

SKOLIAD No. 101

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Please send your solutions to the problems in this edition by **1 October, 2007**. A copy of **MATHEMATICAL MAYHEM Vol. 3** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

This month we present a selection of problems from the 6th Annual CNU Regional High School Mathematics Contest. Thanks go to R. Persky, Christopher Newport University, Newport News, VA.

6th Annual CNU Regional High School Mathematics Contest (2005)

1. There are 8 girls and 6 boys at the Math Club at Central High School. The Club needs to send a delegation to a conference, and the delegation must contain exactly two girls and two boys. The number of possible delegations that can be formed from the membership of the club is

- (A) 480 (B) 420 (C) 576 (D) 1680

4. The remainder of 7^{100} divided by 9 is

- (A) 3 (B) 4 (C) 7 (D) 5

7. When $(x^{\frac{1}{2}} - x^{\frac{2}{3}})^7$ is multiplied out and simplified, one of the terms has the form Kx^4 where K is a constant. Find K .

- (A) 7 (B) -7 (C) 35 (D) -35

8. Two points are picked at random on the unit circle $x^2 + y^2 = 1$. What is the probability the the chord joining the two points has length at least 1?

- (A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$

11. Let m be a constant. The graphs of the lines $y = x - 2$ and $y = mx + 3$ intersect at a point whose x -coordinate and y -coordinate are both positive if and only if

- (A) $m = 1$ (B) $m < 1$ (C) $m > -\frac{3}{2}$ (D) $-\frac{3}{2} < m < 1$

13. Let $f(x)$ be a function such that $f(x) + 2f(-x) = \sin x$ for every real number x . What is the value of $f(\frac{\pi}{2})$?

- (A) -1 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 1

15. $\sqrt{7 + 4\sqrt{3}} - \sqrt{7 - 4\sqrt{3}} =$

- (A) 4 (B) $2\sqrt{3}$ (C) $\sqrt{6}$ (D) 2

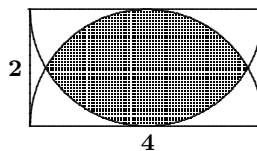
29. One root of $mx^2 - 10x + 3 = 0$ is two thirds of the other root. What is the sum of the roots?

- (A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) $\frac{7}{2}$ (D) $\frac{5}{4}$

33. Calculate the expression $1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n!$.

- (A) $(n^2 + n + 1)n!$ (B) $(n + 1)! - 1$
 (C) $(n + 2)! - n!$ (D) $(n!)^2 - 1$

36. A rectangle has length 4 and height 2. What is the area of the shaded region, which is the intersection of the two semicircles pictured?



- (A) $\frac{4\pi}{3} + 2\sqrt{3}$ (B) $\frac{4\pi}{3} - 2\sqrt{3}$ (C) $\frac{8\pi}{3} - 2\sqrt{3}$ (D) $\frac{8\pi}{3} + 2\sqrt{3}$

6^{ième} Concours CNU Régional de Mathématiques (2005)

1. Il y a 8 filles et 6 garçons au Club de Maths de l'école. Le Club doit former une délégation à envoyer à un congrès, et la délégation doit se composer exactement de deux filles et de deux garçons. Le nombre possible de délégations qui peuvent être formées à partir des membres du Club est

- (A) 480 (B) 420 (C) 576 (D) 1680

4. Le reste de 7^{100} divisé par 9 est

- (A) 3 (B) 4 (C) 7 (D) 5

7. Quand $(x^{\frac{1}{2}} - x^{\frac{2}{3}})^7$ est développé et simplifié, un des termes a la forme Kx^4 où K est une constante. Trouver K .

- (A) 7 (B) -7 (C) 35 (D) -35

8. Deux points sont choisis au hasard sur le cercle unitaire $x^2 + y^2 = 1$. Quelle est la probabilité que la corde joignant les deux points ait une longueur d'au moins 1?

- (A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$

11. Soit m une constante. Les dessins des lignes $y = x - 2$ et $y = mx + 3$ s'intersectent au point d'abscisse et d'ordonnée toutes deux positives si et seulement si

- (A) $m = 1$ (B) $m < 1$ (C) $m > -\frac{3}{2}$ (D) $-\frac{3}{2} < m < 1$

13. Soit $f(x)$ une fonction telle que $f(x) + 2f(-x) = \sin x$ pour tout nombre réel x . Quelle est la valeur de $f(\frac{\pi}{2})$?

- (A) -1 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 1

15. $\sqrt{7 + 4\sqrt{3}} - \sqrt{7 - 4\sqrt{3}} =$

- (A) 4 (B) $2\sqrt{3}$ (C) $\sqrt{6}$ (D) 2

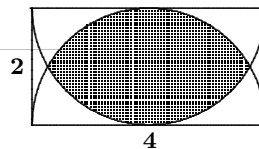
29. Une racine de $mx^2 - 10x + 3 = 0$ est les deux tiers de l'autre racine. Quelle est la somme des racines?

