MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier avril 2007**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

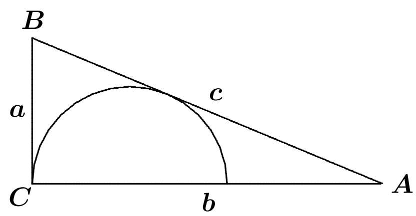
La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M269. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Dans le carré $ABCD$, soit E le point milieu du côté AD, soit F le point sur EB tel que CF soit perpendiculaire à EB, et soit G le point sur EB tel que AG soit perpendiculaire à EB. Montrer que $DF = CG$.

M270. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Les côtés d'un triangle rectangle sont de longueur a et b, son hypoténuse est de longueur c. Un demi-cercle, ayant comme diamètre le côté de longueur b, est tangent aux deux autres côtés. Déterminer le rayon du demi-cercle en fonction de a, b et c.



M271. *Proposé par Yakub N. Aliyev, Université d'Etat de Bakou, Bakou, Azerbaïdjan.*

Sachant que dans un hexagone convexe $ABCDEF$, les côtés BC, DE et FA sont respectivement parallèles aux diagonales AD, CF et EB, on désigne respectivement par K, L et M les intersections des droites AB avec CD, CD avec EF, et EF avec AB ; on désigne enfin par P, Q et R les intersections respectives de CF avec BE, de BE avec AD, et de AD avec CF. Montrer que KP, MR et LQ se coupent en un même point.

M272. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

Soit P un point situé sur le côté AB d'un parallélogramme $ABCD$. Sachant que le rapport de l'aire du triangle ABC et celle du quadrilatère $APCD$ est m/n, déterminer le rapport de AP et PB.

M273. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

Soit A, B, C, D, E, F, G et H des lettres représentant des chiffres de 0 à 9, distincts. Déterminer leur valeur, sachant que les deux produits ci-dessous sont justes. (Noter que le premier chiffre d'un nombre doit être non nul.)

$$\begin{array}{r} ABCD \\ \times E \\ \hline DCBA \end{array} \qquad\qquad \begin{array}{r} BFDG \\ \times G \\ \hline GDFB \end{array}$$

M274. *Proposé par Neven Jurič, Zagreb, Croatie.*

Déterminer l'aire du polygone dont tous les sommets sont sur le cercle d'équation $x^2 + y^2 = 100$, leurs coordonnées étant toutes des entiers.

M275. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

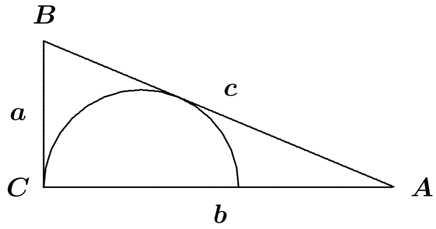
Un triangle pythagorique primitif (TPP en bref) est un triangle rectangle avec, comme longueurs des trois côtés, des entiers dont le plus grand commun diviseur est 1. Parmi les paires de TPPs non congruents possédant des cercles inscrits congruents à rayon entier, trouver une paire de rayon minimal.

M269. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Let $ABCD$ be a square. Let E be the mid-point of the side AD, let F be the point on EB such that CF is perpendicular to EB, and let G be the point on EB such that AG is perpendicular to EB. Show that $DF = CG$.

M270. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A right triangle has legs of lengths a and b and a hypotenuse of length c. A semicircle has its diameter on the side of length b and is tangent to the other two sides. Determine the radius of the semicircle in terms of a, b, and c.



M271. *Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

For the convex hexagon $ABCDEF$, it is known that the sides BC, DE, and FA are parallel to the diagonals AD, CF, and EB, respectively. We denote by K, L, and M the respective intersections of the lines AB with CD, CD with EF, and EF with AB; we further denote by P, Q, and R the respective intersections of CF with BE, BE with AD, and AD with CF. Prove that KP, MR, and LQ intersect at the same point.

M272. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Let $ABCD$ be a parallelogram, and let P be a point situated on AB. If the ratio of the area of triangle ABC to that of quadrilateral $APCD$ is m/n, determine the ratio of AP to PB.

