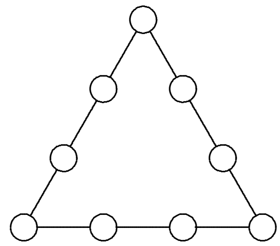
Mayhem Solutions

M219. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

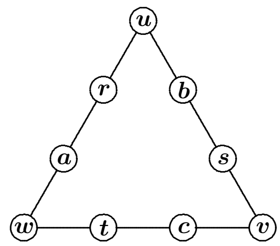
Place each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 in exactly one of the circles in such a way that:



1. the sums of the four numbers on each side of the triangle are equal; and
2. the sums of the squares of the four numbers on each side of the triangle are equal.

Solution by Messiah College Problem Solving Group, Messiah College, Grantham, PA, USA.

Assign variables to the nine positions in the triangle as shown. Since the sum of all nine numbers is 45 and the sums on each side are equal, we have the following equations:



$$a+b+c+r+s+t+u+v+w \;=\; 45\,, \quad (1)$$
$$a+r+w \;=\; b+s+v\,, \quad (2)$$
$$a+r+u \;=\; t+c+v\,. \quad (3)$$

Adding these equations and dividing by 3 gives

$$a \;=\; 15-r+\tfrac{1}{3}\left(-2u+v-2w\right)\,.$$

Since a is an integer, we must have $-2u+v-2w \equiv 0 \pmod 3$, which is equivalent to

$$u+v+w \;\equiv\; 0 \pmod 3\,. \quad (4)$$

We now consider condition 2 of the problem statement modulo 3. Among the squares of the nine numbers 1, 2, ..., 9, the squares of 3, 6, and 9 are congruent to 0 modulo 3 and the others are congruent to 1. Each of the squares of the variables u, v, and w at the corners of the triangle may be congruent to 0 or 1 (modulo 3). But a check of the various cases shows that the three sums of squares cannot be equal, modulo 3, unless either

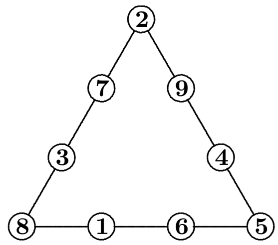
$$u^2 \;\equiv\; v^2 \;\equiv\; w^2 \;\equiv\; 0 \pmod 3 \quad \text{or} \quad u^2 \;\equiv\; v^2 \;\equiv\; w^2 \;\equiv\; 1 \pmod 3\,.$$

These additional constraints, along with (4), leave only three possiblilities for (u,v,w), namely $(1,4,7)$, $(2,5,8)$, and $(3,6,9)$.

Setting $(u,v,w) = (1,4,7)$ in (1), (2), and (3) and simplifying yields $a+r = 11$. The possibilities for $\{a, r\}$ are $\{2, 9\}$, $\{3, 8\}$, and $\{5, 6\}$. None of these yields a solution to the problem.

Setting $(u,v,w) = (3,6,9)$ in (1), (2), and (3) and simplifying yields $a+r = 9$. The possibilities for $\{a, r\}$ are $\{1, 8\}$, $\{2, 7\}$, and $\{4, 5\}$. Again, none of these yields a solution to the problem.

Setting $(u, v, w) = (2, 5, 8)$ in (1), (2), and (3) and simplifying yields $a + r = 10$. The possibilities for $\{a,\ r\}$ are $\{1,\ 9\}$, $\{3,\ 7\}$, and $\{4,\ 6\}$. The set $\{3,\ 7\}$ gives us a solution to the problem, as shown.



Any triangle with 2, 5, and 8 on the corners, and with side numbers $\{2,\ 3,\ 7,\ 8\}$, $\{2,\ 4,\ 5,\ 9\}$, and $\{1,\ 5,\ 6,\ 8\}$ is a solution.

Also solved by MIGUEL MARAÑÓN GRANDES, Grade 12 student, I.E.S. Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and JOSH TREJO and MANDY RODGERS, Angelo State University, San Angelo, TX, USA.

M220. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

Show how to number the faces of an octahedral die using the numbers 1 through 8 in such a way that the sum of the numbers on the four faces joining at each vertex is always the same.

Solution by Messiah College Problem Solving Group, Messiah College, Grantham, PA, USA.

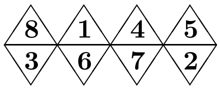
Since the numbers 1 through 8 sum to 36 and each is counted in a sum three times (once at each vertex of its triangular face), the total sum over all 6 vertices is 108, and hence the sum at each vertex must be 18.

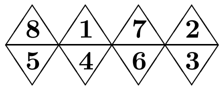
When two faces share an edge, we shall refer to them as *neighbouring faces*. The numbers on a given pair of neighbouring faces are together in two of the vertex sums. Each of these two vertex sums involves two more numbers, which we will refer to as an *adjoining pair* for the given pair of neighbouring faces.

