# Problem of the Month

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This month's problem combines some geometry, some number theory, and some algebra:

**Problem**. A *Pythagorean triple* is a set of three positive integers $(a, b, c)$ such that $a < b < c$ and $a^2 + b^2 = c^2$. Determine all primitive Pythagorean triples that satisfy the equation $a + b - c = 20$.

(A variation of this problem was used at the annual Canadian Mathematics Competitions Mathematics Contests Seminar in June 2006.)

No, "primitive" does not mean that the Pythagorean triples live in caves! A Pythagorean triple $(a, b, c)$ is called *primitive* if the integers $a$, $b$, and $c$ do not all share a common factor greater than 1.

First let's solve this problem in a way that does not assume any prior knowledge on the subject of Pythagorean triples.

*Solution 1:* From the given equation $a + b - c = 20$, we get

$$\begin{aligned} a + b - 20 &= c\,, \\ (a + b - 20)^2 &= c^2\,, \\ a^2 + b^2 + 400 + 2ab - 40a - 40b &= c^2\,, \\ 400 + 2ab - 40a - 40b &= 0 \qquad (\text{since } a^2 + b^2 = c^2), \\ ab - 20a - 20b &= -200\,, \\ ab - 20a - 20b + 400 &= 200\,, \\ (a - 20)(b - 20) &= 200\,. \end{aligned}$$

The factors $a - 20$ and $b - 20$ on the left side of this equation must be integers because $a$ and $b$ are integers. Since $a + b - c = 20$ and $a < b < c$, we have $b > a = 20 + (c - b) > 20$. Thus, the factors $a - 20$ and $b - 20$ are both positive, and $b - 20 > a - 20$.

The easiest thing to do at this stage is to make a table of all possible pairs $a - 20$, $b - 20$.

| $a-20$ | $b-20$ | $a$ | $b$ | $c = a + b - 20$ | Primitive |
|---|---|---|---|---|---|
| 1 | 200 | 21 | 220 | 221 | Yes |
| 2 | 100 | 22 | 120 | 122 | No |
| 4 | 50 | 24 | 70 | 74 | No |
| 5 | 40 | 25 | 60 | 65 | No |
| 8 | 25 | 28 | 45 | 53 | Yes |
| 10 | 20 | 30 | 40 | 50 | No |

Thus, the two primitive Pythagorean triples that work are $(21, 220, 221)$ and $(28, 45, 53)$. (Notice that each triple that we rejected above had a common factor of 2 among the three integers.)

That was a good way to solve this problem. It did not require any advanced machinery.

However, there is some structure to the set of all primitive Pythagorean triples which we could have used. Every primitive Pythagorean triple $(a, b, c)$ with $a < b < c$ can be written in one of the forms $(2mn, m^2 - n^2, m^2 + n^2)$ or $(m^2 - n^2, 2mn, m^2 + n^2)$, where $m$ and $n$ are positive integers of opposite parity (that is, one is even and one is odd) with no common factors greater than 1. We will take this on faith for a little while and come back to the reasoning later.

*Solution 2:* Using the above forms for primitive Pythagorean triples, we rewrite the equation $a + b - c = 20$ as $m^2 - n^2 + 2mn - (m^2 + n^2) = 20$. (Notice that we get the same equation from each of the two possible forms.) Thus, $2mn - 2n^2 = 20$, or $n(m - n) = 10$. Here $m$ and $n$ are positive integers, and therefore $n$ and $m - n$ are positive factors of 10.

Now we make a table showing the possibilities for $n$ and $m - n$.

| $n$ | $m - n$ | $m$ | $2mn$ | $m^2 - n^2$ | $m^2 + n^2$ | $(a, b, c)$ | Primitive |
|---|---|---|---|---|---|---|---|
| 1 | 10 | 11 | 22 | 120 | 122 | $(22, 120, 122)$ | No |
| 2 | 5 | 7 | 28 | 45 | 53 | $(28, 45, 53)$ | Yes |
| 5 | 2 | 7 | 70 | 24 | 74 | $(24, 70, 74)$ | No |
| 10 | 1 | 11 | 220 | 21 | 221 | $(21, 220, 221)$ | Yes |

Therefore, the primitive Pythagorean triples satisfying $a + b - c = 20$ are $(28, 45, 53)$ and $(21, 220, 221)$.

Wonderful! This solution was a fair bit less complicated than Solution 1. Interestingly, the "endgame" was remarkably similar to that in Solution 1.