M273. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The letters A, B, C, D, E, F, G, and H represent distinct digits. Determine their values given that the two products shown are true. (Note that the first digit of a number must be non-zero)

$$\begin{array}{r} ABCD \\ \times E \\ \hline DCBA \end{array} \qquad \begin{array}{r} BFDG \\ \times G \\ \hline GDFB \end{array}$$

M274. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine the area of the polygon whose vertices are all the points on the circle $x^2 + y^2 = 100$ where both coordinates are integers.

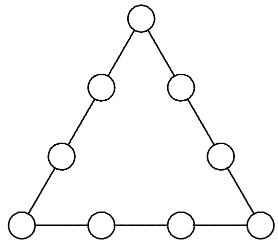
M275. *Proposed by K.R.S. Sastry, Bangalore, India.*

A primitive Pythagorean triangle (PPT) is a right triangle whose sides have lengths which are integers with a greatest common divisor of 1. Among all pairs of non-congruent PPTs which have congruent incircles with an integer radius, find a pair for which this radius is minimized.

Mayhem Solutions

M219. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

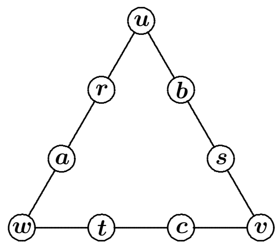
Place each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 in exactly one of the circles in such a way that:



1. the sums of the four numbers on each side of the triangle are equal; and
2. the sums of the squares of the four numbers on each side of the triangle are equal.

Solution by Messiah College Problem Solving Group, Messiah College, Grantham, PA, USA.

Assign variables to the nine positions in the triangle as shown. Since the sum of all nine numbers is 45 and the sums on each side are equal, we have the following equations:



$$a+b+c+r+s+t+u+v+w \;=\; 45\,, \quad (1)$$
$$a+r+w \;=\; b+s+v\,, \quad (2)$$
$$a+r+u \;=\; t+c+v\,. \quad (3)$$

Adding these equations and dividing by 3 gives

$$a \;=\; 15 - r + \tfrac{1}{3}\left(-2u+v-2w\right)\,.$$

Since a is an integer, we must have $-2u+v-2w \equiv 0 \pmod 3$, which is equivalent to

$$u+v+w \;\equiv\; 0 \pmod 3\,. \quad (4)$$

We now consider condition 2 of the problem statement modulo 3. Among the squares of the nine numbers 1, 2, ..., 9, the squares of 3, 6, and 9 are congruent to 0 modulo 3 and the others are congruent to 1. Each of the squares of the variables u, v, and w at the corners of the triangle may be congruent to 0 or 1 (modulo 3). But a check of the various cases shows that the three sums of squares cannot be equal, modulo 3, unless either

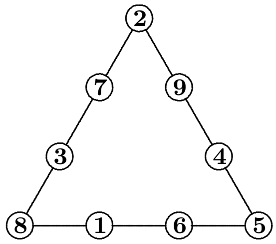
$$u^2 \;\equiv\; v^2 \;\equiv\; w^2 \;\equiv\; 0 \pmod 3 \quad \text{or} \quad u^2 \;\equiv\; v^2 \;\equiv\; w^2 \;\equiv\; 1 \pmod 3\,.$$

These additional constraints, along with (4), leave only three possiblilities for (u,v,w), namely $(1,4,7)$, $(2,5,8)$, and $(3,6,9)$.

Setting $(u,v,w) = (1,4,7)$ in (1), (2), and (3) and simplifying yields $a+r = 11$. The possibilities for $\{a, r\}$ are $\{2, 9\}$, $\{3, 8\}$, and $\{5, 6\}$. None of these yields a solution to the problem.

Setting $(u,v,w) = (3,6,9)$ in (1), (2), and (3) and simplifying yields $a+r = 9$. The possibilities for $\{a, r\}$ are $\{1, 8\}$, $\{2, 7\}$, and $\{4, 5\}$. Again, none of these yields a solution to the problem.

Setting $(u, v, w) = (2, 5, 8)$ in (1), (2), and (3) and simplifying yields $a + r = 10$. The possibilities for $\{a,\ r\}$ are $\{1,\ 9\}$, $\{3,\ 7\}$, and $\{4,\ 6\}$. The set $\{3,\ 7\}$ gives us a solution to the problem, as shown.