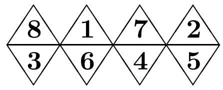
We examine the possibilities for neighbouring faces and their adjoining pairs, where one of the neighbouring faces is the one labelled 1.

Neighbouring Faces	Sum	Sum of Adjoining Pairs	Possible Adjoining Pairs
$(1, 2)$	3	15	$(7, 8)$
$(1, 3)$	4	14	$(6, 8)$
$(1, 4)$	5	13	$(5, 8)$, $(6, 7)$
$(1, 5)$	6	12	$(4, 8)$
$(1, 6)$	7	11	$(3, 8)$, $(4, 7)$
$(1, 7)$	8	10	$(2, 8)$, $(4, 6)$
$(1, 8)$	9	9	$(2, 7)$, $(3, 6)$, $(4, 5)$

In a properly numbered die, it is clear that a pair of neighbouring vertices must have two distinct adjoining pairs. Thus, none of the pairs $(1, 2)$, $(1, 3)$, or $(1, 5)$ can occur as a neighbouring pair. This leaves only four numbers which can be neighbours of 1, namely 4, 6, 7, or 8. Any solution must use three of these four. If 6 is a neighbour of 1, then 4 and 7 cannot both be neighbours of 1, since $(4,7)$ and $(3,8)$ must both be adjoining pairs for $(1,6)$. This leaves only 3 possibilities to check, namely $\{4,\ 6,\ 8\}$, $\{4,\ 7,\ 8\}$, and $\{6,\ 7,\ 8\}$. Each possibility yields a solution, as shown.







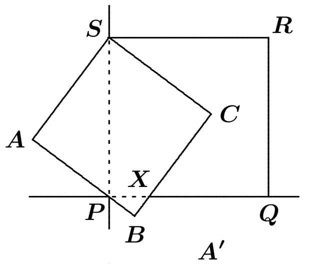
These are the only possibilities up to symmetry.

Also solved by SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College Saratoga Springs, NY, USA; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

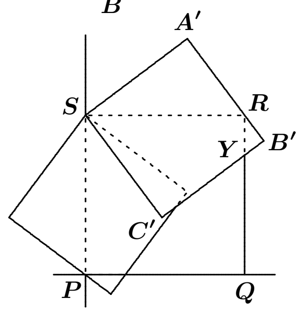
M221. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Prove that a 5×5 square can be covered by three 4×4 squares.

Solution by Skidmore College Problem Solving Group, Skidmore College Saratoga Springs, NY, USA.

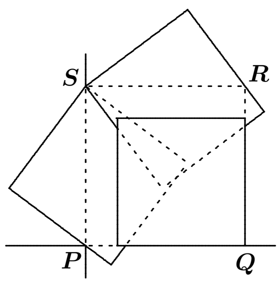


Place the 5×5 square $PQRS$ in the first quadrant with vertices $P(0,0)$, $Q(5,0)$, $R(5,5)$, and $S(0,5)$. Place a 4×4 square $\mathcal{S}_1 = ABCS$ so that the side AB passes through the origin P, with $PA = 3$ and $PB = 1$. By similar triangles, the side BC will intersect the x–axis at the point $X\left(\frac{5}{4},0\right)$.



Place a second 4×4 square $\mathcal{S}_2 = A'B'C'S$ over the 5×5 square $PQRS$ so that the side $A'B'$ passes through the vertex R, with $RA' = 3$ and $RB' = 1$. By symmetry and similar triangles, the side $B'C'$ will intersect RQ at the point $Y\left(5,\ \frac{15}{4}\right)$.

The slopes of the line segments SC and SC' are $-\frac{3}{4}$ and $-\frac{4}{3}$, respectively. Therefore, the two squares $\mathcal{S}_1$ and $\mathcal{S}_2$ overlap each other in the interior of $PQRS$.



Place a third 4×4 square $\mathcal{S}_3$ at the bottom right corner of $PQRS$ so that its vertices are $(1,0)$, $(5,0)$, $(5,4)$, and $(1,4)$. The vertices $(1,0)$ and $(5,4)$ of this third square are covered by the squares $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, because the intersection point $X\left(\frac{5}{4},0\right)$ lies to the right of $(1,0)$ and the intersection point $Y\left(5,\ \frac{15}{4}\right)$ lies below $(5,4)$. The point $(1,4)$ is covered by both squares $\mathcal{S}_1$ and $\mathcal{S}_2$. Therefore, every point in the L–shaped region inside $PQRS$ and outside $\mathcal{S}_3$ is covered by at least one of the squares $\mathcal{S}_1$ and $\mathcal{S}_2$.

