounds such that

(a) Each player plays in each round, and every two players meet at most once.

(b) If player A meets player B in round i, player C meets player D in round i, and player A meets player C in round j, then player B meets player D in round j.

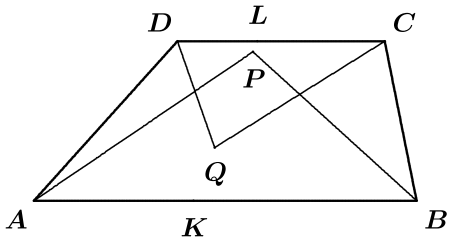
Determine all pairs (n,k) for which there exists an (n,k)-tournament.

C5. A *holey triangle* is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A *diamond* is a unit rhombus with angles of 60° and 120°. Prove that a holey triangle T can be tiled with diamonds if and only if for each $k = 1, 2, \ldots, n$ every upward equilateral triangle of side length k in T contains at most k holes.

C6. Let $\mathcal{P}$ be a convex polyhedron with no parallel edges and no edge parallel to a face other than the two faces it borders. A pair of points on $\mathcal{P}$ are *antipodal* if there exist two parallel planes each containing one of the points and such that $\mathcal{P}$ lies between them. Let A be the number of antipodal pairs of vertices and let B be the number of antipodal pairs of mid-points of edges. Express $A - B$ in terms of the numbers of vertices, edges, and faces of $\mathcal{P}$.

Geometry

G1. Let $ABCD$ be a trapezoid with $AB||CD$ and $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, such that $AK : KB = DL : LC$. Suppose that there are points P and Q on the line segment KL satisfying



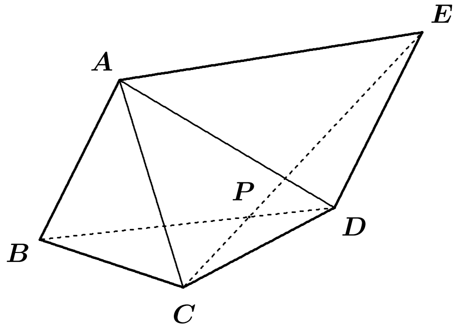
$$\angle APB = \angle BCD \quad \text{and} \quad \angle CQD = \angle ABC\,.$$

Prove that the points P, Q, B, and C are concyclic.

G2. Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE\,;$$
$$\angle ABC = \angle ACD = \angle ADE\,.$$

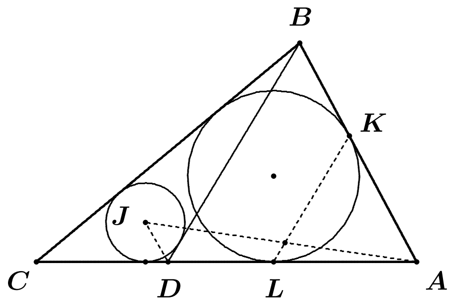
The diagonals BD and CE intersect at P. Prove that the line AP bisects the side CD.



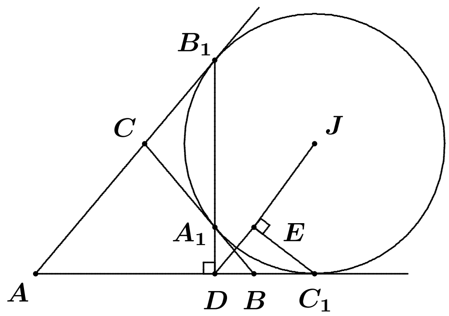
G3. A point D is chosen on the side AC of a triangle ABC with

$$\angle ACB < \angle BAC < 90^\circ$$

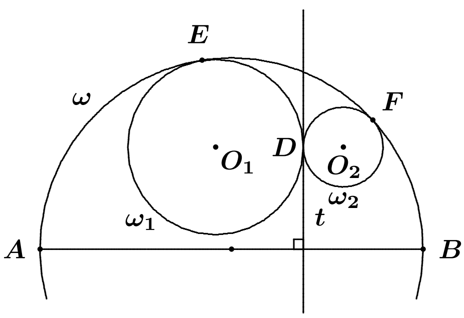
in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incentre of triangle BCD. Prove that the line KL intersects the line segment AJ at its mid-point.