Any triangle with 2, 5, and 8 on the corners, and with side numbers $\{2,\ 3,\ 7,\ 8\}$, $\{2,\ 4,\ 5,\ 9\}$, and $\{1,\ 5,\ 6,\ 8\}$ is a solution.

Also solved by MIGUEL MARANÓN GRANDES, Grade 12 student, I.E.S. Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and JOSH TREJO and MANDY RODGERS, Angelo State University, San Angelo, TX, USA.

M220. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

Show how to number the faces of an octahedral die using the numbers 1 through 8 in such a way that the sum of the numbers on the four faces joining at each vertex is always the same.

Solution by Messiah College Problem Solving Group, Messiah College, Grantham, PA, USA.

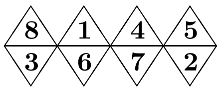
Since the numbers 1 through 8 sum to 36 and each is counted in a sum three times (once at each vertex of its triangular face), the total sum over all 6 vertices is 108, and hence the sum at each vertex must be 18.

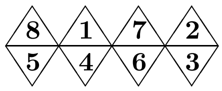
When two faces share an edge, we shall refer to them as *neighbouring faces*. The numbers on a given pair of neighbouring faces are together in two of the vertex sums. Each of these two vertex sums involves two more numbers, which we will refer to as an *adjoining pair* for the given pair of neighbouring faces.

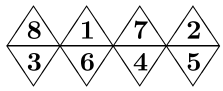
We examine the possibilities for neighbouring faces and their adjoining pairs, where one of the neighbouring faces is the one labelled 1.

Neighbouring Faces	Sum	Sum of Adjoining Pairs	Possible Adjoining Pairs
$(1, 2)$	3	15	$(7, 8)$
$(1, 3)$	4	14	$(6, 8)$
$(1, 4)$	5	13	$(5, 8)$, $(6, 7)$
$(1, 5)$	6	12	$(4, 8)$
$(1, 6)$	7	11	$(3, 8)$, $(4, 7)$
$(1, 7)$	8	10	$(2, 8)$, $(4, 6)$
$(1, 8)$	9	9	$(2, 7)$, $(3, 6)$, $(4, 5)$

In a properly numbered die, it is clear that a pair of neighbouring vertices must have two distinct adjoining pairs. Thus, none of the pairs $(1, 2)$, $(1, 3)$, or $(1, 5)$ can occur as a neighbouring pair. This leaves only four numbers which can be neighbours of 1, namely 4, 6, 7, or 8. Any solution must use three of these four. If 6 is a neighbour of 1, then 4 and 7 cannot both be neighbours of 1, since (4, 7) and (3, 8) must both be adjoining pairs for (1, 6). This leaves only 3 possibilities to check, namely {4, 6, 8}, {4, 7, 8}, and {6, 7, 8}. Each possibility yields a solution, as shown.







These are the only possibilities up to symmetry.

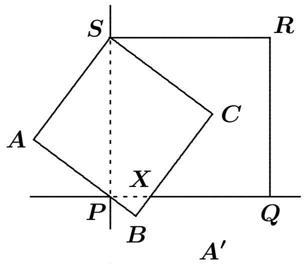
Also solved by SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College Saratoga Springs, NY, USA; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

M221. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

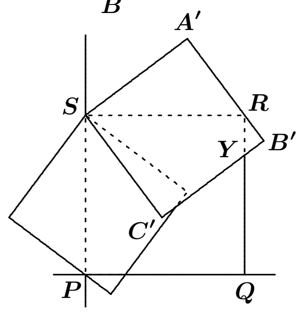
Prove that a 5×5 square can be covered by three 4×4 squares.

Solution by Skidmore College Problem Solving Group, Skidmore College Saratoga Springs, NY, USA.

Place the 5×5 square $PQRS$ in the first quadrant with vertices $P(0,0)$, $Q(5,0)$, $R(5,5)$, and $S(0,5)$. Place a 4×4 square $\mathcal{S}_1 = ABCS$ so that the side AB passes through the origin P, with $PA = 3$ and $PB = 1$. By similar triangles, the side BC will intersect the x–axis at the point $X\left(\frac{5}{4}, 0\right)$.