Also solved by MESSIAH COLLEGE PROBLEM SOLVING GROUP, Messiah College, Grantham, PA, USA; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

M222. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Suppose that $30a + 40b$ and $40a + 30b$ are the sides of a right triangle and that $50a + kb$ is the hypotenuse, where a, b, and k are positive integers. Find the smallest possible values of a, b, and k.

Essentially the same solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain; Richard I. Hess, Rancho Palos Verdes, CA, USA; and Messiah College Problem Solving Group.

By the Pythagorean Theorem, we see that

$$(30a + 40b)^2 + (40a + 30b)^2 \;=\; (50a + kb)^2 \,.$$

Expanding and simplifying, we get $4800ab + 2500b^2 = 100abk + k^2b^2$. Dividing by b (since $b > 0$) and rearranging gives

$$b(50 - k)(50 + k) \;=\; 100a(k - 48) \,.$$

Since a, b, and k are all positive, the factors $50 - k$ and $k - 48$ must have the same sign. This happens only when $k = 49$. Setting $k = 49$, we obtain $99b = 100a$. Since 99 and 100 are relatively prime and we are seeking the smallest values of a and b, we conclude that $a = 99$ and $b = 100$. Thus, the minimum solution is $(a, b, k) = (99, 100, 49)$.

M223. *Proposed by Larry Rice, University of Waterloo, Waterloo, ON.*

The fraction $\frac{3}{10}$ can be written as the sum of two positive rational numbers with numerator 1 in exactly two ways, namely as $\frac{1}{10} + \frac{1}{5}$ and $\frac{1}{20} + \frac{1}{4}$.

Determine the number of ways that $\frac{3}{2006}$ can be expressed as the sum of two positive rational numbers with numerator 1.

Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

Let p and q be positive integers such that $\dfrac{3}{2006} = \dfrac{1}{p} + \dfrac{1}{q}$. Then

$$3pq \;=\; 2006(p + q) \;=\; 2 \cdot 17 \cdot 59(p + q) \,. \tag{1}$$

To find values of p and q, we look at 4 cases.

Case 1: $2 \cdot 17 \cdot 59$ divides p.

Then there exists a positive integer r such that $p = 2 \cdot 17 \cdot 59r$. In this case, (1) takes the form $3qr = 2 \cdot 17 \cdot 59r + q$, or $(3r - 1)q = 2 \cdot 17 \cdot 59r$. Therefore, $q = 2 \cdot 17 \cdot 59r/(3r - 1)$. This last equation has positive integer solutions for q when $3r - 1$ takes the values 2, 17, 59, or $2 \cdot 17 \cdot 59$. [*Ed.:* the values $2 \cdot 17$, $2 \cdot 59$, and $17 \cdot 59$ cannot be expressed in the form $3r - 1$.] The corresponding values of r are 1, 6, 20, and 669, which generate the following (p, q) pairs: $(2006, 1003)$, $(12036, 708)$, $(40120, 680)$ and $(1342014, 669)$.

Case 2: $2 \cdot 17$ divides p, but 59 does not.

Then there exists a positive integer r, not a multiple of 59, such that $p = 2 \cdot 17r$ and $q = 59s$ for some positive integer s. In this case, (1) takes the form $(3r - 59)s = 2 \cdot 17r$, and then $s = 2 \cdot 17r/(3r - 59)$. The values for r, not multiples of 59, which give positive integer solutions for s are 20 and 31 (corresponding to the equations $3r - 59 = 1$ and $3r - 59 = 2 \cdot 17$). Only the second value generates a new (p, q) pair: $(1054, 1829)$.

Case 3: $2 \cdot 59$ divides p, but 17 does not.

Then there exists a positive integer r, not a multiple of 17, such that $p = 2 \cdot 59r$ and $q = 17s$ for some positive integer s. In this case, (1) takes the form $(3r - 17)s = 2 \cdot 59r$, and then $s = 2 \cdot 59r/(3r - 17)$. The values for r, not multiples of 17, which give positive integer solutions for s are 6 and 45 (corresponding to the equations $3r - 17 = 1$ and $3r - 17 = 2 \cdot 59$). Only the second value generates a new (p, q) pair: $(5310, 765)$.

Case 4 : $17 \cdot 59$ divides p, but 2 does not.

Then there exists a positive integer r, not a multiple of 2, such that $p = 17 \cdot 59r$ and $q = 2s$ for some positive integer s. In this case, (1) takes the form $(3r-2)s = 17\cdot 59r$, and then $s = 17\cdot 59r/(3r-2)$. The odd values for r, which give positive integer solutions for s are 1 and 335 (corresponding to the equations $3r - 2 = 1$ and $3r - 2 = 17 \cdot 59$). Only the second value generates a new (p, q) pair: $(336005, 670)$.