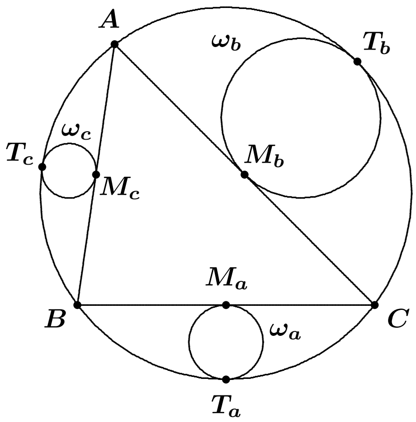
G4. In triangle ABC, let J be the centre of the excircle tangent to side BC at A_1 and to the extensions of sides AC and AB at B_1 and C_1, respectively. Suppose that the lines A_1B_1 and AB are perpendicular and intersect at D. Let E be the foot of the perpendicular from C_1 to line DJ. Determine the angles $\angle BEA_1$ and $\angle AEB_1$.



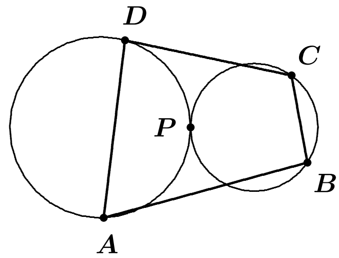
G5. Circles ω_1 and ω_2 with centres O_1 and O_2 are externally tangent at point D and internaly tangent to a circle ω at points E and F, respectively. Line t is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to t, so that A, E, and O_1 are on the same side of t. Prove that the lines AO_1, BO_2, EF, and t are concurrent.



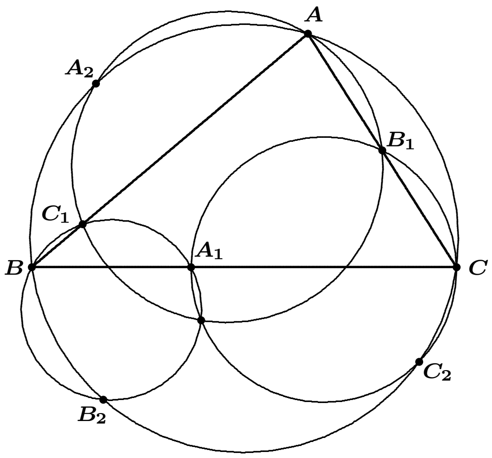
G6. In a triangle ABC, let M_a, M_b, M_c be the respective mid-points of the sides BC, CA, AB and let T_a, T_b, T_c be the mid-points of the arcs BC, CA, AB of the circumcircle of ABC not containing A, B, C, respectively. For each $i \in \{a,b,c\}$, let ω_i be the circle with diameter M_iT_i. Let p_i be the common external tangent to ω_j, ω_k such that $\{i,j,k\} = \{a,b,c\}$ and such that ω_i lies on one side of p_i while ω_j, ω_k lie on the other side. Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude.



G7. Let $ABCD$ be a convex quadrilateral. A circle passing through A and D and a circle passing through B and C are externally tangent at the point P in the interior of the quadrilateral. Prove that if $\angle PAB + \angle PDC \leq 90^\circ$ and $\angle PBA + \angle PCD \leq 90^\circ$, then $AB + CD \geq BC + AD$.



G8. Points A_1, B_1, C_1 are on the sides BC, CA, AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2, respectively (that is, $A_2 \neq A$, $B_2 \neq B$, $C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the mid-points of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.



Number Theory

N1. Given $x \in (0, 1)$ let $y \in (0, 1)$ be the number whose n^{th} digit after the decimal point is the $(2^n)^{\text{th}}$ digit after the decimal point of x. Prove that if x is a rational number, then y is a rational number.

N2. For each positive integer n let

$$f(n) = \frac{1}{n}\left(\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor\right),$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

(a) Prove that $f(n+1) > f(n)$ for infinitely many n.

(b) Prove that $f(n+1) < f(n)$ for infinitely many n.