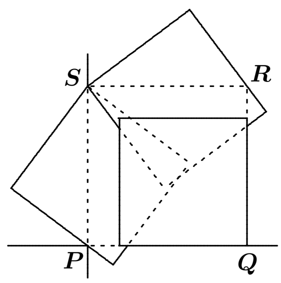


Place a second 4×4 square $\mathcal{S}_2 = A'B'C'S$ over the 5×5 square $PQRS$ so that the side $A'B'$ passes through the vertex R, with $RA' = 3$ and $RB' = 1$. By symmetry and similar triangles, the side $B'C'$ will intersect RQ at the point $Y\left(5, \frac{15}{4}\right)$.



The slopes of the line segments SC and SC' are $-\frac{3}{4}$ and $-\frac{4}{3}$, respectively. Therefore, the two squares $\mathcal{S}_1$ and $\mathcal{S}_2$ overlap each other in the interior of $PQRS$.

Place a third 4×4 square $\mathcal{S}_3$ at the bottom right corner of $PQRS$ so that its vertices are $(1,0)$, $(5,0)$, $(5,4)$, and $(1,4)$. The vertices $(1,0)$ and $(5,4)$ of this third square are covered by the squares $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, because the intersection point $X\left(\frac{5}{4}, 0\right)$ lies to the right of $(1,0)$ and the intersection point $Y\left(5, \frac{15}{4}\right)$ lies below $(5,4)$. The point $(1,4)$ is covered by both squares $\mathcal{S}_1$ and $\mathcal{S}_2$. Therefore, every point in the L–shaped region inside $PQRS$ and outside $\mathcal{S}_3$ is covered by at least one of the squares $\mathcal{S}_1$ and $\mathcal{S}_2$.



Also solved by MESSIAH COLLEGE PROBLEM SOLVING GROUP, Messiah College, Grantham, PA, USA; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

M222. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Suppose that $30a + 40b$ and $40a + 30b$ are the sides of a right triangle and that $50a + kb$ is the hypotenuse, where a, b, and k are positive integers. Find the smallest possible values of a, b, and k.

Essentially the same solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain; Richard I. Hess, Rancho Palos Verdes, CA, USA; and Messiah College Problem Solving Group.

By the Pythagorean Theorem, we see that

$$(30a + 40b)^2 + (40a + 30b)^2 \;=\; (50a + kb)^2 \,.$$

Expanding and simplifying, we get $4800ab + 2500b^2 = 100abk + k^2b^2$. Dividing by b (since $b > 0$) and rearranging gives

$$b(50 - k)(50 + k) \;=\; 100a(k - 48)\,.$$

Since a, b, and k are all positive, the factors $50 - k$ and $k - 48$ must have the same sign. This happens only when $k = 49$. Setting $k = 49$, we obtain $99b = 100a$. Since 99 and 100 are relatively prime and we are seeking the smallest values of a and b, we conclude that $a = 99$ and $b = 100$. Thus, the minimum solution is $(a, b, k) = (99, 100, 49)$.

M223. *Proposed by Larry Rice, University of Waterloo, Waterloo, ON.*

The fraction $\frac{3}{10}$ can be written as the sum of two positive rational numbers with numerator 1 in exactly two ways, namely as $\frac{1}{10} + \frac{1}{5}$ and $\frac{1}{20} + \frac{1}{4}$.

Determine the number of ways that $\frac{3}{2006}$ can be expressed as the sum of two positive rational numbers with numerator 1.

Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

Let p and q be positive integers such that $\dfrac{3}{2006} = \dfrac{1}{p} + \dfrac{1}{q}$. Then

$$3pq \;=\; 2006(p + q) \;=\; 2 \cdot 17 \cdot 59(p + q)\,. \qquad (1)$$

To find values of p and q, we look at 4 cases.

Case 1: $2 \cdot 17 \cdot 59$ divides p.