[*Ed:* The remaining possible cases, where one of the primes 2, 17, and 59 divides p and the other two do not, simply exchange the roles of p and q, and thus do not lead to any new pairs.]

In conclusion, there are exactly seven ways in which 3/2006 can be expressed as the sum of two positive rational numbers with numerator 1:

$$\frac{1}{669} + \frac{1}{1342014}, \quad \frac{1}{670} + \frac{1}{336005}, \quad \frac{1}{680} + \frac{1}{40120}, \quad \frac{1}{708} + \frac{1}{12036},$$
$$\frac{1}{765} + \frac{1}{5310}, \quad \frac{1}{1003} + \frac{1}{2006}, \quad \frac{1}{1054} + \frac{1}{1829}.$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and Messiah College Problem Solving Group. One incomplete solution was submitted.

M224. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

Let $A(-1, 1)$ and $B(3, 9)$ be two points on the parabola $y = x^2$. Take another point $M(m, m^2)$ on the parabola lying between A and B. Let H be the point on the line segment joining A to B that has the same x–coordinate as M.

Show that if the length of MH is k units, then triangle AMB has area $2k$ square units. Does this relationship still hold if M is not between A and B?

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

The equation for the line through points $A(-1,1)$ and $B(3,9)$ can easily be found to be $y = 2x + 3$. Let $M(m, m^2)$ be any point on the parabola (not necessarily between A and B). The length of the line segment joining points $M(m, m^2)$ and $H(m, 2m+3)$ is $k = |m^2 - 2m - 3|$, and the area of triangle ABM can be calculated as follows:

$$\begin{aligned}[ABC] &= \tfrac{1}{2}\left\|\overrightarrow{AM} \times \overrightarrow{AB}\right\| \;=\; \tfrac{1}{2}\|\langle m+1, m^2-1, 0\rangle \times \langle 4, 8, 0\rangle\| \\ &= \tfrac{1}{2}\|\langle 0, 0, -4m^2 + 8m + 12\rangle\| \;=\; 2|m^2 - 2m - 3| \;=\; 2k\,.\end{aligned}$$

Thus, the given relationship holds no matter whether M is between A and B or not.

Also solved by ESTHER MARÍA GARCÍA-CABALLERO, Universidad de Jaén, Jaén, Spain.

M225. *Proposed by Zun Shan, Normal University, China; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Define a sequence $\{x_n\}$ by $x_1 = 1/2005$ and $x_{n+1} = x_n + x_n^2$ for $n \geq 1$. Set

$$S \;=\; \frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{2005}}{x_{2006}}\,.$$

Determine $\lfloor S \rfloor$, the greatest integer not exceeding S.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

It is easy to prove, by induction, that

$$x_n \;=\; x_1(1+x_1)(1+x_1+x_1^2)\cdots(1+x_1+x_1^2+\cdots+x_1^{n-1})\,.$$

Then,

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_{n+1}} \;=\; \frac{1}{1+x_1} + \frac{1}{1+x_1+x_1^2} + \cdots + \frac{1}{\sum\limits_{i=1}^{n+1} x_1^{i-1}}\,.$$

Setting $x_1 = 1/2005$ and $n = 2005$, we see that

$$\begin{aligned}S \;&=\; \frac{1}{1+\frac{1}{2005}} + \frac{1}{1+\frac{1}{2005}+\left(\frac{1}{2005}\right)^2} + \cdots \\ &\qquad + \frac{1}{1+\frac{1}{2005}+\left(\frac{1}{2005}\right)^2+\cdots+\left(\frac{1}{2005}\right)^{2005}} \;=\; \sum_{n=1}^{2005}\frac{1}{T_n}\,,\end{aligned}$$

where $T_n = \sum\limits_{i=0}^{n}\left(\frac{1}{2005}\right)^i$.

Since $T_n < \sum\limits_{i=0}^{\infty}\left(\frac{1}{2005}\right)^i = \frac{2005}{2004}$, we have $\frac{1}{T_n} > \frac{2004}{2005}$, and therefore, $S > \sum\limits_{n=1}^{2005}\frac{2004}{2005} = 2004$. But we also have $\frac{1}{T_n} \leq \frac{1}{1+\frac{1}{2005}} = \frac{2005}{2006}$ for each n.

Therefore,

$$S \;<\; \sum_{n=1}^{2005} \frac{2005}{2006} \;=\; \frac{2005^2}{2006} \;<\; 2005\,.$$

Now, since $2004 < S < 2005$, we see that $\lfloor S \rfloor = 2004$.

One incorrect solution was received.