Problem of the Month

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This month, we give some thought about repeating decimals (decimals, decimals, decimals, decimals...)

Problem #1 (2006 Canadian Open Mathematics Challenge)
Suppose n and D are integers with n positive and $0 \le D \le 9$. Determine n if $\frac{n}{810} = 0.\overline{9D5} = 0.9D59D59D5\ldots$.

I knew that I should have paid more attention in elementary school! If you're like me, you probably remember that $\frac{1}{3} = 0.33333\ldots$ which can also be written as $0.\overline{3}$. Maybe you remember that $\frac{7}{9} = 0.\overline{7} = 0.7777\ldots$. How about $\frac{1}{7}$? Do you know that this equals $0.\overline{142857}$?

To solve Problem #1, it would be helpful to convert the repeating decimal to a fraction. But how do we do this? Let's look at two different ways.

In the first approach, we set $x = 0.\overline{9D5} = 0.9D59D59D5\ldots$. Then $1000x = 9D5.9D59D5\ldots = 9D5.\overline{9D5}$. Thus,

$$\begin{aligned} 999x &= 1000x - x = 9D5.\overline{9D5} - 0.\overline{9D5} = 9D5 \qquad \text{(an integer!)} \\ x &= \frac{9D5}{999}\,. \end{aligned}$$

Does this method look familiar? It may, if you have ever tried to prove that $0.\overline{9}$ actually equals 1.

For a second approach, we rewrite $0.9D59D59D5\ldots$ as

$$\frac{9D5}{10^3} + \frac{9D5}{10^6} + \frac{9D5}{10^9} + \cdots$$

This is an infinite geometric series with first term $a = \frac{9D5}{10^3}$ and common ratio $r = \frac{1}{10^3}$; thus, its sum is

$$\frac{a}{1-r} = \frac{\frac{9D5}{10^3}}{1 - \frac{1}{10^3}} = \frac{9D5}{1000-1} = \frac{9D5}{999}\,.$$

It is reassuring to get the same answer in two different ways. Try using one or both of these methods to show that $0.\overline{1234} = \frac{1234}{9999}$ and $0.\overline{abc} = \frac{abc}{999}$. Can you come up with some general rules for converting repeating decimals to fractions?

Now we are ready to solve Problem #1.

Solution to Problem #1: From the given information and our comments above, we know that $\frac{n}{810} = 0.\overline{9D5} = \frac{9D5}{999}$. Clearing the fractions yields

$999n = 810(9D5)$. We can simplify this by dividing both sides by 27, giving $37n = 30(9D5)$. Since 37 is a factor of the left side, it must be a factor of the right side. Since 37 and 30 have no common factors, then 37 must divide exactly into $9D5$. How can we determine D? One way would be to get out a calculator and try to find a multiple of 37 that is between 900 and 1000 and ends with a 5. This wouldn't be too hard.

Here is another approach. We first note that

$$37 \times 20 \;=\; 740 \;<\; 9D5 \;<\; 1120 \;=\; 37 \times 30\,.$$

Hence, $9D5 = 37 \times 25$ because no other number in the permissible range when multiplied by 37 will end in 5. Therefore, $D = 2$.

But we want the value of n. Recall that $37n = 30(9D5) = 30(925)$. Hence, $n = 30(925)/37 = 750$.

As with many problems involving a repeating decimal, the decimal gets converted to a fraction. So the amount of knowledge of repeating decimals that we need is not enormous.

Here is another such problem to keep you busy over the next month:

Problem #2 (1992 AIME) Let S be the set of all rational numbers r with $0 < r < 1$, that have a repeating decimal expansion of the form $0.\overline{abc}$, where the digits a, b, and c are not necessarily distinct. To write the elements of S as fractions in lowest terms, how many different numerators are required?

Good luck! I've put a few hints at the end.

In February's Problem of the Month, we looked at a problem involving determining the average number of "change points" in sequences of 0s and 1s. This involved counting the total number of change points over all such sequences in a clever way.

Imagine my surprise on the first Saturday in December (about the time I was writing the February column), when I saw the following problem on the 2006 William Lowell Putnam Mathematical Competition. (I have modified this problem slightly to remove the special case of $n = 2$ and to remove some of the more technical notation.)

Problem #3. A permutation π of $\{1, 2, \ldots, n\}$ (with $n \geq 3$) has a local maximum at position k if the two neighbouring numbers (or, in case $k = 1$ or $k = n$, the one neighbouring number) are both smaller than the number in position k. (For example, if $n = 5$, then $2, 1, 4, 5, 3$ has local maxima in positions 1 and 4.) What is the average number of local maxima of a permutation of $\{1, 2, \ldots, n\}$, averaging over all such permutations?

We will try to solve this problem by the same technique that we used in February: fixing a position and counting the total number of permutations with a local maximum in that position.

Solution to Problem #3: First consider position 1. How many permutations have a local maximum in position 1? Whether or not there is a local maximum at position 1 depends on the numbers in positions 1 and 2. Any pair of numbers can give a local maximum at position 1 if they are arranged with the larger number first. (For example, the pair 3 and 5 gives a local maximum in position 1 if the 5 comes before the 3.)

There are $\binom{n}{2}$ possible pairs of numbers that can be placed in positions 1 and 2. There is only one way to arrange a given pair to get a local maximum in position 1. There are then $(n-2)!$ ways of filling out the rest of the permutation. Thus, there are

$$\binom{n}{2}(n-2)! \;=\; \frac{n!}{(n-2)!2!}(n-2)! \;=\; \frac{n!}{2}$$

permutations with a local maximum in position 1. In other words, among all such permutations, there are $\frac{1}{2}n!$ local maxima in position 1. By a similar argument, there are $\frac{1}{2}n!$ local maxima in position n.

Now consider a position k with $1 < k < n$. How many local maxima are there at position k? Whether there is a local maximum at position k depends on the numbers in positions $k-1$, k, and $k+1$. Any triple of numbers can be arranged to form a local maximum at position k in two ways. For example, if we choose 1, 3, 7, then a local maximum occurs in the middle if (and only if) they are arranged as 1, 7, 3 or 3, 7, 1. There are $\binom{n}{3}$ ways of choosing the three numbers that will go in positions $k-1$ through $k+1$, two ways of arranging these numbers to form a local maximum at position k, and $(n-3)!$ ways to arrange the remaining $n-3$ numbers in the permutation. Thus, there are

$$2\binom{n}{3}(n-3)! \;=\; \frac{2n!}{(n-3)!3!}(n-3)! \;=\; \frac{n!}{3}$$

permutations with a local maximum at position k. In other words, there are $\frac{1}{3}n!$ local maxima at position k among all such permutations. (Remember that there are $n-2$ values for k that we have to keep track of in this case.)

Hence, the total number of local maxima over all such permutations is

$$\tfrac{1}{2}n! + \tfrac{1}{2}n! + (n-2)\left(\tfrac{1}{3}n!\right) \;=\; \tfrac{1}{3}(n+1)n!\,.$$

Since the total number of permutations of $\{1, 2, \ldots, n\}$ is $n!$, the average number of local maxima is $\frac{1}{3}(n+1)$.

It's always neat to see an old technique come in handy. That's part of the reason why we practice solving problems—the more we practice, the more techniques we learn, and the more likely we are to think, "Hey, wait a second! I know what to do here."

Hints for Problem #2:

- Convert the repeating decimal to a fraction.
- When is this fraction irreducible? How many of these cases are there?
- If the given fraction is reducible, what happens? What are the possible denominators when reduced? What are the possible numerators?