Then there exists a positive integer r such that $p = 2 \cdot 17 \cdot 59r$. In this case, (1) takes the form $3qr = 2 \cdot 17 \cdot 59r + q$, or $(3r - 1)q = 2 \cdot 17 \cdot 59r$. Therefore, $q = 2 \cdot 17 \cdot 59r/(3r - 1)$. This last equation has positive integer solutions for q when $3r - 1$ takes the values 2, 17, 59, or $2 \cdot 17 \cdot 59$. [*Ed.:* the values $2 \cdot 17$, $2 \cdot 59$, and $17 \cdot 59$ cannot be expressed in the form $3r - 1$.] The corresponding values of r are 1, 6, 20, and 669, which generate the following (p, q) pairs: $(2006, 1003)$, $(12036, 708)$, $(40120, 680)$ and $(1342014, 669)$.

Case 2: $2 \cdot 17$ divides p, but 59 does not.

Then there exists a positive integer r, not a multiple of 59, such that $p = 2 \cdot 17r$ and $q = 59s$ for some positive integer s. In this case, (1) takes the form $(3r - 59)s = 2 \cdot 17r$, and then $s = 2 \cdot 17r/(3r - 59)$. The values for r, not multiples of 59, which give positive integer solutions for s are 20 and 31 (corresponding to the equations $3r - 59 = 1$ and $3r - 59 = 2 \cdot 17$). Only the second value generates a new (p, q) pair: $(1054, 1829)$.

Case 3: $2 \cdot 59$ divides p, but 17 does not.

Then there exists a positive integer r, not a multiple of 17, such that $p = 2 \cdot 59r$ and $q = 17s$ for some positive integer s. In this case, (1) takes the form $(3r - 17)s = 2 \cdot 59r$, and then $s = 2 \cdot 59r/(3r - 17)$. The values for r, not multiples of 17, which give positive integer solutions for s are 6 and 45 (corresponding to the equations $3r - 17 = 1$ and $3r - 17 = 2 \cdot 59$). Only the second value generates a new (p, q) pair: $(5310, 765)$.

Case 4 : $17 \cdot 59$ divides p, but 2 does not.

Then there exists a positive integer r, not a multiple of 2, such that $p = 17 \cdot 59r$ and $q = 2s$ for some positive integer s. In this case, (1) takes the form $(3r - 2)s = 17 \cdot 59r$, and then $s = 17 \cdot 59r/(3r - 2)$. The odd values for r, which give positive integer solutions for s are 1 and 335 (corresponding to the equations $3r - 2 = 1$ and $3r - 2 = 17 \cdot 59$). Only the second value generates a new (p, q) pair: $(336005, 670)$.

[*Ed:* The remaining possible cases, where one of the primes 2, 17, and 59 divides p and the other two do not, simply exchange the roles of p and q, and thus do not lead to any new pairs.]

In conclusion, there are exactly seven ways in which $3/2006$ can be expressed as the sum of two positive rational numbers with numerator 1:

$$\frac{1}{669} + \frac{1}{1342014}, \quad \frac{1}{670} + \frac{1}{336005}, \quad \frac{1}{680} + \frac{1}{40120}, \quad \frac{1}{708} + \frac{1}{12036},$$
$$\frac{1}{765} + \frac{1}{5310}, \quad \frac{1}{1003} + \frac{1}{2006}, \quad \frac{1}{1054} + \frac{1}{1829}.$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and Messiah College Problem Solving Group. One incomplete solution was submitted.

M224. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

Let $A(-1, 1)$ and $B(3, 9)$ be two points on the parabola $y = x^2$. Take another point $M(m, m^2)$ on the parabola lying between A and B. Let H be the point on the line segment joining A to B that has the same x–coordinate as M.

Show that if the length of MH is k units, then triangle AMB has area $2k$ square units. Does this relationship still hold if M is not between A and B?

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

The equation for the line through points $A(-1,1)$ and $B(3,9)$ can easily be found to be $y = 2x+3$. Let $M(m,m^2)$ be any point on the parabola (not necessarily between A and B). The length of the line segment joining points $M(m,m^2)$ and $H(m,2m+3)$ is $k = |m^2-2m-3|$, and the area of triangle ABM can be calculated as follows:

$$\begin{aligned}[ABC] &= \tfrac{1}{2}\left\|\overrightarrow{AM}\times\overrightarrow{AB}\right\| = \tfrac{1}{2}\|\langle m+1,m^2-1,0\rangle\times\langle 4,8,0\rangle\| \\ &= \tfrac{1}{2}\|\langle 0,0,-4m^2+8m+12\rangle\| = 2|m^2-2m-3| = 2k\,.\end{aligned}$$

Thus, the given relationship holds no matter whether M is between A and B or not.