N3. Find all solutions in integers of the equation

$$\frac{x^7 - 1}{x - 1} = y^5 - 1.$$

N4. Let a and b be relatively prime integers with $1 < b < a$. Define the *weight* of an integer c, denoted by $w(c)$, to be the minimum possible value of $|x| + |y|$ taken over all pairs of integers x and y such that

$$ax + by = c.$$

An integer c is called a *local champion* if $w(c) \geq \max\{w(c \pm a), w(c \pm b)\}$. Find all local champions and determine their number.

N5. Prove that for every positive integer n, there exists an integer m such that $2^m + m$ is divisible by n.

A second problem set is the 2005/06 Swedish Mathematical Contest. My thanks go to Robert Morewood, Canadian Team Leader at the IMO, for collecting them for our use.

SWEDISH MATHEMATICAL CONTEST 2005/2006

Final Round

November 19, 2005 (Time: 5 hours)

1. Find all solutions in integers x and y of the equation

$$(x + y^2)(x^2 + y) = (x + y)^3.$$

2. A queue in front of a counter consists of 12 persons. The counter is then closed because of a technical problem and the 12 people are redirected to another one. In how many different ways can the new queue be formed if each person maintains the same position as before, or is one step closer to the front, or is one step farther from the front?

3. In the triangle ABC the angle bisector from A intersects the side BC in the point D and the angle bisector from C intersects the side AB in the point E. The angle at B is greater than $60°$. Prove that $AE + CD < AC$.

4. The polynomial $f(x)$ is of degree four. The zeroes of f are real and form an arithmetic progression, that is, the zeroes are a, $a+d$, $a+2d$, and $a+3d$ where a and d are real numbers. Prove that the three zeroes of $f'(x)$ also form an arithmetic progression.

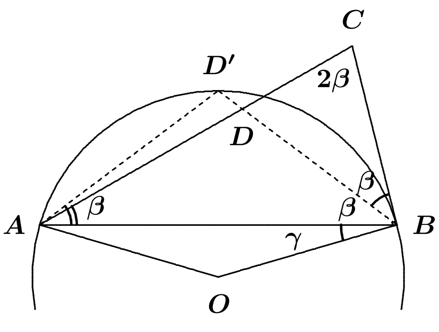
5. Each square on a 2005×2005 chessboard is painted either black or white. This is done in such a way that each 2×2 "sub-chessboard" contains an odd number of black squares. Show that the number of black squares among the four corner squares is even. In how many different ways can the chessboard be painted so that the above condition is satisfied?

6. All the edges of a regular tetrahedron are of length 1. The tetrahedron is projected orthogonally into a plane. Determine the largest possible area and the least possible area of the image.

Next we look at an alternate solution to problem 2 of the 17th Irish Mathematical Olympiad discussed at [2007 : 151; 2008 : 88-89].

2. Let A and B be distinct points on a circle T. Let C be a point distinct from B such that $|AB| = |AC|$ and such that BC is tangent to T at B. Suppose that the bisector of $\angle ABC$ meets AC at a point D inside T. Show that $\angle ABC > 72°$.

Alternate Solution by Luyan Zhong-Qiao, Columbia International College, Hamilton, ON.



Let $D' \neq B$ be the second point of intersection of BD with the circle T. Let $\angle ABD = \angle DBC = \beta$. Since $AB = AC$, we have $\angle ACB = 2\beta$. Also, BC is tangent to the circle and AB is a chord, hence $\angle D'AB = \beta$. Let $\angle ABO = \gamma$.

Since BC is tangent to T, we have $2\beta + \gamma = 90^\circ$. Adding the angles in the isosceles triangle ABC yields $4\beta + \angle CAB = 180^\circ$. From these two equations it follows that $\angle CAB = 2\gamma$. Since D is inside T we have

$$\beta = \angle D'AB > \angle CAB = 2\gamma ,$$

and therefore $\frac{\beta}{2} > \gamma$. This last inequality together with $2\beta + \gamma = 90^\circ$ yields $2\beta + \frac{\beta}{2} > 90^\circ$, or $\frac{5\beta}{2} > 90^\circ$. Hence, $\beta > 36^\circ$ and $\angle ABC = 2\beta > 72^\circ$.

Next we look at a comment on the solution to problem 6 of the 2007 Italian Olympiad [2007: 149–150; 208: 84–85].