The problem can be interpreted geometrically. Each Pythagorean triple $(a, b, c)$ corresponds to a right triangle whose sides have integer lengths $a$, $b$, and $c$. The equation $a^2 + b^2 = c^2$ is the Pythagorean Theorem (which, of course, is why we call $(a, b, c)$ a Pythagorean triple). The diameter of the inscribed circle of such a triangle is $a+b-c$. (Can you show this?) A primitive Pythagorean triple corresponds to a right triangle which is not similar to any smaller right triangle with integer side lengths. The problem asks us to find all such triangles which have an inscribed circle of diameter 20.

**Postscript.**

Before wrapping up this article, we should look at the formulae for generating primitive Pythagorean triples, which we will abbreviate as PPTs.

First, we check that any triple of the form $(2mn, m^2 - n^2, m^2 + n^2)$ or $(m^2 - n^2, 2mn, m^2 + n^2)$ is, in fact, a Pythagorean triple. This requires squaring each of the three terms and checking that the sum of the first two squares equals the third. I'll let you think about why such a triple must be primitive, given that $m$ and $n$ are positive, have opposite parity, and have no common factors greater than 1.

Next, we check that every PPT is of one of these two forms. We will use one of the well-known methods to do this and break the procedure into a number of steps. (There are certainly other methods that can be used to derive these formulae.)

*Step 0:* Odd and even perfect squares.

If $a$ is even, then $a = 2k$ for some integer $k$; thus, $a^2 = 4k^2$ which gives a remainder of 0 when divided by 4.

If $a$ is odd, then $a = 2k+1$ for some integer $k$; thus, $a^2 = 4k^2+4k+1$ which gives a remainder of 1 when divided by 4.

This doesn't seem relevant immediately, but hang on!

*Step 1:* Parities of $a$, $b$, $c$.

Suppose $(a, b, c)$ is a PPT. If $a$ and $b$ were both even, then $c$ would be even, since $c^2 = a^2 + b^2$. Then $a$, $b$, and $c$ would would all be divisible by 2, contradicting our assumption that the Pythagorean triple $(a, b, c)$ is primitive.

If $a$ and $b$ were both odd, then $c$ would be even. In this case, the remainder upon dividing $c^2$ by 4 would be 0 and the remainder upon dividing $a^2 + b^2$ by 4 would be 2, leading to a contradiction.

Therefore, having eliminated all of the other cases, we are left with the case where one of $a$ or $b$ is even and the other is odd (implying that $c$ is odd). Assume that $a$ is odd and $b$ is even.

*Step 2:* Some algebra.

Since $a^2 + b^2 = c^2$, then $b^2 = c^2 - a^2 = (c-a)(c+a)$. Dividing both sides by 4, we get $\left(\frac{1}{2}b\right)^2 = \left(\frac{1}{2}(c-a)\right)\left(\frac{1}{2}(c+a)\right)$. Note that $\frac{1}{2}b$ is an integer, because $b$ is even, and $\frac{1}{2}(c-a)$ and $\frac{1}{2}(c+a)$ are integers, because $a$ and $c$ are both odd (making $c-a$ and $c+a$ even).

*Step 3:* Analysis of the factors $\frac{1}{2}(c-a)$ and $\frac{1}{2}(c+a)$.

If $\frac{1}{2}(c-a)$ and $\frac{1}{2}(c+a)$ had a common factor larger than 1, then this factor would also be a factor of their sum (which equals $c$) and of their difference (which equals $a$). Then $b$ would also share this common factor, since $b^2 = c^2 - a^2$. This can't happen, because $a$, $b$, and $c$ have no common factor. Also, $\frac{1}{2}(c-a)$ and $\frac{1}{2}(c+a)$ cannot both be odd. If they were odd, then their sum $c$ and their difference $a$ would be even (and we know that they are odd).

So $\frac{1}{2}(c-a)$ and $\frac{1}{2}(c+a)$ have no common factors, one is even and one is odd, and their product is a perfect square.

*Step 4:* Introduction of $m$ and $n$.

It must be the case, therefore, that each of $\frac{1}{2}(c-a)$ and $\frac{1}{2}(c+a)$ is a perfect square itself. (Think about this.) We can write $\frac{1}{2}(c-a) = n^2$ and $\frac{1}{2}(c+a) = m^2$, with $m$ and $n$ positive integers.

Then $m$ and $n$ cannot have a common factor larger than 1 (since $m^2$ and $n^2$ don't), and $m$ and $n$ must have opposite parity (since $m^2$ and $n^2$ do).

*Step 5:* The big finish.

Since $\frac{1}{2}(c-a) = n^2$ and $\frac{1}{2}(c+a) = m^2$, then $c = m^2 + n^2$ (adding) and $a = m^2 - n^2$ (subtracting). Also, $\left(\frac{1}{2}b\right)^2 = m^2n^2$; that is, $b^2 = 4m^2n^2$. Then $b = 2mn$, since $b$, $m$, and $n$ are positive.

Thus, each PPT has exactly the form that we had hoped.