Also solved by ESTHER MARÍA GARCÍA-CABALLERO, Universidad de Jaén, Jaén, Spain.

M225. *Proposed by Zun Shan, Normal University, China; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Define a sequence $\{x_n\}$ by $x_1 = 1/2005$ and $x_{n+1} = x_n + x_n^2$ for $n \geq 1$. Set

$$S = \frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{2005}}{x_{2006}}\,.$$

Determine $\lfloor S \rfloor$, the greatest integer not exceeding S.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

It is easy to prove, by induction, that

$$x_n = x_1(1+x_1)(1+x_1+x_1^2)\cdots(1+x_1+x_1^2+\cdots+x_1^{n-1})\,.$$

Then,

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_{n+1}} = \frac{1}{1+x_1} + \frac{1}{1+x_1+x_1^2} + \cdots + \frac{1}{\sum\limits_{i=1}^{n+1} x_1^{i-1}}\,.$$

Setting $x_1 = 1/2005$ and $n = 2005$, we see that

$$\begin{aligned}S &= \frac{1}{1+\frac{1}{2005}} + \frac{1}{1+\frac{1}{2005}+\left(\frac{1}{2005}\right)^2} + \cdots \\ &\qquad + \frac{1}{1+\frac{1}{2005}+\left(\frac{1}{2005}\right)^2+\cdots+\left(\frac{1}{2005}\right)^{2005}} = \sum_{n=1}^{2005}\frac{1}{T_n}\,,\end{aligned}$$

where $T_n = \sum\limits_{i=0}^{n}\left(\frac{1}{2005}\right)^i$.

Since $T_n < \sum\limits_{i=0}^{\infty}\left(\frac{1}{2005}\right)^i = \frac{2005}{2004}$, we have $\frac{1}{T_n} > \frac{2004}{2005}$, and therefore, $S > \sum\limits_{n=1}^{2005}\frac{2004}{2005} = 2004$. But we also have $\frac{1}{T_n} \leq \frac{1}{1+\frac{1}{2005}} = \frac{2005}{2006}$ for each n.

Therefore,

$$S < \sum_{n=1}^{2005} \frac{2005}{2006} = \frac{2005^2}{2006} < 2005\,.$$

Now, since $2004 < S < 2005$, we see that $\lfloor S \rfloor = 2004$.

One incorrect solution was received.

Problem of the Month

Ian VanderBurgh

This month's problem combines some geometry, some number theory, and some algebra:

Problem. A *Pythagorean triple* is a set of three positive integers (a, b, c) such that $a < b < c$ and $a^2 + b^2 = c^2$. Determine all primitive Pythagorean triples that satisfy the equation $a + b - c = 20$.

(A variation of this problem was used at the annual Canadian Mathematics Competitions Mathematics Contests Seminar in June 2006.)

No, "primitive" does not mean that the Pythagorean triples live in caves! A Pythagorean triple (a, b, c) is called *primitive* if the integers a, b, and c do not all share a common factor greater than 1.

First let's solve this problem in a way that does not assume any prior knowledge on the subject of Pythagorean triples.

Solution 1: From the given equation $a + b - c = 20$, we get

$$\begin{aligned}
a + b - 20 &= c\,, \\
(a + b - 20)^2 &= c^2\,, \\
a^2 + b^2 + 400 + 2ab - 40a - 40b &= c^2\,, \\
400 + 2ab - 40a - 40b &= 0 \qquad (\text{since } a^2 + b^2 = c^2), \\
ab - 20a - 20b &= -200\,, \\
ab - 20a - 20b + 400 &= 200\,, \\
(a - 20)(b - 20) &= 200\,.
\end{aligned}$$

The factors $a - 20$ and $b - 20$ on the left side of this equation must be integers because a and b are integers. Since $a + b - c = 20$ and $a < b < c$, we have $b > a = 20 + (c - b) > 20$. Thus, the factors $a - 20$ and $b - 20$ are both positive, and $b - 20 > a - 20$.