6. Let P be a point inside the triangle ABC. Say that the lines AP, BP, and CP meet the sides of ABC at A', B', and C', respectively. Let

$$x = \frac{AP}{PA'} , \quad y = \frac{BP}{PB'} , \quad z = \frac{CP}{PC'} .$$

Prove that $xyz = x + y + z + 2$.

Comment by J. Chris Fisher, University of Regina, Regina, SK.

The result is a theorem of Euler featured in the ***Crux Mathematicorum*** article "Euler's Triangle Theorem" by G.C. Shephard in [1999 : 148–153]. Euler's proof is very neat, even nicer than any of the solutions found in ***Crux Mathematicorum***. It can be located in Edward Sandifer's book, *How Euler Did It*, and on his webpage: `http://www.maa.org/news/howeulerdidit.html` click on "19th Century Triangle Geometry".

Now we turn to solutions from our readers to problems given in the February 2008 number of the *Corner*. First a solution to a problem of the 11th Form, Final Round, XXXI Russian Mathematical Olympiad given in the *Corner* at [2008: 20–21].

5. *(N. Agakhanov)* Does there exist a bounded function $f : \mathbb{R} \to \mathbb{R}$ with $f(1) > 0$ such that

$$f^2(x+y) \geq f^2(x) + 2f(xy) + f^2(y)$$

for all x, $y \in \mathbb{R}$?

Solved by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Suppose that f is such a function. Let $a_0 = 1$ and $a_n = a_{n-1} + \frac{1}{a_{n-1}}$ for $n \geq 1$. Then

$$f^2(a_1) \geq f^2(a_0) + 2f(1) + f^2\left(\frac{1}{a_0}\right) \geq 2f(1) .$$

As an induction step, assume that $f^2(a_n) \geq 2nf(1)$ for some $n \geq 1$. Then

$$\begin{aligned} f^2(a_{n+1}) &= f^2\left(a_n + \frac{1}{a_n}\right) \\ &\geq f^2(a_n) + 2f(1) + f^2\left(\frac{1}{a_n}\right) \\ &\geq f^2(a_n) + 2f(1) \geq 2(n+1)f(1)\,, \end{aligned}$$

completing the induction. Hence $f^2(a_n) \geq 2nf(1)$ for all $n \geq 1$, contradicting the facts that $f(1) > 0$ and f is bounded.

And to complete our files for the *Corner*, we look at a problem of the Taiwan Mathematical Olympiad, Selected Problems 2005, given in [2008: 21–22].

1. A $\triangle ABC$ is given with side lengths a, b, and c. A point P lies inside $\triangle ABC$, and the distances from P to the three sides are p, q, and r, respectively. Prove that

$$R \leq \frac{a^2+b^2+c^2}{18\sqrt[3]{pqr}}\,,$$

where R is the circumradius of $\triangle ABC$. When does equality hold?

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and George Tsapakidis, Agrinio, Greece. We give Bataille's write-up.

Let F denote the area of $\triangle ABC$. We have the well-known relation $2F = \dfrac{abc}{2R}$, but also from the definition of p, q, and r we have the equation $2F = ap + bq + cr$. Thus, the proposed inequality is equivalent to

$$\frac{abc}{2(ap+bq+cr)} \leq \frac{a^2+b^2+c^2}{18\sqrt[3]{pqr}}$$

or

$$(a^2+b^2+c^2)(ap+bq+cr) \geq 9abc\sqrt[3]{pqr}\,. \qquad (1)$$

By the AM–GM Inequality,

$$a^2+b^2+c^2 \geq 3\sqrt[3]{a^2b^2c^2} \quad \text{and} \quad ap+bq+cr \geq 3\sqrt[3]{abcpqr}\,,$$

and the inequality (1) now follows from

$$(a^2+b^2+c^2)(ap+bq+cr) \geq 9\sqrt[3]{a^2b^2c^2}\cdot\sqrt[3]{abc}\cdot\sqrt[3]{pqr}\,.$$

That completes the *Corner* for this number, and this Volume. As Joanne Canape, who has been translating my scribbles into beautiful LaTeX has decided that twenty-plus years is enough, I want to thank her too for all the help over the time we've worked together.