- (A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) $\frac{7}{2}$ (D) $\frac{5}{4}$

33. Que vaut l'expression $1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n!$.

- (A) $(n^2 + n + 1)n!$ (B) $(n + 1)! - 1$
 (C) $(n + 2)! - n!$ (D) $(n!)^2 - 1$

36. Un rectangle a une longueur de 4 et une hauteur de 2. Quelle est l'aire de la région hachurée, qui est l'intersection des deux demi-cercles dessinés?



- (A) $\frac{4\pi}{3} + 2\sqrt{3}$ (B) $\frac{4\pi}{3} - 2\sqrt{3}$ (C) $\frac{8\pi}{3} - 2\sqrt{3}$ (D) $\frac{8\pi}{3} + 2\sqrt{3}$

Next we give the official solutions to the 22nd W.J. Blundon contest [2006 : 354-356].

1. An automobile went up a hill at an average speed of 30 km/hr and down the same distance at an average speed of 60 km/hr. What was the average speed for the trip?

Official solution.

Let d be the distance one way, t_1 the time going up the hill, and t_2 the time going down. Since $30t_1 = d = 60t_2$, then $t_1 = 2t_2$. The required speed is $\frac{2d}{t_1 + t_2} = \frac{120t_2}{2t_2 + t_2} = 40$ km/hr.

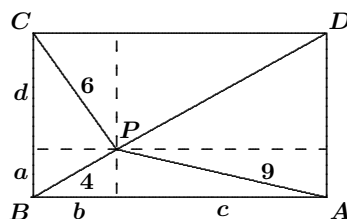
- 2.** Let P be a point in the interior of rectangle $ABCD$. If $PA = 9$, $PB = 4$, and $PC = 6$, find PD .

Official solution.

Since $PD^2 = c^2 + d^2$, $c^2 = 9^2 - a^2$, and $d^2 = 6^2 - b^2$, we have

$$\begin{aligned} PD^2 &= 9^2 - a^2 + 6^2 - b^2 \\ &= 117 - (a^2 + b^2) \\ &= 117 - 16 = 101. \end{aligned}$$

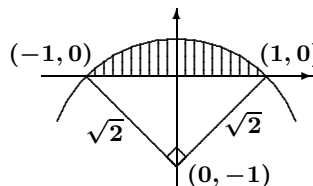
Hence, $PD = \sqrt{101}$.



- 3.** Find the area of the region above the x -axis and below the graph of $x^2 + (y + 1)^2 = 2$.

Official solution.

The graph of the equation $x^2 + (y + 1)^2 = 2$ is a circle of radius $\sqrt{2}$ with centre at $(0, -1)$. The circle intersects the x -axis at $(\pm 1, 0)$. The area of the required region is clearly a quarter of the circle of radius $\sqrt{2}$ minus the area of the triangle with base length $\sqrt{2}$ and height $\sqrt{2}$. That is,

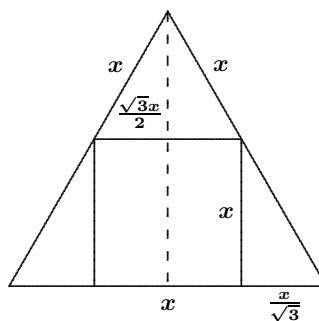


$$\text{area of the region} = \frac{1}{4}\pi(\sqrt{2})^2 - \frac{1}{2}(\sqrt{2})^2 = \frac{\pi}{2} - 1.$$

- 4.** A square is inscribed in an equilateral triangle. Find the ratio of the area of the square to the area of the triangle.

Official solution.

Let x be the length of each side of the square. Note that the top triangle is equilateral and all the right triangles are 30° - 60° - 90° triangles. Using the values of $\tan 60^\circ$ and $\sin 60^\circ$, the sides of the right triangles are calculated as shown. The base of the equilateral triangle is $x + \frac{2x}{\sqrt{3}}$ and the height is $x + \frac{\sqrt{3}x}{2}$.



$$\text{The required ratio is } \frac{x^2}{\frac{1}{2}\left(x + \frac{2x}{\sqrt{3}}\right)\left(x + \frac{\sqrt{3}x}{2}\right)} = 28\sqrt{3} - 48.$$

- 5.** Find the number of solutions to the equation $2x + 5y = 2005$ for which both x and y are positive integers.

Official solution, expanded by the editor.