The easiest thing to do at this stage is to make a table of all possible pairs $a - 20$, $b - 20$.

$a-20$	$b-20$	a	b	$c=a+b-20$	Primitive
1	200	21	220	221	Yes
2	100	22	120	122	No
4	50	24	70	74	No
5	40	25	60	65	No
8	25	28	45	53	Yes
10	20	30	40	50	No

Thus, the two primitive Pythagorean triples that work are $(21, 220, 221)$ and $(28, 45, 53)$. (Notice that each triple that we rejected above had a common factor of 2 among the three integers.)

That was a good way to solve this problem. It did not require any advanced machinery.

However, there is some structure to the set of all primitive Pythagorean triples which we could have used. Every primitive Pythagorean triple (a, b, c) with $a < b < c$ can be written in one of the forms $(2mn, m^2 - n^2, m^2 + n^2)$ or $(m^2 - n^2, 2mn, m^2 + n^2)$, where m and n are positive integers of opposite parity (that is, one is even and one is odd) with no common factors greater than 1. We will take this on faith for a little while and come back to the reasoning later.

Solution 2: Using the above forms for primitive Pythagorean triples, we rewrite the equation $a + b - c = 20$ as $m^2 - n^2 + 2mn - (m^2 + n^2) = 20$. (Notice that we get the same equation from each of the two possible forms.) Thus, $2mn - 2n^2 = 20$, or $n(m - n) = 10$. Here m and n are positive integers, and therefore n and $m - n$ are positive factors of 10.

Now we make a table showing the possibilities for n and $m - n$.

n	$m-n$	m	$2mn$	m^2-n^2	m^2+n^2	(a,b,c)	Primitive
1	10	11	22	120	122	$(22, 120, 122)$	No
2	5	7	28	45	53	$(28, 45, 53)$	Yes
5	2	7	70	24	74	$(24, 70, 74)$	No
10	1	11	220	21	221	$(21, 220, 221)$	Yes

Therefore, the primitive Pythagorean triples satisfying $a + b - c = 20$ are $(28, 45, 53)$ and $(21, 220, 221)$.

Wonderful! This solution was a fair bit less complicated than Solution 1. Interestingly, the "endgame" was remarkably similar to that in Solution 1.

The problem can be interpreted geometrically. Each Pythagorean triple (a, b, c) corresponds to a right triangle whose sides have integer lengths a, b, and c. The equation $a^2 + b^2 = c^2$ is the Pythagorean Theorem (which, of course, is why we call (a, b, c) a Pythagorean triple). The diameter of the inscribed circle of such a triangle is $a+b-c$. (Can you show this?) A primitive Pythagorean triple corresponds to a right triangle which is not similar to any smaller right triangle with integer side lengths. The problem asks us to find all such triangles which have an inscribed circle of diameter 20.

Postscript.

Before wrapping up this article, we should look at the formulae for generating primitive Pythagorean triples, which we will abbreviate as PPTs.

First, we check that any triple of the form $(2mn, m^2 - n^2, m^2 + n^2)$ or $(m^2 - n^2, 2mn, m^2 + n^2)$ is, in fact, a Pythagorean triple. This requires squaring each of the three terms and checking that the sum of the first two squares equals the third. I'll let you think about why such a triple must be primitive, given that m and n are positive, have opposite parity, and have no common factors greater than 1.

Next, we check that every PPT is of one of these two forms. We will use one of the well-known methods to do this and break the procedure into a number of steps. (There are certainly other methods that can be used to derive these formulae.)

Step 0: Odd and even perfect squares.

If a is even, then $a = 2k$ for some integer k; thus, $a^2 = 4k^2$ which gives a remainder of 0 when divided by 4.

If a is odd, then $a = 2k+1$ for some integer k; thus, $a^2 = 4k^2+4k+1$ which gives a remainder of 1 when divided by 4.

This doesn't seem relevant immediately, but hang on!

Step 1: Parities of a, b, c.

Suppose (a, b, c) is a PPT. If a and b were both even, then c would be even, since $c^2 = a^2 + b^2$. Then a, b, and c would would all be divisible by 2, contradicting our assumption that the Pythagorean triple (a, b, c) is primitive.