The given equation can be rewritten as $2x = 5(401 - y)$. If x and y are integers satisfying the equation, then x must be divisible by 5; that is, $x = 5t$ for some integer t . Then $10t = 5(401 - y)$, which simplifies to $2t = 401 - y$. If $y > 0$, then $2t < 401$ and hence $t \leq 200$ (since t is an integer). We also want $t > 0$ to get $x > 0$. Thus, $1 \leq t \leq 200$.

Each integer t such that $1 \leq t \leq 200$ gives positive integers $x = 5t$ and $y = 401 - 2t$ which are solutions of the original equation. Hence, there are exactly 200 solutions which are positive integers.

6. For what values of a does the equation $4x^2 + 4ax + a + 6 = 0$ have real solutions?

Official solution, modified by the editor.

A quadratic equation has real solutions if and only if its discriminant is non-negative. The discriminant of the given equation is

$$\Delta = (4a)^2 - 4(4)(a + 6) = 16(a^2 - a - 6) = 16(a - 3)(a + 2).$$

We see that $\Delta \geq 0$ if and only if $a \geq 3$ or $a \leq -2$.

7. Ace runs with constant speed and Flash runs x times as fast, $x > 1$. Flash gives Ace a head start of y metres, and, at a given signal, they start off in the same direction. Find the distance Flash must run to catch Ace.

Official solution.

Let d be the distance Flash must travel to catch Ace, let v be Ace's speed, and let t be the time needed to catch up. Then we have $d = vxt$ and also $d - y = vt$. Eliminating v , we have $d - y = \frac{d}{x}$. Hence, $d = \frac{xy}{x - 1}$.

8. Show that $3^n - 2n - 1$ is divisible by 4 for any positive integer n .

Official solution.

We consider two cases.

For n even, we write $n = 2m$. Then

$$\begin{aligned} 3^n - 2n - 1 &= 3^{2m} - 2(2m) - 1 = 3^{2m} - 4m - 1 \\ &= (3^m - 1)(3^m + 1) - 4m. \end{aligned}$$

Clearly, $3^m - 1$ and $3^m + 1$ are even; whence, 4 divides $(3^m - 1)(3^m + 1)$. Thus, 4 divides $3^n - 2n - 1$.

For n odd, we write $n = 2m + 1$. Then

$$\begin{aligned} 3^n - 2n - 1 &= 3^{2m+1} - 2(2m + 1) - 1 = 3^{2m+1} - 3 - 4m \\ &= 3(3^m - 1)(3^m + 1) - 4m. \end{aligned}$$

As above, 4 divides $(3^m - 1)(3^m + 1)$. Thus, 4 divides $3^n - 2n - 1$.

9. If the polynomial $P(x) = x^3 - x^2 + x - 2$ has the three zeroes a , b , and c , find $a^3 + b^3 + c^3$.

Official solution.

We have

$$\begin{aligned} P(a) &= a^3 - a^2 + a - 2 = 0, \\ P(b) &= b^3 - b^2 + b - 2 = 0, \\ P(c) &= c^3 - c^2 + c - 2 = 0. \end{aligned}$$

Summing these three equations, we get

$$a^3 + b^3 + c^3 - (a^2 + b^2 + c^2) + (a + b + c) - 6 = 0.$$

Since $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$, we get

$$a^3 + b^3 + c^3 = (a + b + c)^2 - 2(ab + bc + ca) - (a + b + c) + 6.$$

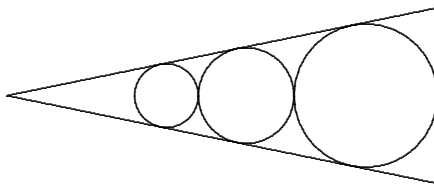
But we also have

$$\begin{aligned} x^3 - x^2 + x - 2 &= (x - a)(x - b)(x - c) \\ &= x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc, \end{aligned}$$

which implies that $a + b + c = 1$ and $ab + bc + ca = 1$. Therefore,

$$a^3 + b^3 + c^3 = 1^2 - 2(1) - 1 + 6 = 4.$$

10. A circle of radius 2 is tangent to both sides of an angle. A circle of radius 3 is tangent to the first circle and both sides of the angle. A third circle is tangent to the second circle and both sides of the angle. Find the radius of the third circle.



Official solution.

Let x be the radius of the third circle, and let a be the shortest distance from the vertex of the angle to the first circle. By similar triangles, we have $\frac{a+2}{2} = \frac{a+7}{3}$, and hence $a = 8$. By similar triangles again, we have $\frac{a+10+x}{x} = \frac{a+2}{2}$, implying that $\frac{18+x}{x} = 5$. Hence, $x = \frac{9}{2}$.

That brings us to the end of another issue. Please send in solutions! We had no readers' solutions to feature this month.