If a and b were both odd, then c would be even. In this case, the remainder upon dividing c^2 by 4 would be 0 and the remainder upon dividing $a^2 + b^2$ by 4 would be 2, leading to a contradiction.

Therefore, having eliminated all of the other cases, we are left with the case where one of a or b is even and the other is odd (implying that c is odd). Assume that a is odd and b is even.

Step 2: Some algebra.

Since $a^2 + b^2 = c^2$, then $b^2 = c^2 - a^2 = (c-a)(c+a)$. Dividing both sides by 4, we get $\left(\frac{1}{2}b\right)^2 = \left(\frac{1}{2}(c-a)\right)\left(\frac{1}{2}(c+a)\right)$. Note that $\frac{1}{2}b$ is an integer, because b is even, and $\frac{1}{2}(c-a)$ and $\frac{1}{2}(c+a)$ are integers, because a and c are both odd (making $c-a$ and $c+a$ even).

Step 3: Analysis of the factors $\frac{1}{2}(c-a)$ and $\frac{1}{2}(c+a)$.

If $\frac{1}{2}(c-a)$ and $\frac{1}{2}(c+a)$ had a common factor larger than 1, then this factor would also be a factor of their sum (which equals c) and of their difference (which equals a). Then b would also share this common factor, since $b^2 = c^2 - a^2$. This can't happen, because a, b, and c have no common factor. Also, $\frac{1}{2}(c-a)$ and $\frac{1}{2}(c+a)$ cannot both be odd. If they were odd, then their sum c and their difference a would be even (and we know that they are odd).

So $\frac{1}{2}(c-a)$ and $\frac{1}{2}(c+a)$ have no common factors, one is even and one is odd, and their product is a perfect square.

Step 4: Introduction of m and n.

It must be the case, therefore, that each of $\frac{1}{2}(c-a)$ and $\frac{1}{2}(c+a)$ is a perfect square itself. (Think about this.) We can write $\frac{1}{2}(c-a) = n^2$ and $\frac{1}{2}(c+a) = m^2$, with m and n positive integers.

Then m and n cannot have a common factor larger than 1 (since m^2 and n^2 don't), and m and n must have opposite parity (since m^2 and n^2 do).

Step 5: The big finish.

Since $\frac{1}{2}(c-a) = n^2$ and $\frac{1}{2}(c+a) = m^2$, then $c = m^2 + n^2$ (adding) and $a = m^2 - n^2$ (subtracting). Also, $\left(\frac{1}{2}b\right)^2 = m^2n^2$; that is, $b^2 = 4m^2n^2$. Then $b = 2mn$, since b, m, and n are positive.

Thus, each PPT has exactly the form that we had hoped.

Mayhem Year End Wrap Up

Shawn Godin

Another year has come and gone. For me, on this cold, windy Sunday morning in October, this is one of my last tasks as Editor of the Mayhem section. This task is bitter-sweet for me. For the last six years I have worked with some great people and reshaped Mayhem to better fit inside its mother journal, ***Crux Mathematicorum***. The job has consumed a big chunk of my meager free time, so the freedom will be a welcome change. Having said that, I must add that I will miss the wonderful problems, solutions, and articles sent in by our readers and staff members. It has been an honour to be associated with such a well-respected and world-class journal.

At this point, I need to thank a couple of members of the staff, without whom Mayhem would not be. First, I must thank Mayhem Assistant Editor and future Editor, JEFF HOOPER. Jeff's thoughtful suggestions always help deliver a better issue. Secondly, I must thank IAN VANDERBURGH. Ian continues to present our readers with great material in his regular column, The Problem of the Month.

I also need to thank those people behind the scenes: ED BARBEAU, ROBERT BILINSKI, MARK BREDIN, RICHARD HOSHINO, MONIKA KHEBEIS, RON LANCASTER, JOHN GRANT McLOUGHLIN, PAUL OTTAWAY, LARRY RICE, ERIC ROBERT, BRUCE SHAWYER, and GRAHAM WRIGHT. They were there when I needed them and their contributions were always appreciated, although not always acknowledged. Thanks everyone!

All the best of the season to all our readers and contributors! I hope you have a great year in 2007. Thank you for making the last six years so enjoyable. Happy problem solving!